Title: Quantum Information Lecture

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Mutual Information

Amount of ignorance removed from a random variable Y.

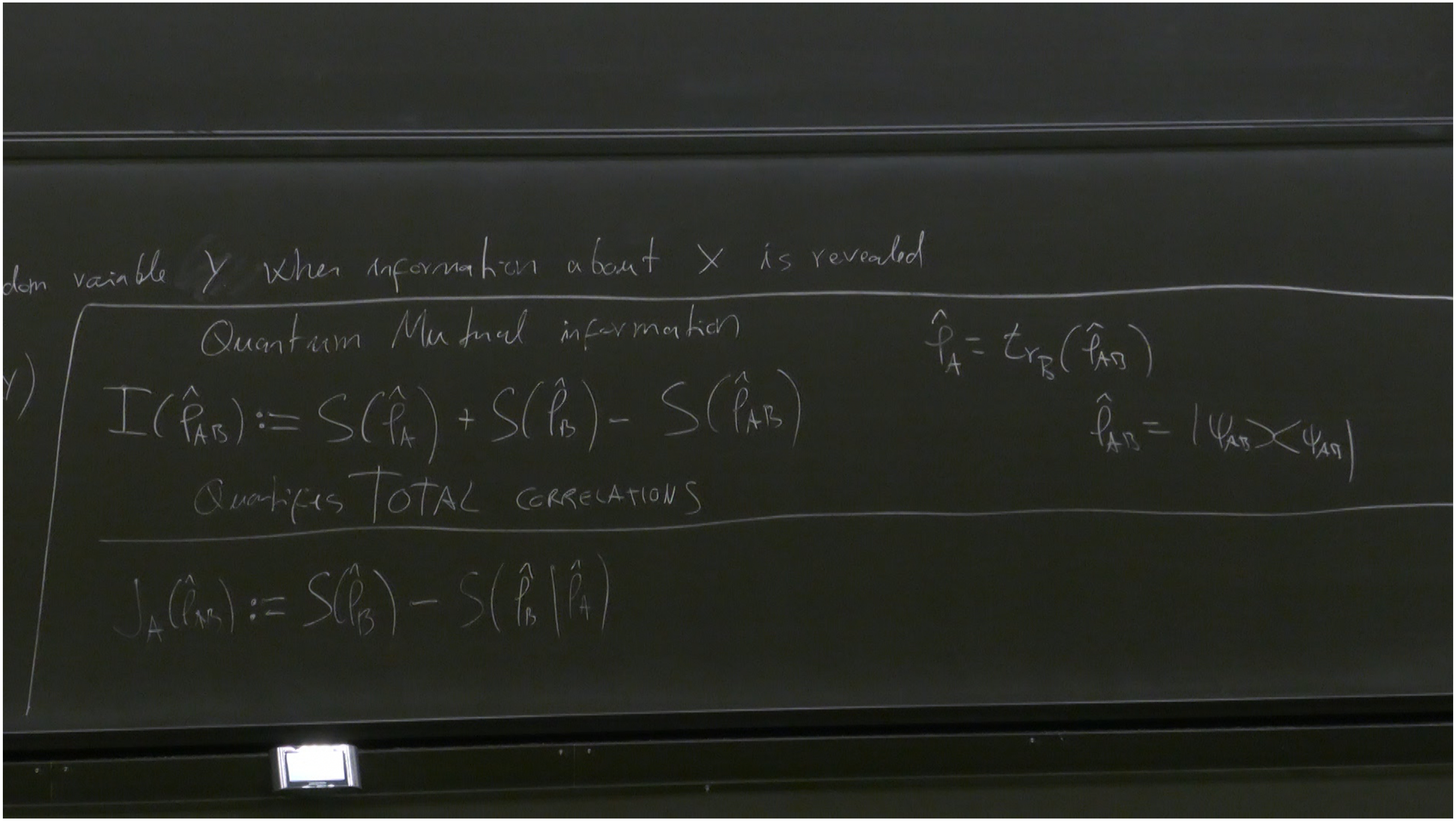
$H(X|Y)$ conditional entropy

$$I(X,Y) := H(X) + H(Y) - H(X,Y)$$

$$\Updownarrow$$

$$I(X,Y) := H(X) - H(X|Y)$$

where $$H(X|Y) = -\sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(y)}\right)$$


dom variable Y when information about X is revealed
Quantum Mutual information
$I(\hat{\rho}_{AB}) := S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB})$
Quantifies TOTAL CORRELATIONS
$\hat{\rho}_A = tr_B(\hat{\rho}_{AB})$
$\hat{\rho}_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$
$J_A(\hat{\rho}_{AB}) := S(\hat{\rho}_B) - S(\hat{\rho}_B | \hat{\rho}_A)$

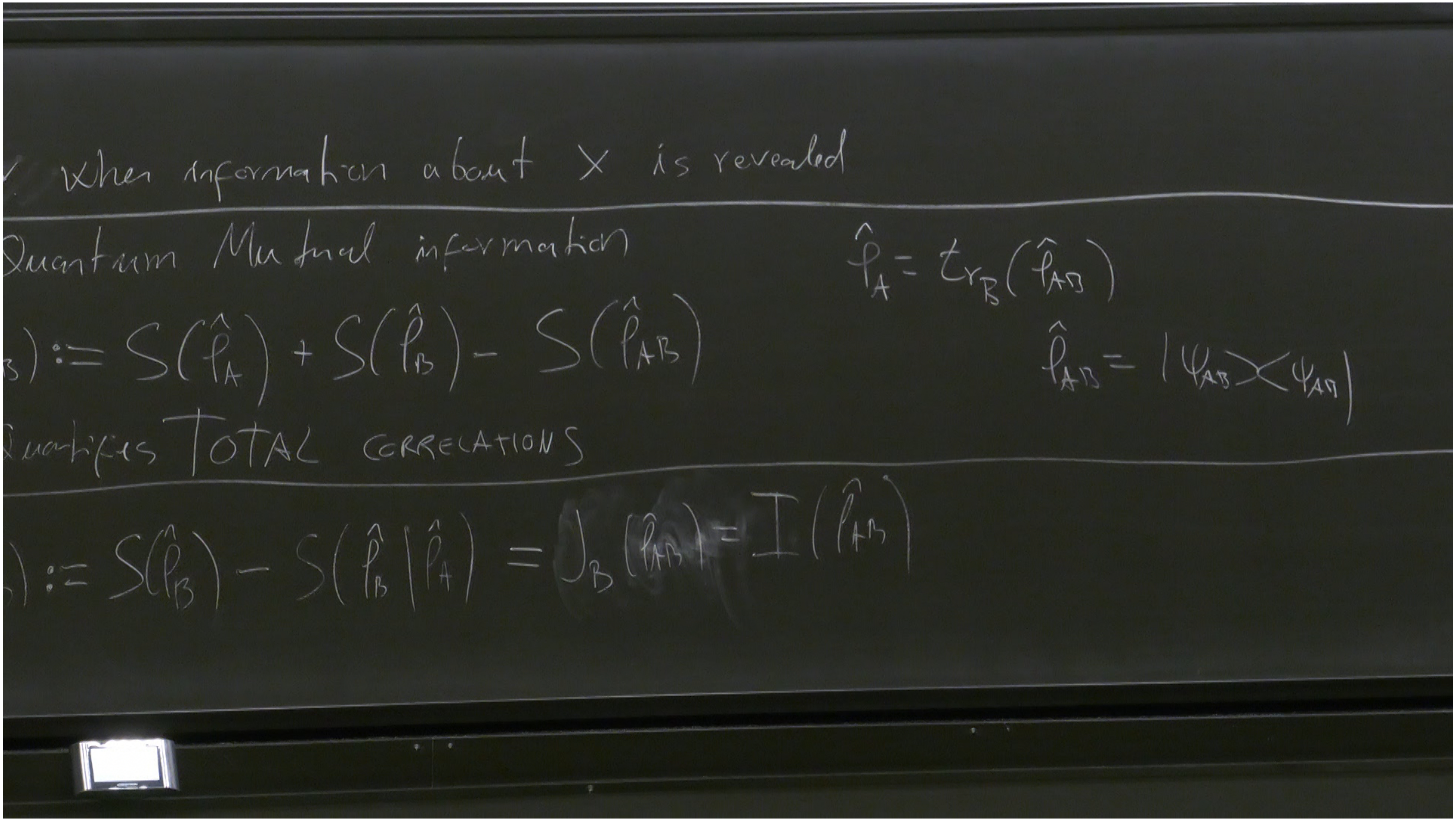
$$\hat{\rho}_{AB} = \frac{1}{2}\left(|10\rangle\langle 10| + |01\rangle\langle 01| \right)$$

$$\hat{P}_{B=0} = \mathbb{1}_A \otimes |0\rangle_B\langle 0|$$

$$\rightarrow \quad \hat{\rho}'_{AB} = |10\rangle\langle 10|$$

$$\hat{\rho}_A = \mathrm{tr}_B\, \hat{\rho}'_{AB} = |1\rangle\langle 1|$$

$$S(\rho_A | \hat{P}_{B=0}) = 0$$


when information about X is revealed
Quantum Mutual information
ρ̂_A = tr_B(ρ̂_AB)
B) := S(ρ̂_A) + S(ρ̂_B) − S(ρ̂_AB)
ρ̂_AB = |ψ_AB⟩⟨ψ_AB|
Quantifies TOTAL CORRELATIONS
) := S(ρ̂_B) − S(ρ̂_B | ρ̂_A) = J_B(ρ̂_AB) = I(ρ̂_AB)

Quantum conditional entropy

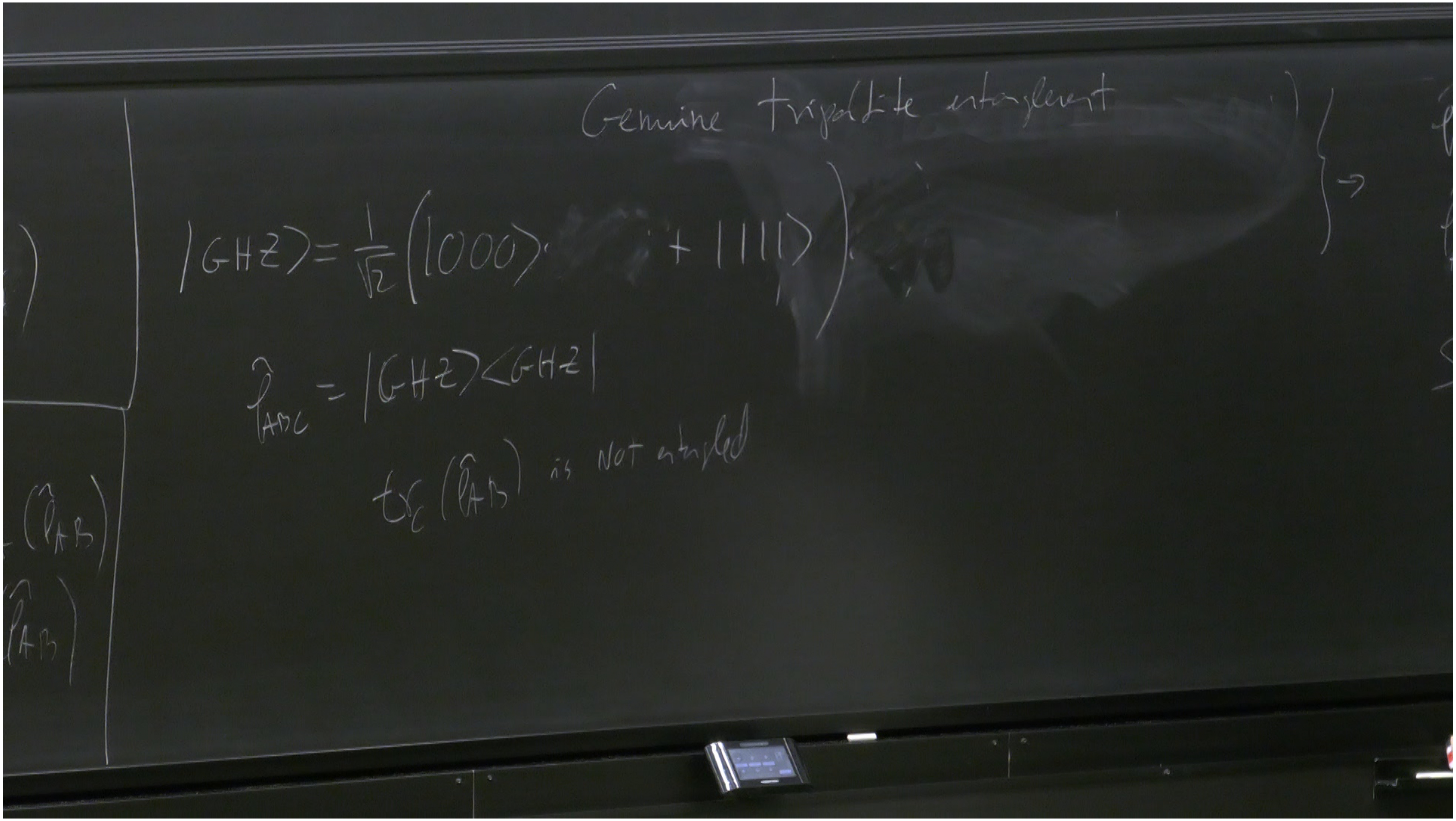
$$S(\hat{\rho}_B | \hat{\rho}_A) = \min_{\{\Pi_j^A\}} S(\hat{\rho}_B | \Pi_j^A)$$

$\hat{\rho}_{AB} =$

Quantum Discord

$$D_A(\hat{\rho}_{AB}) := I(\hat{\rho}_{AB}) - J_A(\hat{\rho}_{AB})$$

$$D_B(\hat{\rho}_{AB}) := I(\hat{\rho}_{AB}) - J_B(\hat{\rho}_{AB})$$


Genuine tripartite entanglement
|GHZ> = 1/√2 (|000> + |111>)
ρ̂_ABC = |GHZ><GHZ|
tr_C (ρ̂_AB) is NOT entangled

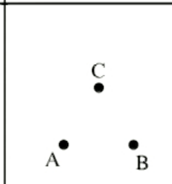
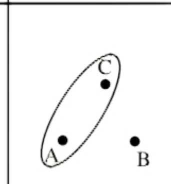
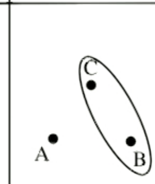
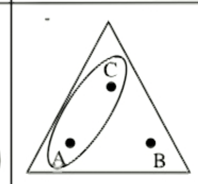
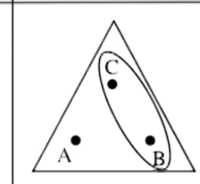
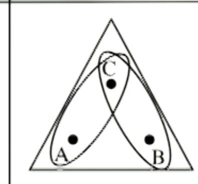
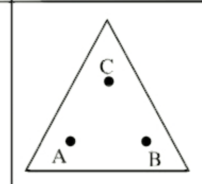
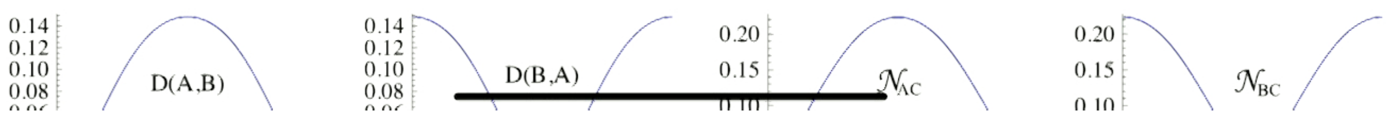
$\vert\alpha\rangle$	$\pm\vert0\rangle$	$\neq\pm\vert0\rangle$	$\pm\vert0\rangle$	$\pm\vert1\rangle$	$\neq\pm\{\vert0\rangle,\vert1\rangle\}$	$\neq\pm\{\vert0\rangle,\vert1\rangle\}$	$\pm\vert1\rangle$
$\vert\beta\rangle$	$\pm\vert0\rangle$	$\pm\vert0\rangle$	$\neq\pm\vert0\rangle$	$\neq\pm\{\vert0\rangle,\vert1\rangle\}$	$\pm\vert1\rangle$	$\neq\pm\{\vert0\rangle,\vert1\rangle\}$	$\pm\vert1\rangle$
Entanglement structure							
$D(A,B)$	0	0	0	>0	0	>0	0
$D(B,A)$	0	0	0	0	>0	>0	0

FIG. 1. The relationship between the entanglement structure of $|\psi\rangle_{ABC}$ and the discord in ρ_{AB}. For given conditions on $|\alpha\rangle$ and $|\beta\rangle$ we display the resulting entanglement structure and the results for the discords $D(A,B)$ and $D(B,A)$. In the structure diagrams an ellipse represents the presence of bipartite entanglement while a triangle represents the presence of tripartite entanglement.



VI. CONCLUSIONS

We studied the relationship between the discord in an unentangled bipartite system AB and the bipartite and tripartite entanglement found in its purification ABC.

We found that both purely tripartite entanglement and bipartite entanglement between AC (or BC) are necessary to have nonzero discord $D(A, B)$ (or $D(B, A)$). In fact, tripartite entanglement in the purification is required for *any* correlations between A and B, either quantum or classical. The further addition of bipartite entanglement is what then allows these correlations to take on a quantum nature. While simple, this realization has significant explanatory power. For example, we found that there is a trade-off between the two directions of discord between A and B. Both cannot be large at the same time because their strength relies on the strength of the AC or BC entanglement respectively. However, the AC and BC entanglements cannot be strong simultaneously because of entanglement monogamy. We have therefore shown that the asymmetry between $D(A, B)$ or $D(B, A)$ stems from entanglement monogamy.

While our primary result does not in general lead to an implication in the opposite direction (namely the lack of discord in AB does not imply that the purification ABC on the level of 2-partite quantum correlations as discord. It is tempting to conjecture that the relationship between discord and purified entanglement will continue to play a central role in the study of higher-party entanglement and discord.

To this end, it should be very interesting to generalize our strategy of purifying discord to more than three quantum systems. Also, it should be possible and very interesting, also for practical purposes, to investigate the corresponding Hamiltonians, i.e., to study which types of interactions give rise to the structures of discord and its purification that we consider in this program.

VII. ACKNOWLEDGEMENTS

The authors would like to give many thanks to Nicolai Friis and Marcus Huber for their very helpful feedback and input that has greatly improved the quality of the paper. The authors gratefully acknowledge support through the Banting, Discovery and Canada Research Chairs programmes of NSERC. AK is grateful to the University of Queensland for its hospitality during his sabbatical visit.

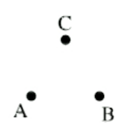
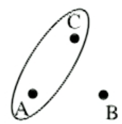
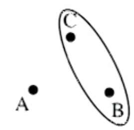
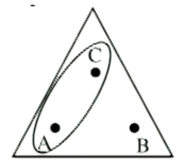
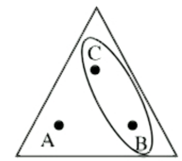
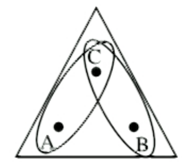
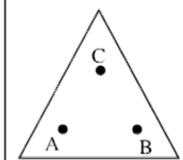
$\vert\alpha\rangle$	$\pm\vert 0\rangle$	$\neq\pm\vert 0\rangle$	$\pm\vert 0\rangle$	$\pm\vert 1\rangle$	$\neq\pm\{\vert 0\rangle,\vert 1\rangle\}$	$\neq\pm\{\vert 0\rangle,\vert 1\rangle\}$	$\pm\vert 1\rangle$
$\vert\beta\rangle$	$\pm\vert 0\rangle$	$\pm\vert 0\rangle$	$\neq\pm\vert 0\rangle$	$\neq\pm\{\vert 0\rangle,\vert 1\rangle\}$	$\pm\vert 1\rangle$	$\neq\pm\{\vert 0\rangle,\vert 1\rangle\}$	$\pm\vert 1\rangle$
Entanglement structure							
$D(A,B)$	0	0	0	>0	0	>0	0
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FIG. 1. The relationship between the entanglement structure of $|\psi\rangle_{ABC}$ and the discord in ρ_{AB}. For given conditions on $|\alpha\rangle$ and $|\beta\rangle$ we display the resulting entanglement structure and the results for the discords $D(A,B)$ and $D(B,A)$. In the structure diagrams an ellipse represents the presence of bipartite entanglement while a triangle represents the presence of tripartite entanglement.

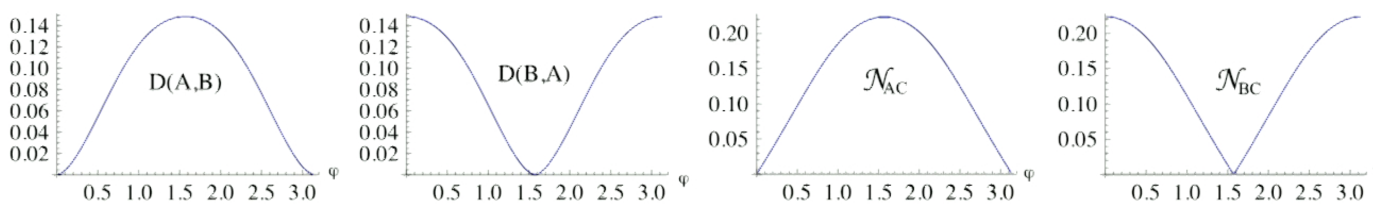

FIG. 2. The behavior of discord and bipartite negativity as we move in the (α,β) plane along a trajectory of constant tripartite entanglement $\pi_{ABC}=0.2$. $\varphi\in[0,2\pi)$ is a va[redacted]ory through (α,β) space.

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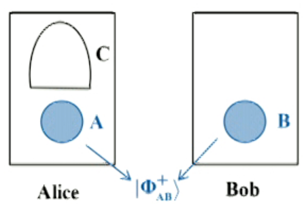
1. Alice an Bob prepare their qubits A and B in a Bell pair and B goes away.
2. Alice makes a joint measurement of both her qubits C (the input that is to be teleported) and A (her half of the maximally entangled bipartite state). This generates 2 bits of information and, thus, the states C and A are destroyed. The measurement is done in the Bell basis.
3. Then, Alice sends the result of her measurement to Bob via a classical channel.
4. Bob is able to recover the input state by using the information from Alice and the quatum correlations contained in the originally entangled state. Bob uses the classical information received to decide how to locally act on B to transform it into C.

So now, let us see how the protocol works in more detail.

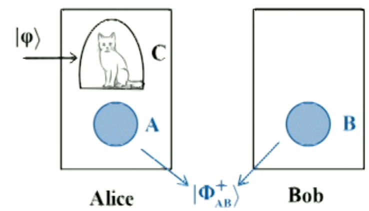
1. Firstly, Alice and Bob prepare a Bell state (maximally entangled) for their qubits A, B. Let us assume that this state is the following Bell state:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right). \tag{94}$$



[11]Versions of the protocol with non-maximally entangled states do exist. Teleportation would happen with less fidelity in those cases.

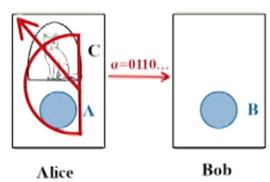
$$|\varphi\rangle_C = \alpha|0\rangle + \beta|1\rangle \tag{95}$$



The full state of the system of all 3 qubits can be written as:

$$\begin{aligned}
|\Psi_{ABC}\rangle &= |\varphi\rangle_C \otimes |\Phi^+\rangle_{AB} \\
&= \frac{1}{\sqrt{2}}(\alpha|0\rangle_C + \beta|1\rangle_C) \otimes (|00\rangle_{AB} + |11\rangle_{AB}) \\
&= \frac{\alpha}{\sqrt{2}}(|000\rangle_{CAB} + |011\rangle_{CAB}) + \frac{\beta}{\sqrt{2}}(|100\rangle_{CAB} + |111\rangle_{CAB}) .
\end{aligned} \tag{96}$$

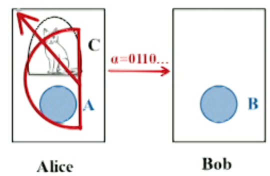
2. Alice makes a joint measurement for her qubits, C, A, in the Bell basis, so, we manipulate the full state (96) to be expressed in this basis.



The Bell basis is:

$$
\begin{aligned}
|\Psi_{\rm ABC}\rangle &= |\varphi\rangle_{\rm C} \otimes |\Phi^+\rangle_{\rm AB} \\
&= \frac{1}{\sqrt{2}}(\alpha|0\rangle_{\rm C} + \beta|1\rangle_{\rm C}) \otimes (|00\rangle_{\rm AB} + |11\rangle_{\rm AB}) \\
&= \frac{\alpha}{\sqrt{2}}\left(|000\rangle_{\rm CAB} + |011\rangle_{\rm CAB}\right) + \frac{\beta}{\sqrt{2}}\left(|100\rangle_{\rm CAB} + |111\rangle_{\rm CAB}\right). \qquad (96)
\end{aligned}
$$

2. Alice makes a joint measurement for her qubits, C, A, in the Bell basis, so, we manipulate the full state (96) to be expressed in this basis.



The Bell basis is:

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right), \qquad (97)
\end{aligned}
$$

which yields for the original states:

$$
\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right), \qquad |11\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right) \\
|01\rangle &= \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right), \qquad |10\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle - |\Psi^-\rangle\right). \qquad (98)
\end{aligned}
$$

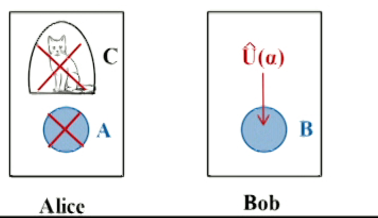
So, we can express the full state of the 3 qubits as:

$$
\begin{aligned}
|\Psi_{\text{ABC}}\rangle = \; & \frac{1}{2}|\Phi^+\rangle_{\text{CA}} \otimes (\alpha|0\rangle_{\text{B}} + \beta|1\rangle_{\text{B}}) \\
& + \frac{1}{2}|\Phi^-\rangle_{\text{CA}} \otimes (\alpha|0\rangle_{\text{B}} - \beta|1\rangle_{\text{B}}) \\
& + \frac{1}{2}|\Psi^+\rangle_{\text{CA}} \otimes (\alpha|1\rangle_{\text{B}} + \beta|0\rangle_{\text{B}}) \\
& + \frac{1}{2}|\Psi^-\rangle_{\text{CA}} \otimes (\alpha|1\rangle_{\text{B}} - \beta|0\rangle_{\text{B}})\,. \qquad (99)
\end{aligned}
$$

Now when Alice carries out the Bell measurement, given the structure of the joint state, The resulting state of Bob is some Bloch sphere rotation of the original state of C, and the exact rotation is one to one related to the outcome of the measurement. If Bob gets the information about which of the 4 states Alice measures, he will know how to alter his state to recover the input state. For each outcome, Bob will have one of the following states:

$$
|\varphi\rangle_{\text{B}} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} a \\ -b \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} b \\ a \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -b \\ a \end{pmatrix}, \qquad (100)
$$

where we assume $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. All that Bob needs is the initial information about measurement of Alice so that he can act on his state with the appropriate local operator and recover the input state $|\varphi\rangle_{\text{C}} = \begin{pmatrix} a \\ b \end{pmatrix}$.



The local operators corresponding to each measurement are summarised in the following table:

A measurement	B state	Local operation used	B final state
$\lvert\Phi^+\rangle_{\mathrm{CA}}$	$\lvert\varphi\rangle_{\mathrm{B}} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$	$\mathbb{1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\lvert\varphi\rangle_{\mathrm{C}} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$
$\lvert\Phi^-\rangle_{\mathrm{CA}}$	$\lvert\varphi\rangle_{\mathrm{B}} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$	$\hat{\sigma}_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\lvert\varphi\rangle_{\mathrm{C}} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$
$\lvert\Psi^+\rangle_{\mathrm{CA}}$	$\lvert\varphi\rangle_{\mathrm{B}} = \begin{pmatrix}\beta\\ \alpha\end{pmatrix}$	$\hat{\sigma}_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\lvert\varphi\rangle_{\mathrm{C}} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$
$\lvert\Psi^-\rangle_{\mathrm{CA}}$	$\lvert\varphi\rangle_{\mathrm{B}} = \begin{pmatrix}-\beta\\ \alpha\end{pmatrix}$	$\mathrm{i}\hat{\sigma}_y = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\lvert\varphi\rangle_{\mathrm{C}} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

Remark: Note that what is being teleported is not the state itself, but the quantum information about the state. The state itself is destroyed when Alice performs the measurement.

7.5 Stinespring's dilation theorem

According to Stinespring's theorem, we are always allowed to interpret mixedness as entanglement, if we encode our system in a higher dimensional Hilbert space. Let us make this statement more rigorous. First we will define the notion of the purification of a state.

Definition 7.5 [redacted]