Title: QFT III Lecture

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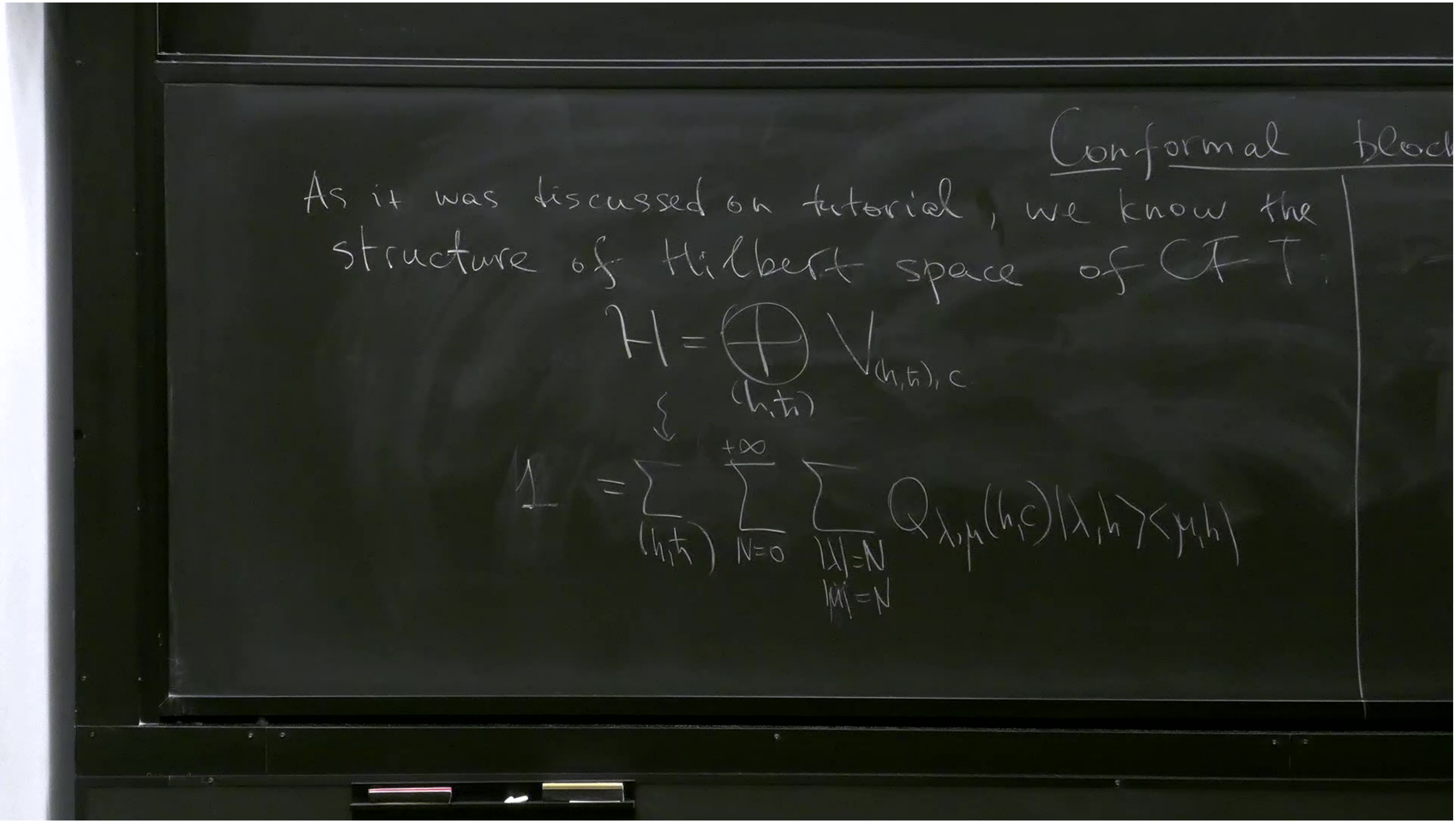
Collection: QFT III 2023/24

Date: March 27, 2024 - 10:15 AM

URL: https://pirsa.org/24030039



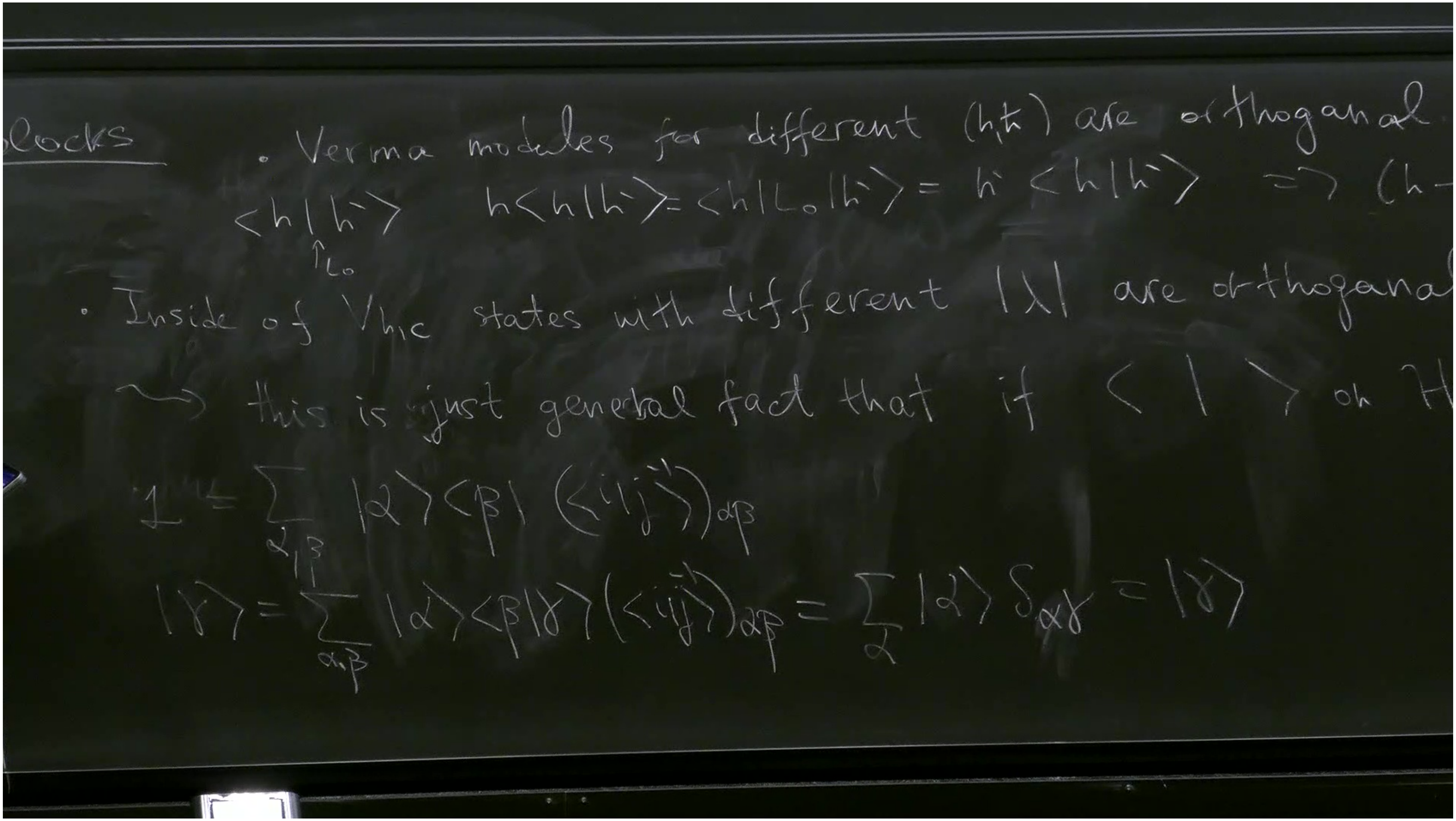
Conformal bloc

As it was discussed on tutorial, we know the structure of Hilbert space of CFT:

$$\mathcal{H} = \bigoplus_{(h,\bar{h})} V_{(h,\bar{h}),c}$$

$$\Big\downarrow$$

$$\mathbb{1} = \sum_{(h,\bar{h})} \sum_{N=0}^{+\infty} \sum_{\substack{|\lambda|=N \\ |\mu|=N}} Q_{\lambda,\mu}(h,c) |\lambda,h\rangle\langle\mu,h|$$



locks

- Verma modules for different $(h,\bar{h})$ are orthoganal

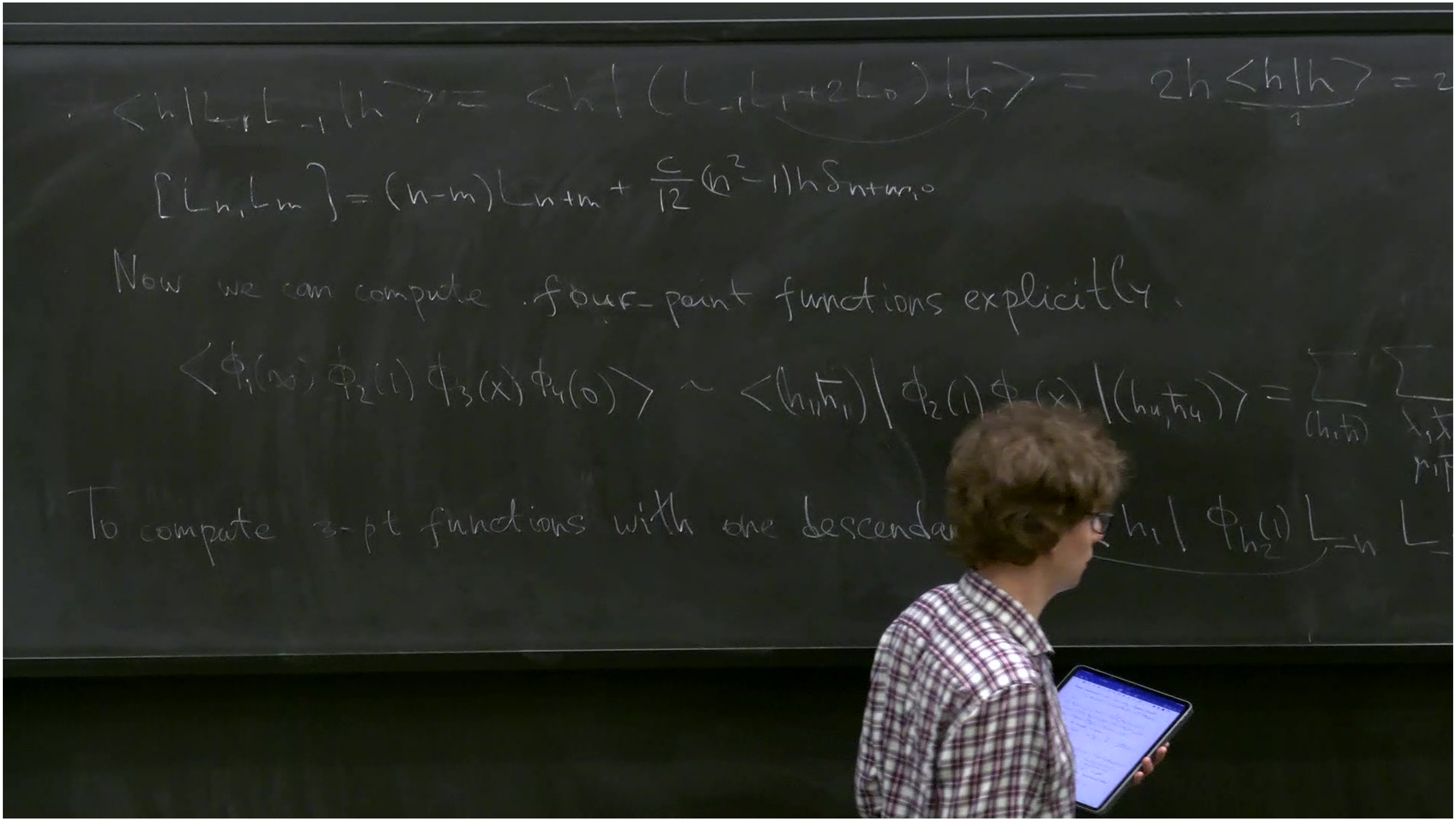
$\langle h | h' \rangle$ $\uparrow L_0$ $\quad h\langle h|h'\rangle = \langle h|L_0|h'\rangle = h'\langle h|h'\rangle \Rightarrow (h-$

- Inside of $V_{h,c}$ states with different $|\lambda|$ are orthoganal

⟶ this is just general fact that if $\langle \ | \ \rangle$ on H

$$1 = \sum_{\alpha,\beta} |\alpha\rangle\langle\beta| \left(\langle i|j\rangle^{-1}\right)_{\alpha\beta}$$

$$|\gamma\rangle = \sum_{\alpha,\beta} |\alpha\rangle\langle\beta|\gamma\rangle\left(\langle i|j\rangle^{-1}\right)_{\alpha\beta} = \sum_{\alpha} |\alpha\rangle\,\delta_{\alpha\gamma} = |\gamma\rangle$$



$\langle h|L_1 L_{-1}|h\rangle = \langle h|(L_{-1}L_1+2L_0)|h\rangle = 2h\underbrace{\langle h|h\rangle}_{1} = 2$

$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^2-1)n\,\delta_{n+m,0}$

Now we can compute four-point functions explicitly.

$\langle \phi_1(\infty)\phi_2(1)\phi_3(x)\phi_4(0)\rangle \sim \langle (h_1,\bar{h}_1)|\phi_2(1)\phi(x)|(h_4,\bar{h}_4)\rangle = \sum_{(h,\bar{h})}\sum_{\lambda,\bar{\lambda},\mu,\bar{\mu}}$

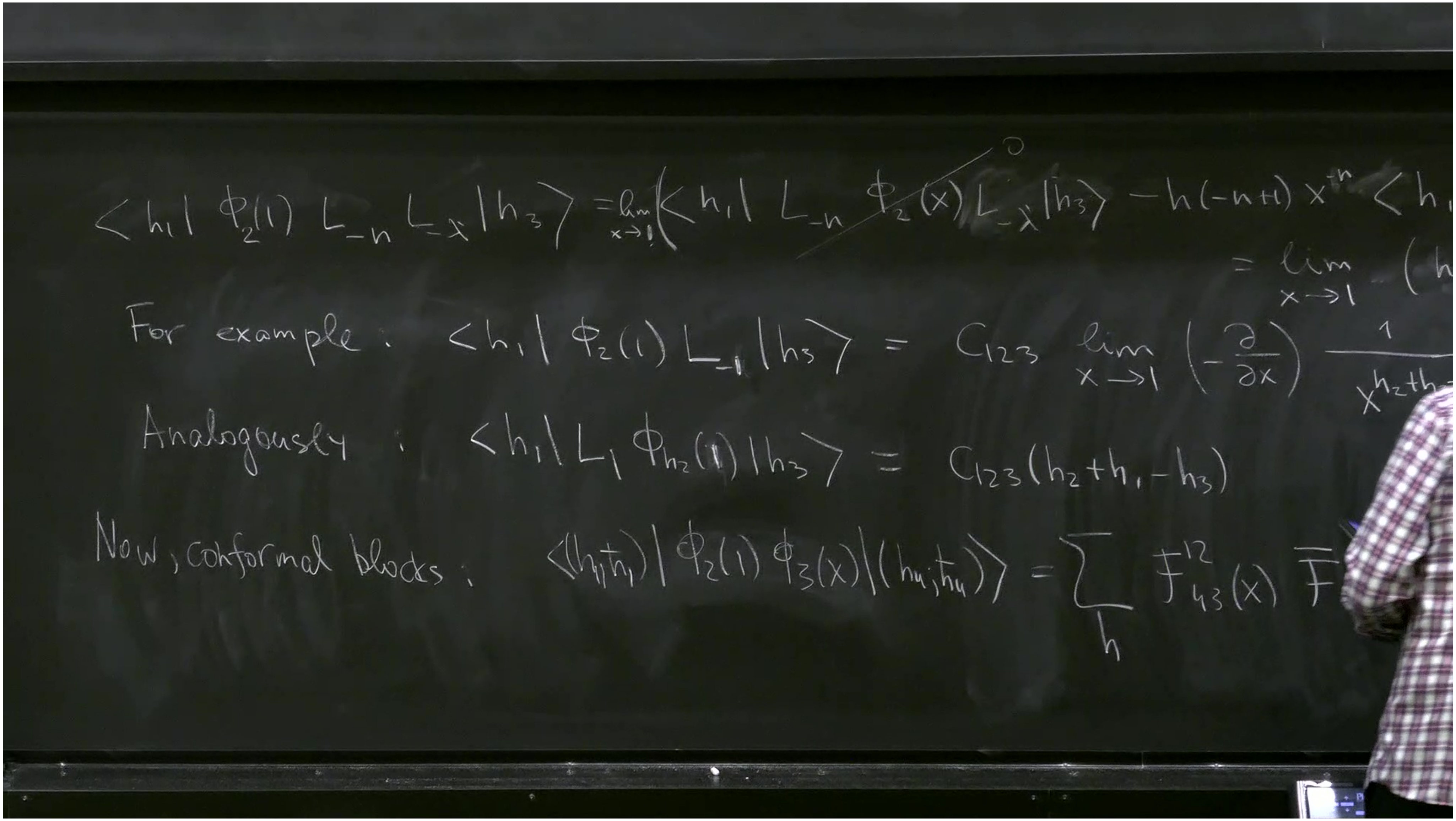
To compute 3-pt functions with one descendan $h_1|\phi_{h_2}(1)L_{-n}L$

$$\underbrace{n|h\rangle}_{1} = 2h \rightsquigarrow Q_{\Box,\Box}(h,c) = \frac{1}{2h}$$

$$= \sum_{(h,\bar{h})} \sum_{\substack{\lambda,\bar{\lambda}\\ \mu,\bar{\mu}}} \langle (h_1,\bar{h}_1)| \phi_2(1) | \lambda,\bar{\lambda},(h,\bar{h})\rangle Q_{\lambda,\mu}(h,c)\, Q_{\bar{\lambda},\bar{\mu}}(\bar{h},c) \langle \mu,\bar{\mu},(h,\bar{h}) | \phi_3(x) | (h_4,\bar{h}_4)\rangle$$

$${}_2(1)\, L_{-n} L_{-\lambda} |h_3\rangle$$

$$[L_n, \phi(w,\bar{w})] = h(n+1) w^n \phi(w,\bar{w}) + w^{n+1} \partial\phi(w,\bar{w})$$



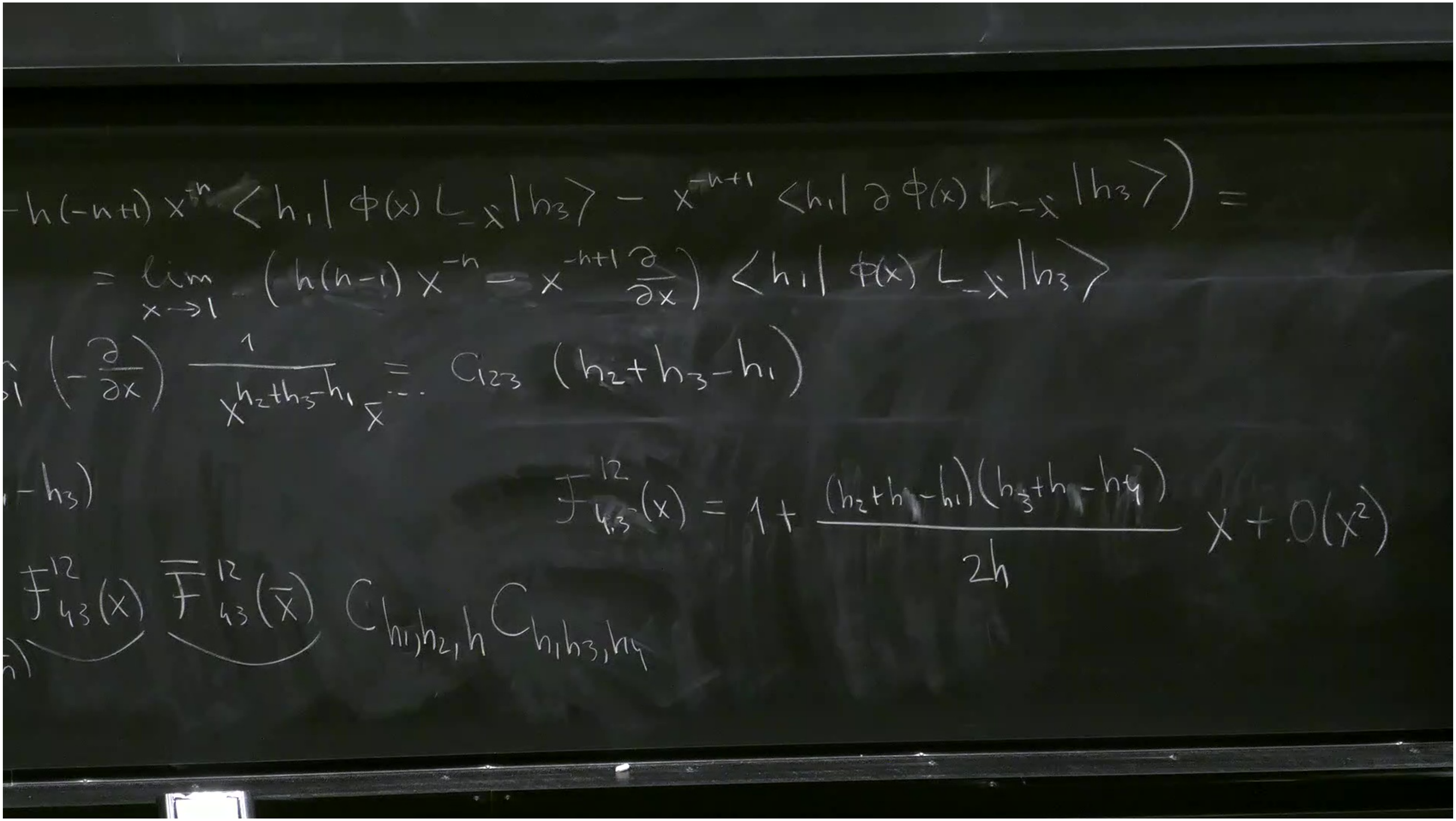
$$\langle h_1 | \phi_2(1) L_{-n} L_{-\lambda} | h_3 \rangle = \lim_{x \to 1} \Big(\langle h_1 | \cancelto{0}{L_{-n} \phi_2(x)} L_{-\lambda} | h_3 \rangle - h(-n+1) x^{-n} \langle h$$

$$= \lim_{x \to 1} \big(h$$

For example: $\langle h_1 | \phi_2(1) L_{-1} | h_3 \rangle = C_{123} \lim_{x \to 1} \left(-\frac{\partial}{\partial x} \right) \frac{1}{x^{h_2 + h}}$

Analogously: $\langle h_1 | L_1 \phi_{h_2}(1) | h_3 \rangle = C_{123} (h_2 + h_1 - h_3)$

Now, conformal blocks: $\langle (h_1, \bar{h}_1) | \phi_2(1) \phi_3(x) | (h_4, \bar{h}_4) \rangle = \sum_h \mathcal{F}^{12}_{43}(x) \bar{\mathcal{F}}$



$$-h(-n+1)\,x^{-n}\langle h_1|\,\phi(x)\,L_{-\dot{x}}|h_3\rangle - x^{-n+1}\langle h_1|\,\partial\phi(x)\,L_{-\dot{x}}|h_3\rangle\Big) =$$

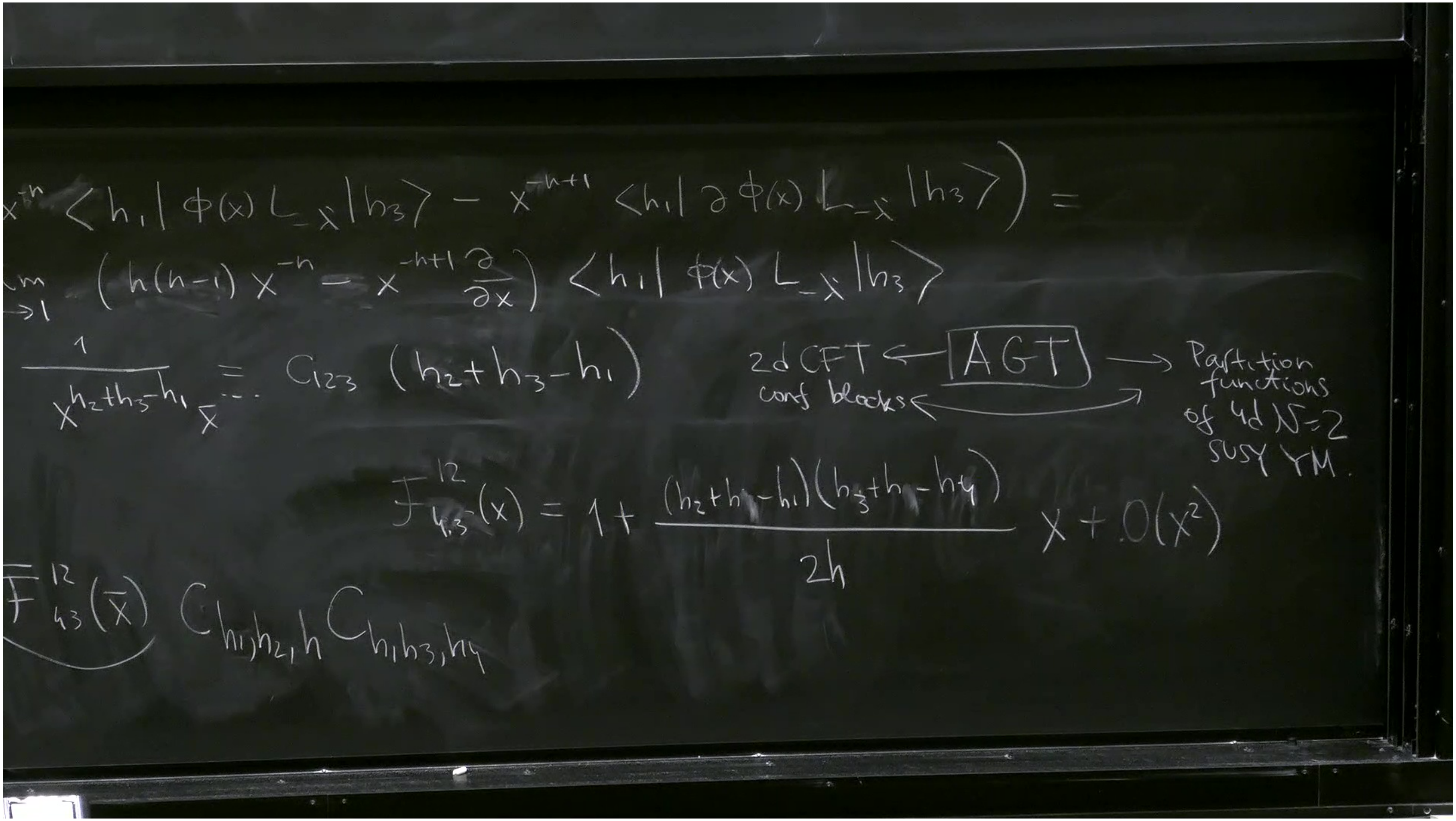
$$= \lim_{x\to 1}\left(h(n-1)\,x^{-n} - x^{-n+1}\frac{\partial}{\partial x}\right)\langle h_1|\,\phi(x)\,L_{-\dot{x}}|h_3\rangle$$

$$1\left(-\frac{\partial}{\partial x}\right)\frac{1}{x^{h_2+h_3-h_1}\,\bar{x}^{\cdots}} = C_{123}\,(h_2+h_3-h_1)$$

$$-h_3)$$

$$\mathcal{F}^{12}_{43}(x) = 1 + \frac{(h_2+h-h_1)(h_3+h-h_4)}{2h}\,x + O(x^2)$$

$$\underbrace{\mathcal{F}^{12}_{43}(x)}\;\underbrace{\bar{\mathcal{F}}^{12}_{43}(\bar{x})}\;C_{h_1,h_2,h}\,C_{h_1,h_3,h_4}$$



$$x^{-n} \langle h_1 | \phi(x) L_{-\lambda} | h_3 \rangle - x^{-n+1} \langle h_1 | \partial \phi(x) L_{-\lambda} | h_3 \rangle \Big) =$$

$$\lim_{\to 1} \left(h(h-1) x^{-n} - x^{-n+1} \frac{\partial}{\partial x} \right) \langle h_1 | \phi(x) L_{-\lambda} | h_3 \rangle$$

$$\frac{1}{x^{h_2+h_3-h_1} \bar{x}^{\cdots}} = C_{123} \left(h_2 + h_3 - h_1 \right)$$

2d CFT ⟵ $\boxed{\text{AGT}}$ ⟶ Partition functions of 4d $\mathcal{N}=2$ SUSY YM.

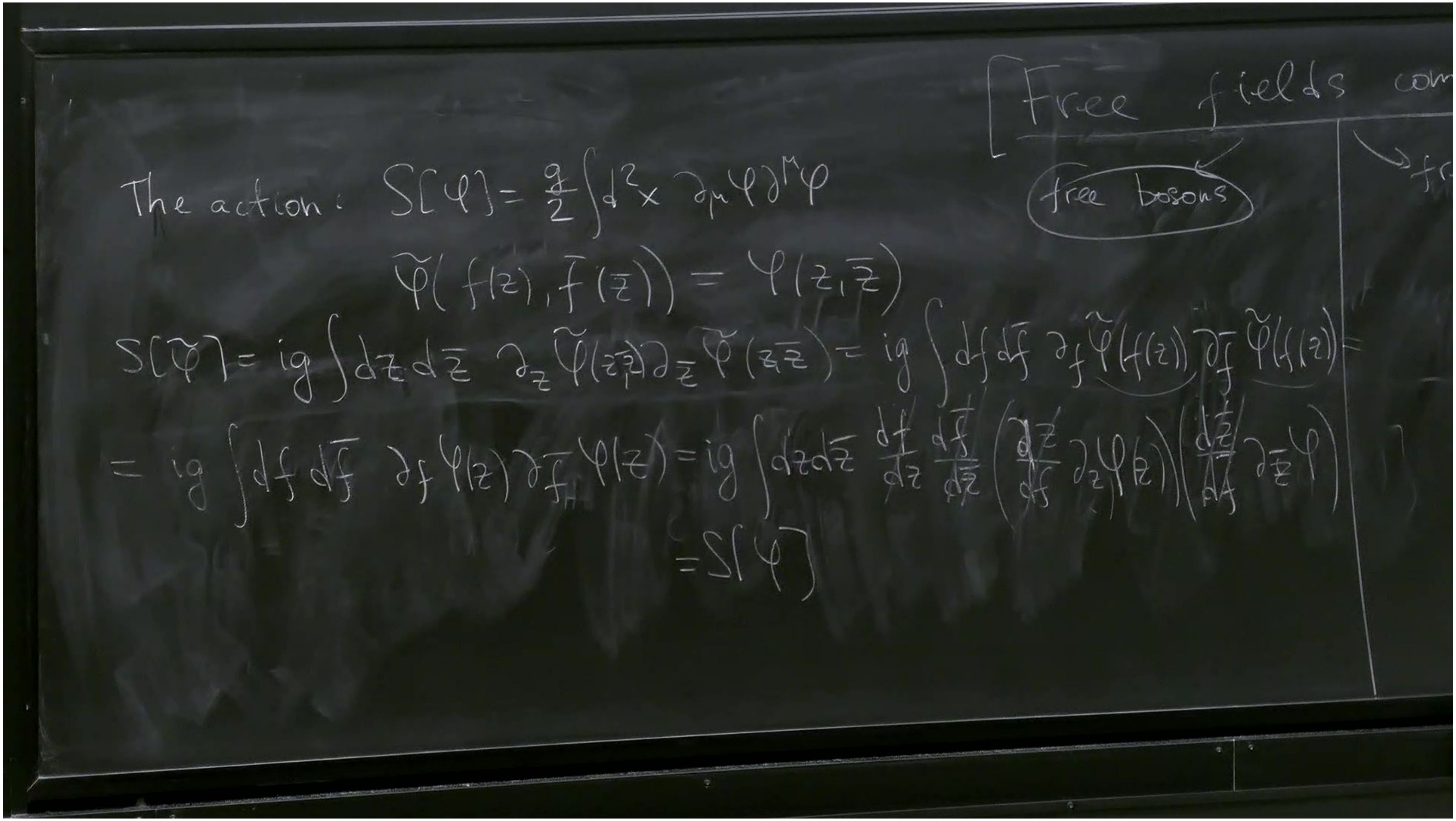
conf blocks ⟵⟶

$$\mathcal{F}^{12}_{43}(x) = 1 + \frac{(h_2+h-h_1)(h_3+h-h_4)}{2h} x + O(x^2)$$

$$\mathcal{F}^{12}_{43}(\bar{x})\, C_{h_1,h_2,h}\, C_{h,h_3,h_4}$$

tion
ctions
d $\mathcal{N}=2$
Y YM.

$$x^2 : Q = \begin{pmatrix} \langle h|(L_1)^2(L_{-1})^2|h\rangle & \langle h|(L_1)^2 L_{-2}|h\rangle \\ \langle h|L_2(L_{-1})^2|h\rangle & \langle h|L_2 L_{-2}|h\rangle \end{pmatrix}^{-1}$$



The action: $S[\varphi] = \frac{g}{2}\int d^2x\, \partial_\mu \varphi \partial^\mu \varphi$

$$\tilde{\varphi}(f(z), \bar{f}(\bar{z})) = \varphi(z, \bar{z})$$

$$S[\tilde{\varphi}] = ig\int dz\, d\bar{z}\; \partial_z \tilde{\varphi}(z,\bar{z})\partial_{\bar{z}} \tilde{\varphi}(z\bar{z}) = ig\int df\, d\bar{f}\; \partial_f \tilde{\varphi}(f(z))\, \partial_{\bar{f}} \tilde{\varphi}(f(z)) =$$

$$= ig\int df\, d\bar{f}\; \partial_f \varphi(z)\, \partial_{\bar{f}} \varphi(\bar{z}) = ig\int dz\, d\bar{z}\; \cancel{\frac{df}{dz}}\, \cancel{\frac{d\bar{f}}{d\bar{z}}} \left(\cancel{\frac{dz}{df}}\, \partial_z \varphi(z)\right)\left(\cancel{\frac{d\bar{z}}{d\bar{f}}}\, \partial_{\bar{z}} \varphi\right)$$

$$= S[\varphi]$$

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free bosons

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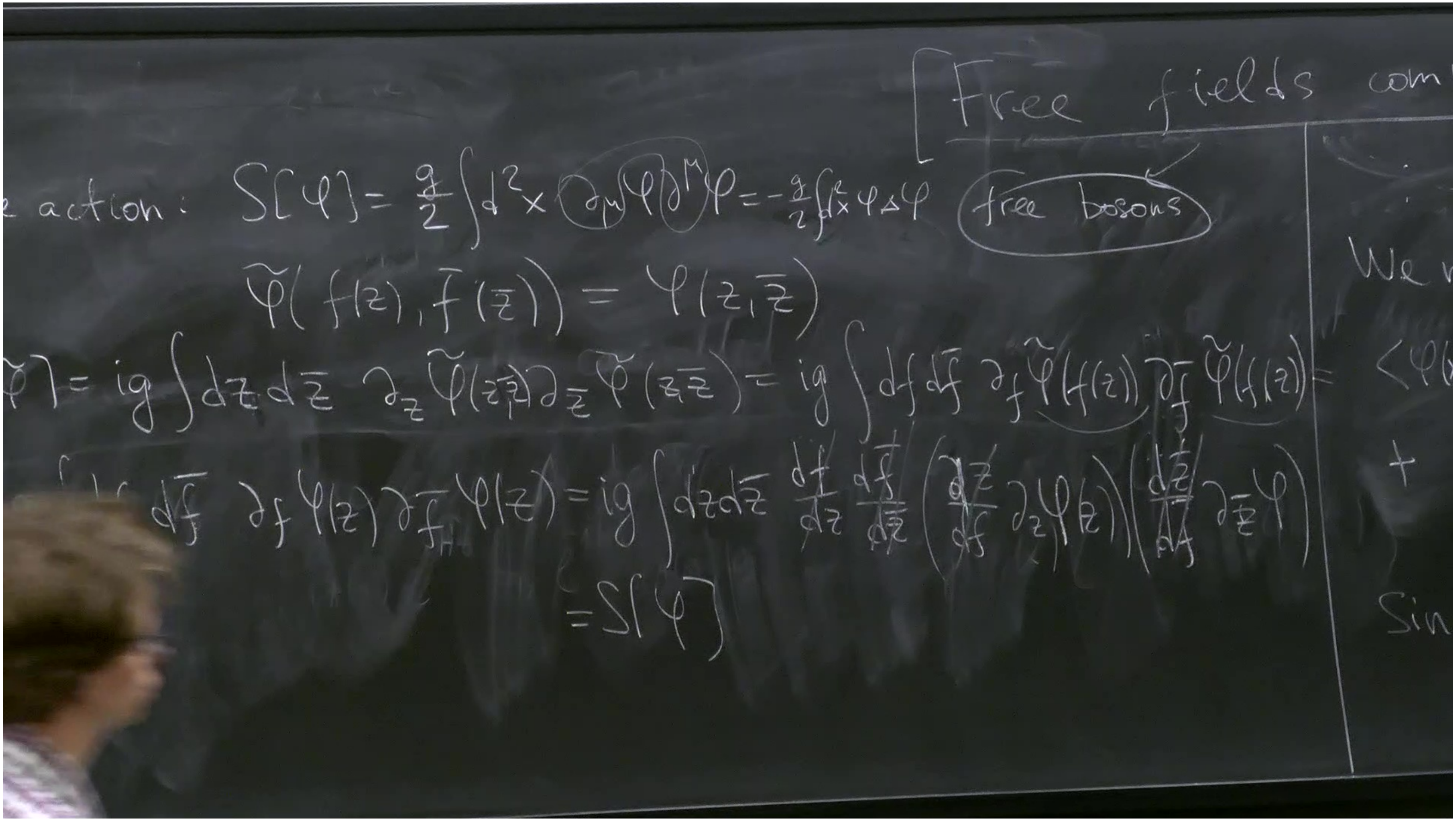

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e action: $S[\varphi] = \frac{g}{2}\int d^2x\, \partial_\mu\varphi\,\partial^\mu\varphi = -\frac{g}{2}\int d^2x\, \varphi\Delta\varphi$ free bosons

$$\tilde{\varphi}(f(z), \bar{f}(\bar{z})) = \varphi(z, \bar{z})$$

$$\tilde{\varphi}] = ig\int dz\, d\bar{z}\; \partial_z\tilde{\varphi}(z,\bar{z})\,\partial_{\bar{z}}\tilde{\varphi}(z\bar{z}) = ig\int df\, d\bar{f}\; \partial_f\tilde{\varphi}(f(z))\,\partial_{\bar{f}}\tilde{\varphi}(f(z)) =$$

$$d\bar{f}\; \partial_f\varphi(z)\,\partial_{\bar{f}}\varphi(\bar{z}) = ig\int dz\, d\bar{z}\; \frac{\cancel{df}}{dz}\,\frac{\cancel{d\bar{f}}}{d\bar{z}}\left(\frac{\partial z}{\cancel{\partial f}}\partial_z\varphi(z)\right)\left(\frac{d\bar{z}}{\cancel{d\bar{f}}}\partial_{\bar{z}}\varphi\right)$$

$$= S[\varphi]$$

We

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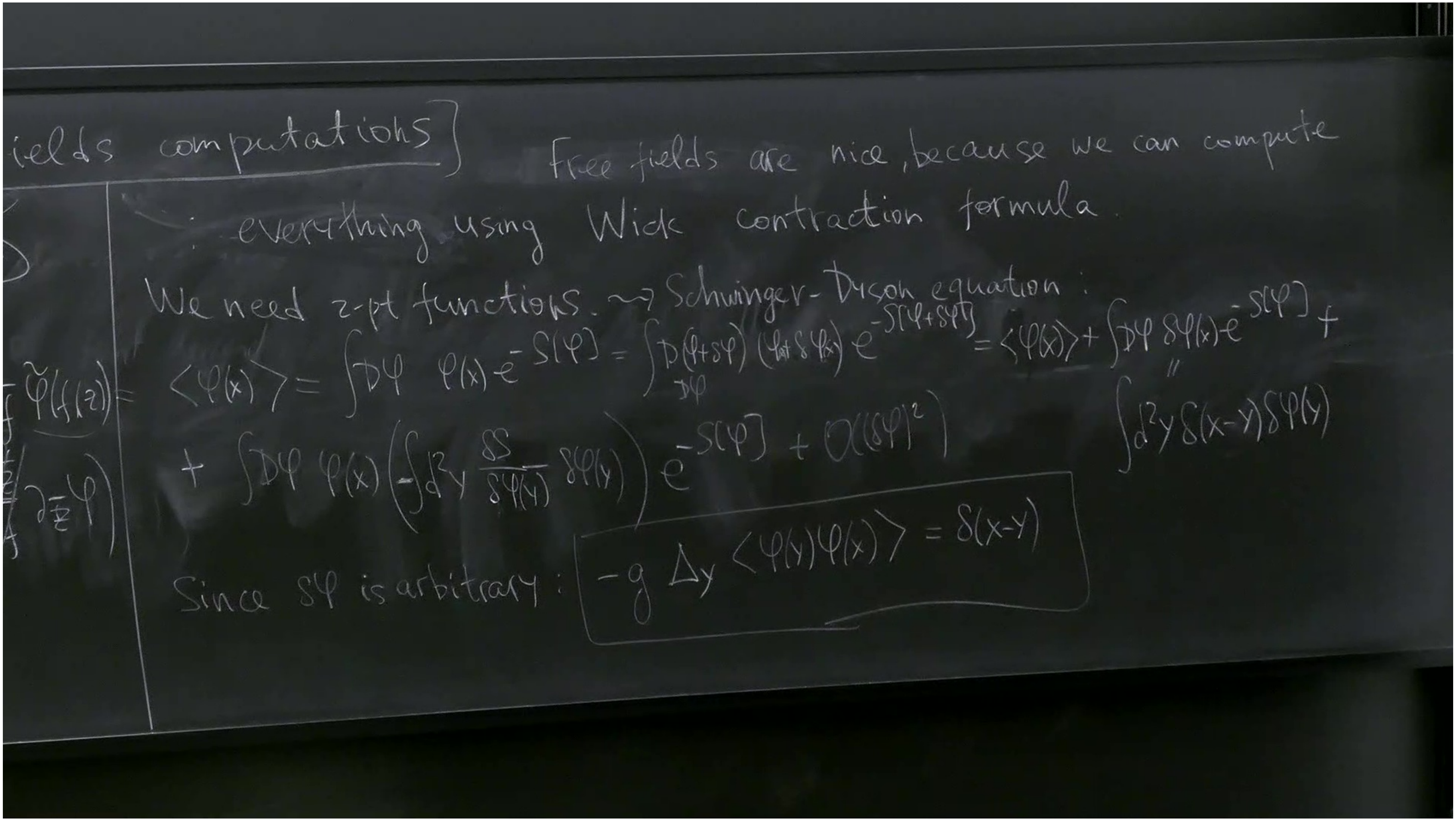


ields computations]

$\tilde{\varphi}(f(z))$ =

$\partial_{\bar{z}}\varphi)$

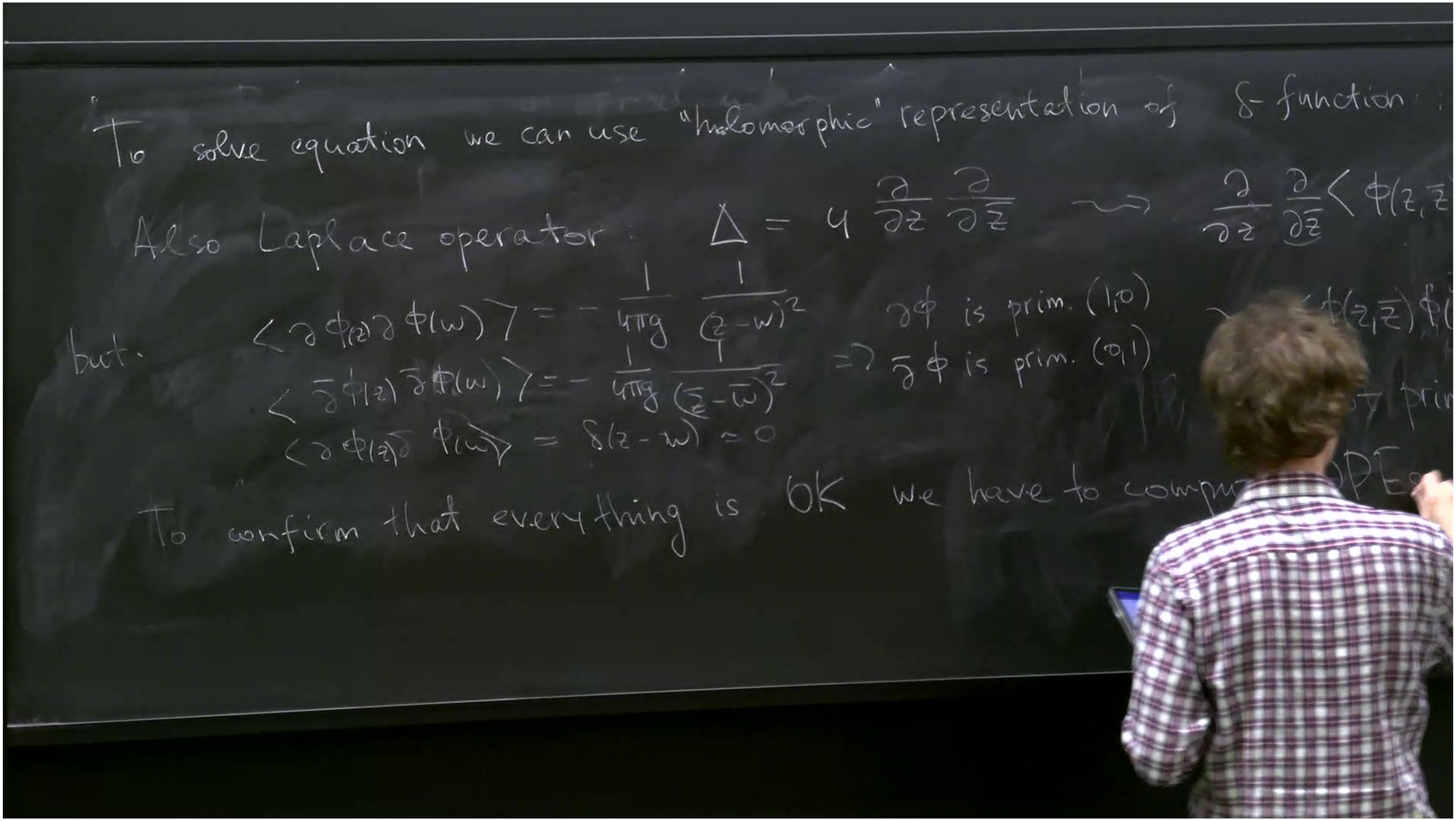
Free fields are nice, because we can compute everything using Wick contraction formula.

We need 2-pt functions. ⇝ Schwinger-Dyson equation:

$$\langle\varphi(x)\rangle = \int D\varphi\ \varphi(x)\, e^{-S[\varphi]} = \int \underset{D\varphi}{D(\varphi+\delta\varphi)}\, (\varphi(x)+\delta\varphi(x))\, e^{-S[\varphi+\delta\varphi]} = \langle\varphi(x)\rangle + \int D\varphi\ \underset{\int d^2y\, \delta(x-y)\delta\varphi(y)}{\underset{\shortparallel}{\delta\varphi(x)}}\, e^{-S[\varphi]} +$$

$$+ \int D\varphi\ \varphi(x)\left(-\int d^2y\, \frac{\delta S}{\delta\varphi(y)}\, \delta\varphi(y)\right) e^{-S[\varphi]} + O((\delta\varphi)^2)$$

Since $\delta\varphi$ is arbitrary: $\boxed{-g\ \Delta_y \langle\varphi(y)\varphi(x)\rangle = \delta(x-y)}$



To solve equation we can use "holomorphic" representation of δ-function :

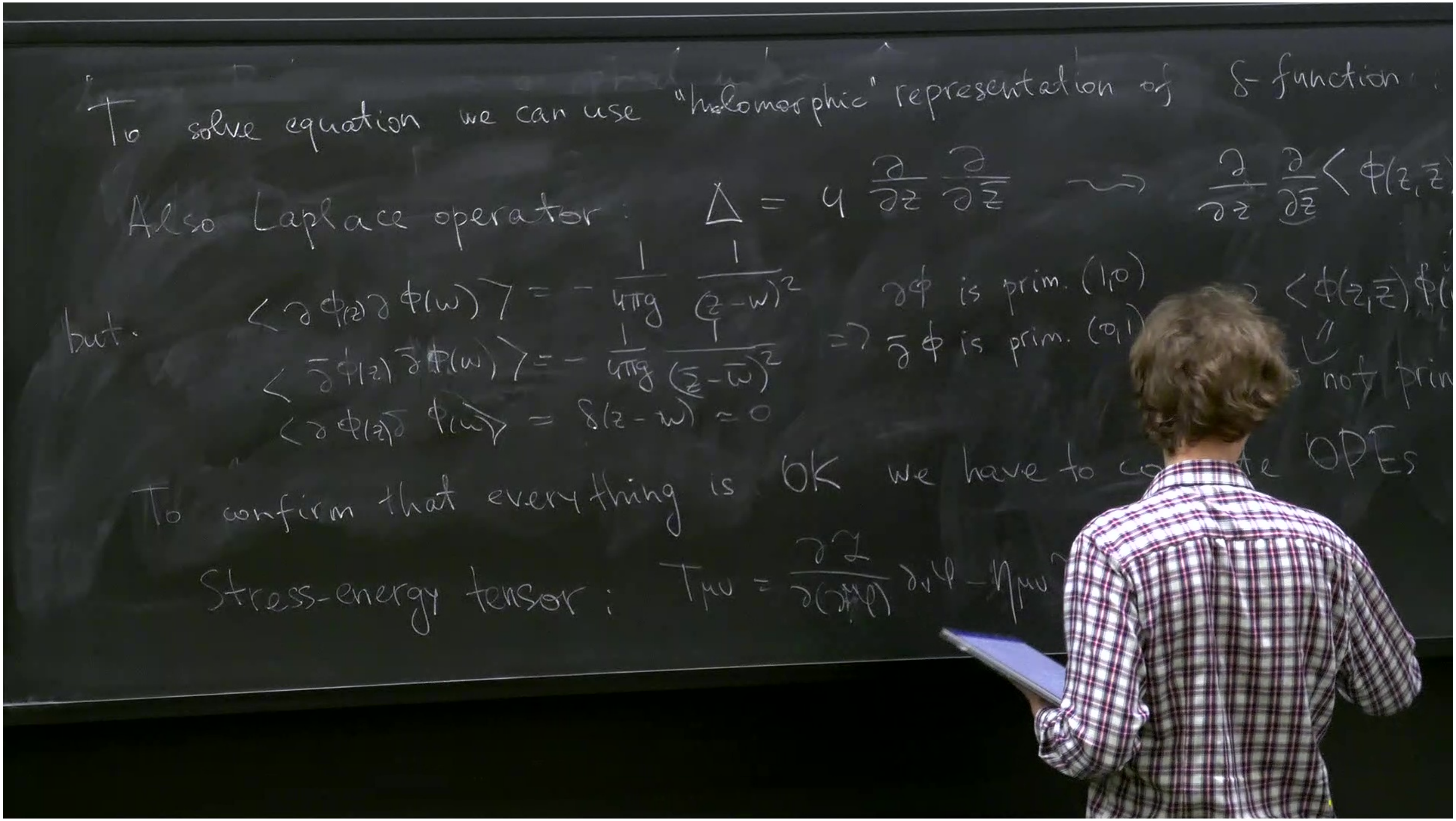
Also Laplace operator : $\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \rightsquigarrow \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \langle \phi(z, \bar{z}$

but. $\langle \partial \phi(z) \partial \phi(w) \rangle = -\frac{1}{4\pi g} \frac{1}{(z-w)^2}$ $\quad \partial\phi$ is prim. (1,0) $\quad \to \langle \phi(z,\bar{z}) \phi($

$\langle \bar{\partial} \phi(z) \bar{\partial} \phi(w) \rangle = -\frac{1}{4\pi g} \frac{1}{(\bar{z}-\bar{w})^2} \Rightarrow \bar{\partial}\phi$ is prim. (0,1) $\quad$ prim

$\langle \partial \phi(z) \bar{\partial} \phi(w) \rangle = \delta(z-w) = 0$

To confirm that everything is OK we have to compu OPEs



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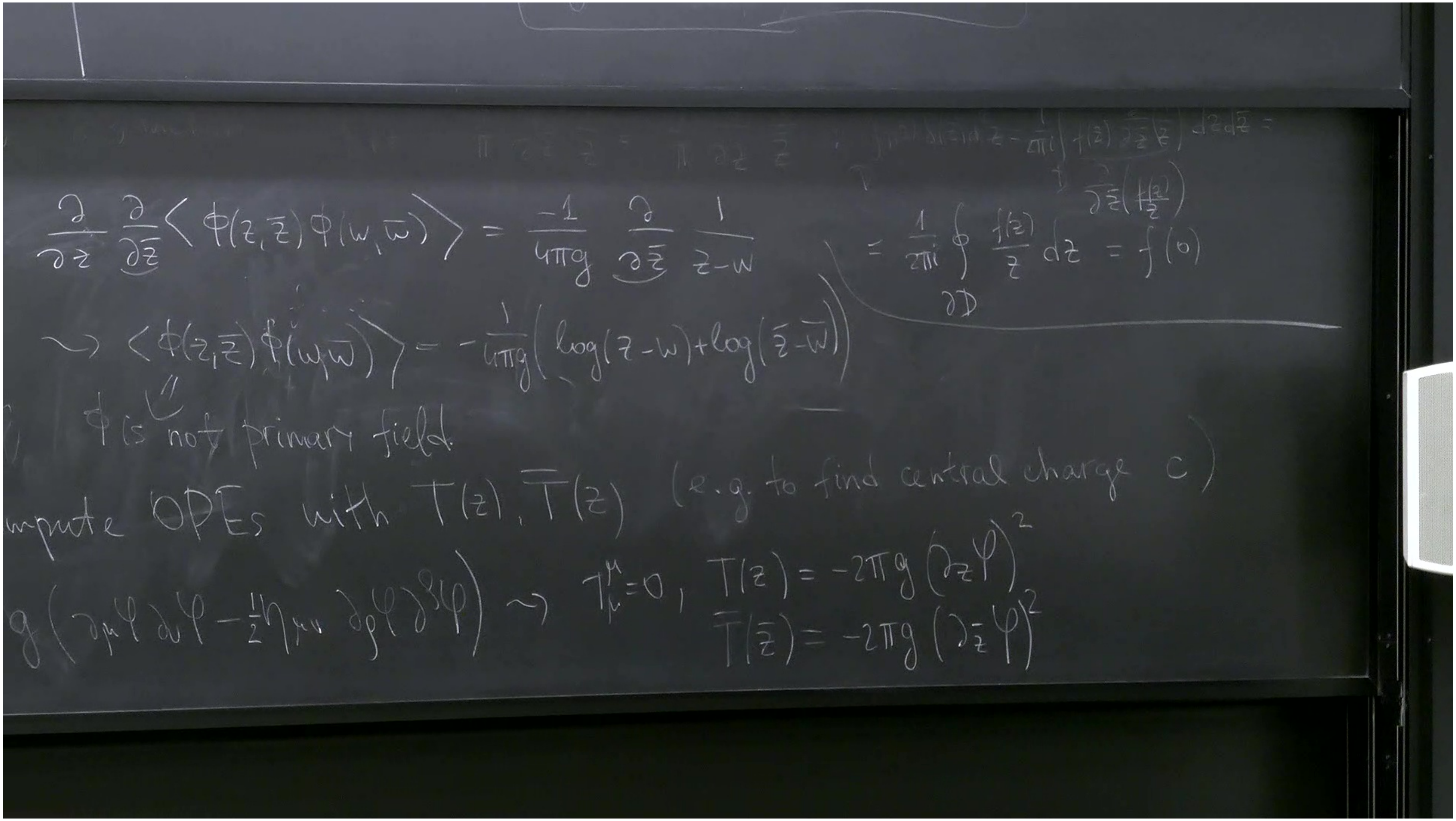
$\langle \bar\partial\phi(z)\bar\partial\phi(w)\rangle = -\frac{1}{4\pi g}\frac{1}{(\bar z-\bar w)^2}$ $\Rightarrow$ $\bar\partial\phi$ is prim. (0,1) //

$\langle \partial\phi(z)\bar\partial\phi(w)\rangle = \delta(z-w) \approx 0$ not prin

To confirm that everything is OK we have to co te OPEs

Stress-energy tensor : $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \eta_{\mu\nu}$



$$\frac{1}{\pi i}\int f(z)\,\frac{\partial}{\partial \bar z}\left(\frac{1}{z}\right) dz\,d\bar z =$$

$$\frac{\partial}{\partial \bar z}\left(\frac{f(z)}{z}\right)$$

$$= \frac{1}{2\pi i}\oint_{\partial D} \frac{f(z)}{z}\,dz = f(0)$$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\left\langle \phi(z,\bar z)\,\phi(w,\bar w)\right\rangle = \frac{-1}{4\pi g}\,\underbrace{\frac{\partial}{\partial \bar z}}\,\frac{1}{z-w}$$

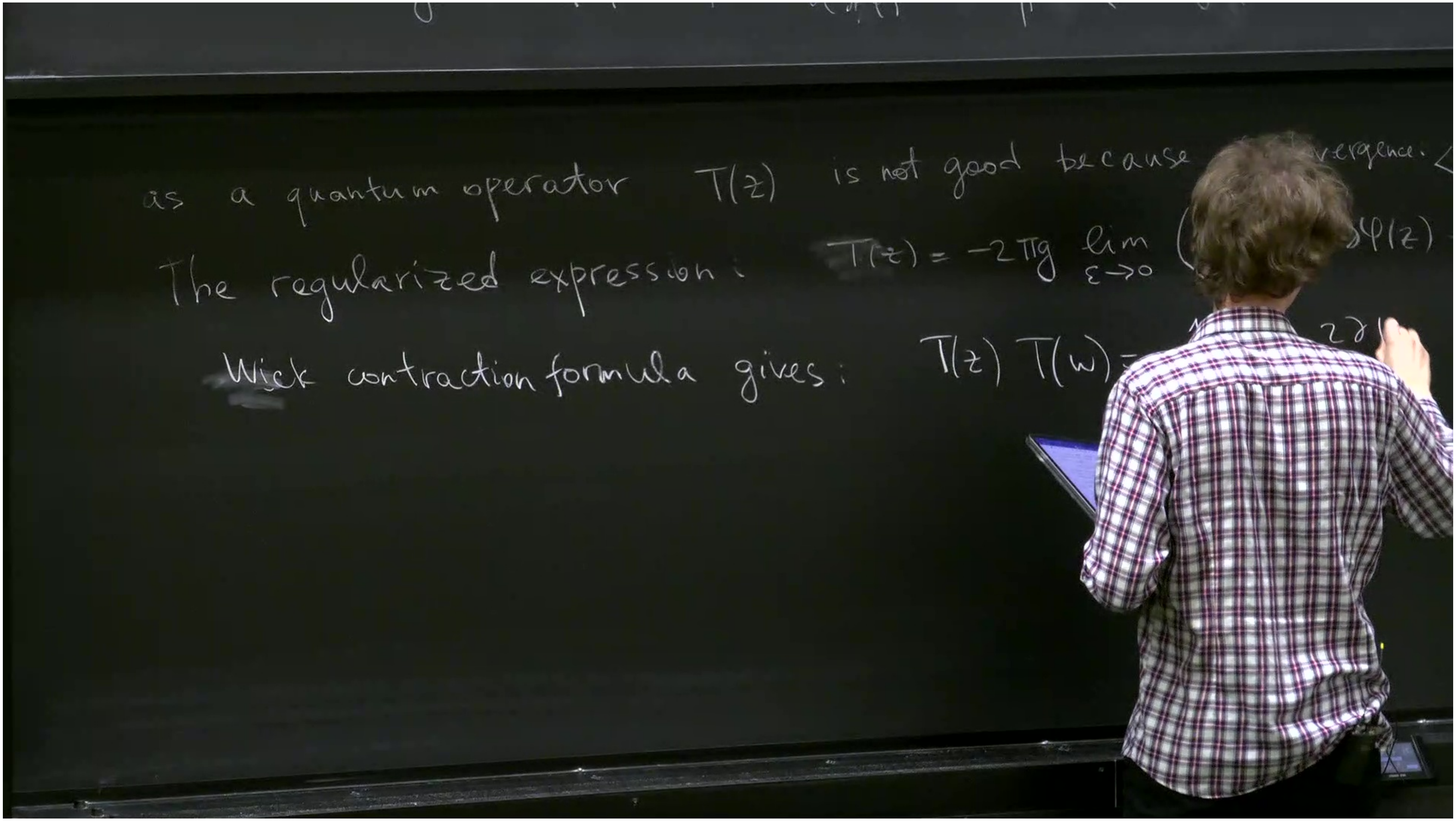
$$\rightsquigarrow \left\langle \phi(z,\bar z)\,\phi(w,\bar w)\right\rangle = -\frac{1}{4\pi g}\left(\log(z-w)+\log(\bar z-\bar w)\right)$$

ϕ is not primary field.

mpute OPEs with $T(z), \bar T(z)$ (e.g. to find central charge c)

$$g\left(\partial_\mu\varphi\,\partial_\nu\varphi - \tfrac{1}{2}\eta_{\mu\nu}\,\partial_\rho\varphi\,\partial^\rho\varphi\right) \rightsquigarrow T^\mu_\mu = 0,\quad T(z) = -2\pi g\left(\partial_z\varphi\right)^2$$

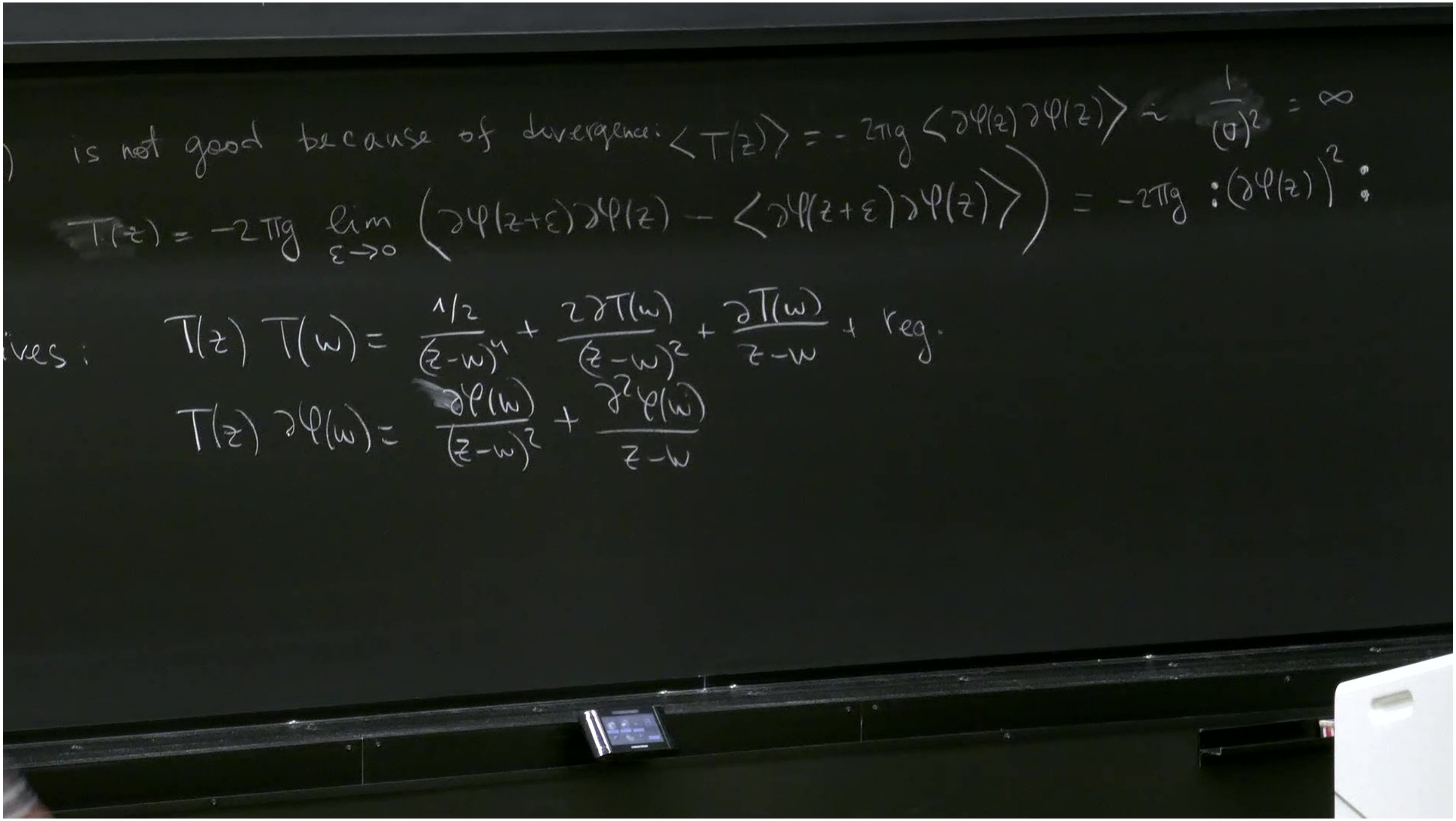
$$\bar T(\bar z) = -2\pi g\left(\partial_{\bar z}\varphi\right)^2$$



as a quantum operator $T(z)$ is not good because vergence: $\langle$

The regularized expression: $T(z) = -2\pi g \lim_{\varepsilon \to 0} ($ $\partial\varphi(z)$

Wick contraction formula gives: $T(z)\,T(w) =$ 2∂



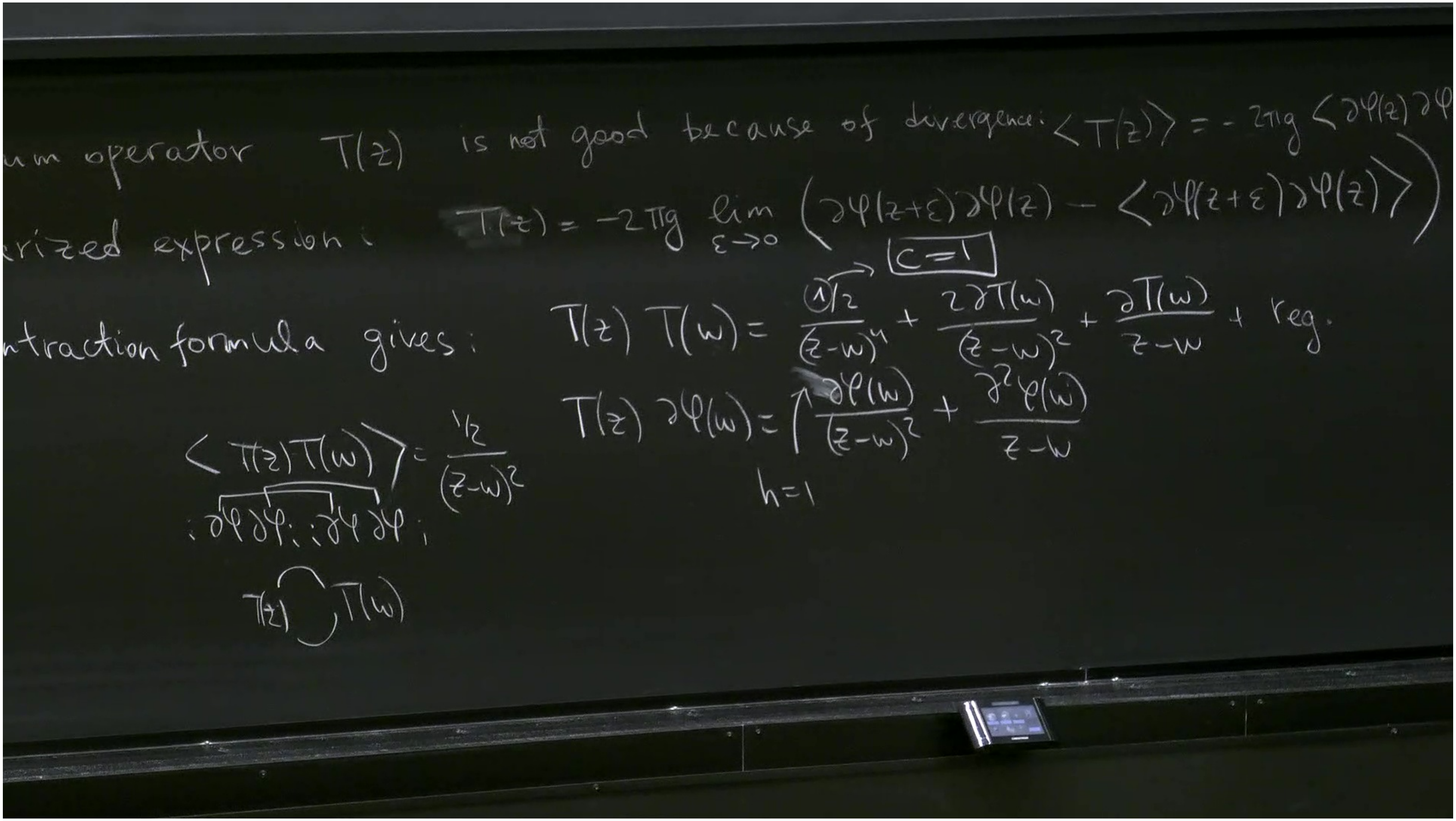
) is not good because of divergence: $\langle T(z)\rangle = -2\pi g \langle \partial\varphi(z)\partial\varphi(z)\rangle \sim \frac{1}{(0)^2} = \infty$

$$T(z) = -2\pi g \lim_{\varepsilon\to 0}\Big(\partial\varphi(z+\varepsilon)\partial\varphi(z) - \langle\partial\varphi(z+\varepsilon)\partial\varphi(z)\rangle\Big) = -2\pi g :(\partial\varphi(z))^2:$$

ives :

$$T(z)\,T(w) = \frac{1/2}{(z-w)^4} + \frac{2\partial T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

$$T(z)\,\partial\varphi(w) = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}$$



um operator $T(z)$ is not good because of divergence: $\langle T(z)\rangle = -2\pi g \langle \partial\varphi(z)\partial\varphi$

rized expression: $T(z) = -2\pi g \lim_{\varepsilon\to 0}\Big(\partial\varphi(z+\varepsilon)\partial\varphi(z) - \langle\partial\varphi(z+\varepsilon)\partial\varphi(z)\rangle\Big)$

$\boxed{c=1}$

ntraction formula gives: $T(z)\,T(w) = \frac{\textcircled{1}/2}{(z-w)^4} + \frac{2\partial T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$

$T(z)\,\partial\varphi(w) = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}$

$h=1$

$\langle T(z)T(w)\rangle = \frac{1/2}{(z-w)^2}$

$:\partial\varphi\partial\varphi:\,:\partial\varphi\partial\varphi:$

$T(z)\quad T(w)$