Title: QFT III Lecture

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Collection: QFT III 2023/24

Date: March 22, 2024 - 10:15 AM

URL: https://pirsa.org/24030037

$$T(z)\,\phi_h(w) = \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \text{reg.}$$

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

How to go to operators? $\rightsquigarrow$ Let's start from Minkowski theory on a cylinder!

Operator formaliz

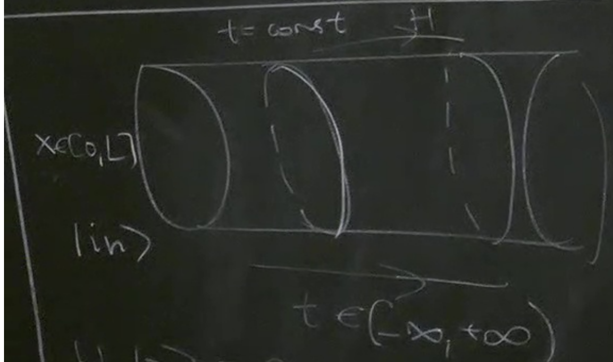
$x \in [0, L]$

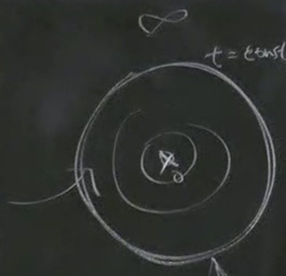
$|in\rangle$

H

ϕ

...lizm in 2d CFTs





We want to work in Euclidean time.

- in Minkowski: $(\partial_t^2 - \partial_x^2)\psi(x,t) = 0$

  $\Rightarrow \psi_n \sim e^{i\frac{2\pi}{L}(x \pm t)n}$

- in Euclidean: $\psi(x,t) \sim e^{-\frac{2\pi}{L}(t+ix)n}$

$\boxed{\text{radial quantization}}$

$z = e^{\frac{2\pi}{L}(t+ix)}$ $\quad \xi = r + i\varphi$

$, \; z^n$

$\partial / \partial \bar{z} \; \psi = 0$

$z = e^{\frac{2\pi}{L}\xi}$

$H|0\rangle = 0$ - some vacuum.

ϕ - some operator, $|\phi_{in}\rangle = \lim_{t \to -\infty} \phi(x,t)|0\rangle$

$\langle \phi_{out}| = \lim_{t \to +\infty} \langle 0|\phi(x,t)$

- $t \to -\infty \rightsquigarrow z = 0$
- $t \to +\infty \rightsquigarrow z \to \infty$

Reasons: • ϕ : $\langle \phi_{out} | \phi_{in} \rangle = \lim_{z \to 0} \lim_{w \to \infty} \langle 0 | \phi(w,\bar{w}) \, w^{2h} (\bar{w})^{2\bar{h}} \, \phi(z,\bar{z}) | 0 \rangle =$

$$= \lim_{w \to \infty} w^{2h} (\bar{w})^{2\bar{h}} \frac{C_{\phi,\phi}}{(w \;\;)^{2h} (\bar{w} \;\;)^{2\bar{h}}} = C_{\phi,\phi}$$

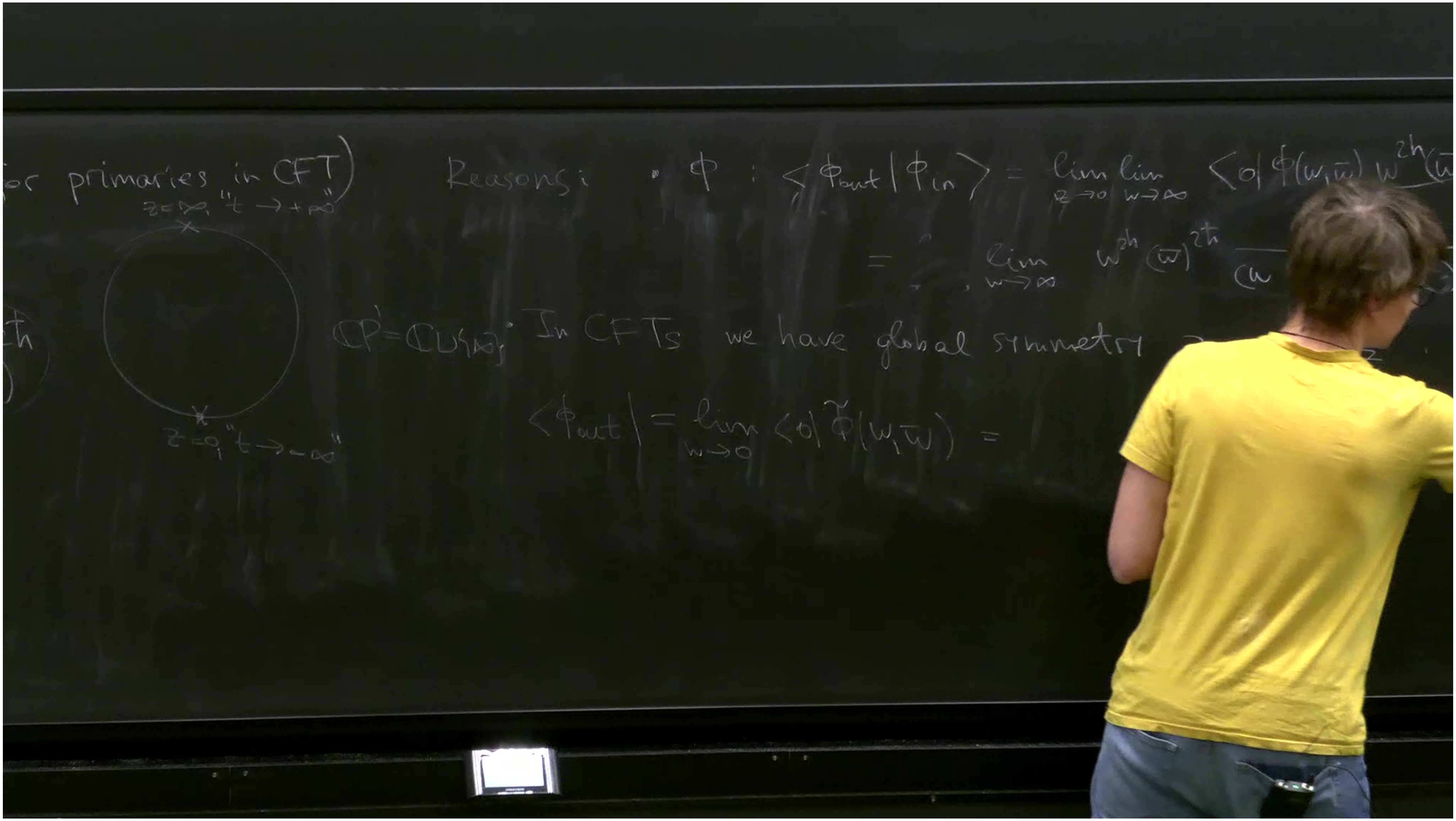
• In CFTs we have global symmetry $z \mapsto w = \frac{1}{z}$

The operator-state correspondence. (for primaries in CFT)

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle$$

$$\langle\phi_{out}| = \lim_{z,\bar{z}\to\infty} \langle 0|\phi(z,\bar{z})\, z^{2h}(\bar{z})^{2\bar{h}}$$

Re



for primaries in CFT)
Reasons:
ϕ : $\langle \phi_{out} | \phi_{in} \rangle = \lim_{z \to 0} \lim_{w \to \infty} \langle 0 | \phi(w, \bar{w}) \, w^{2h}$
$= \lim_{w \to \infty} w^{2h} (\bar{w})^{2\bar{h}}$
$z = \infty$, "$t \to +\infty$"
$z = 0$, "$t \to -\infty$"
$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$
In CFTs we have global symmetry
$\langle \phi_{out} | = \lim_{w \to 0} \langle 0 | \tilde{\phi}(w, \bar{w}) =$

easons:

• ϕ : $\langle \phi_{out} | \phi_{in} \rangle = \lim_{z\to 0} \lim_{w \to \infty} \langle 0 | \phi(w,\bar w)\, w^{2h} (\bar w)^{2\bar h}\, \phi(z,\bar z) | 0 \rangle =$

$$= \lim_{w\to\infty} w^{2h} (\bar w)^{2\bar h} \frac{C_{\phi,\phi}}{(w \ \ )^{2h} (\bar w \ \ )^{2\bar h}} = C_{\phi,\phi}$$

• In CFTs we have global symmetry $z \mapsto w = \frac{1}{z}$. $\tilde\phi(w,\bar w) = (-w^{-2})^{h} (-\bar w^{-2})^{\bar h} \phi\left(\frac{1}{w}, \frac{1}{\bar w}\right)$

$$\langle \phi_{out} | = \lim_{w\to 0} \langle 0 | \tilde\phi(w,\bar w) = \lim_{w\to 0} w^{-2h} \bar w^{-2\bar h} \langle 0 | \phi\left(\frac{1}{w}, \frac{1}{\bar w}\right) = \lim_{z\to\infty} z^{2h} \bar z^{2\bar h} \langle 0 | \phi(z,\bar z)$$

$$= \lim_{w\to\infty} w^{2h} (\bar{w})^{2\bar{h}} \frac{C_{\phi,\phi}}{(w\ \ )^{2h}(\bar{w}\ \ )^{2\bar{h}}} = C_{\phi,\phi}$$

$\mathbb{C}\cup\{\infty\}$ • In CFTs we have global symmetry $z \mapsto w = \frac{1}{z}$. $\tilde{\phi}(w,\bar{w}) = (-w^{-2})^{h}(-\bar{w}^{-2})^{\bar{h}}\phi(\frac{1}{w},\frac{1}{\bar{w}})$

$$\langle\phi_{out}| = \lim_{w\to 0}\langle 0|\tilde{\phi}(w,\bar{w}) = \lim_{w\to 0} w^{-2h}\,\bar{w}^{-2\bar{h}}\langle 0|\phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right) = \lim_{z\to\infty} z^{2h}\,\bar{z}^{2\bar{h}}\langle 0|\phi(z,\bar{z})$$

Remark What about Hermitian conjugation?

$$\langle\phi_{out}| = (|\phi_{in}\rangle)^{\dagger} \Rightarrow [\phi(z,\bar{z})]^{\dagger} = \phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\bar{z}^{-2h}\,z^{-2\bar{h}}$$

$z^{*} = \bar{z}$

The mode expansions (for primary) is: $\phi(z,\bar{z}) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} z^{-m-h} \bar{z}^{-}$

$[\phi(z,\bar{z})]^{\dagger} = \sum (\bar{z})^{m+h} (z)^{n+\bar{h}} \phi_{m}$

naive: $(\phi(z,\bar{z}))^{\dagger} = \sum_{m,n} \bar{z}^{-m-h} z^{-n}$

similarly, for $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_{n}$

s : $\phi(z,\bar{z}) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}$

$[\phi(z,\bar{z})]^\dagger = \sum (\bar{z})^{m+h} (z)^{n+\bar{h}} \phi_{m,n} \cdot \bar{z}^{-2h} z^{-2\bar{h}} = \sum_{m,n\in\mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi_{-m,-n}$

naive : $(\phi(z,\bar{z}))^\dagger = \sum_{m,n} \bar{z}^{-m-h} z^{-n-\bar{h}} (\phi_{m,n})^\dagger$

similarly, for $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ we have $\boxed{(L_n)^\dagger = L_{-n}}$

$$\langle \phi_{out}| = \lim_{w\to 0} \langle 0|\phi(w,\bar w) = \lim_{w\to 0} w^{-2h}\, \bar w^{-2\bar h} \langle 0|\phi\left(\frac{1}{w},\frac{1}{\bar w}\right) = \lim_{z\to\infty} z^{2h}\, \bar z^{2\bar h} \langle 0|\phi(z,\bar z)$$

ark What about Hermitian conjugation?

$$\langle\phi_{out}| = (|\phi_{in}\rangle)^{\dagger} \Rightarrow [\phi(z,\bar z)]^{\dagger} = \phi\left(\frac{1}{\bar z},\frac{1}{z}\right) \bar z^{-2h}\, z^{-2\bar h}$$

$$z^{*} = \bar z \qquad z \sim \frac{1}{\bar z} \qquad z = e^{\frac{2\pi}{L}(t+ix)}, \quad \frac{1}{\bar z} = e^{\frac{2\pi}{L}(-t+ix)}$$

$$m,n \in \mathbb{Z}$$

$$\bar z^{-m-h}\, z^{-n-\bar h} (\phi_{m,n})^{\dagger}$$

$$z^{-n-2} L_n \quad \text{we have} \quad \boxed{(L_n)^{\dagger} = L_{-n}}$$

What about Hermitian conjugation?

$\langle \phi_{out}| = (|\phi_{in}\rangle)^\dagger \Rightarrow [\phi(z,\bar z)]^\dagger = \phi\left(\frac{1}{\bar z}, \frac{1}{z}\right) \bar z^{-2h} z^{-2\bar h}$

$z^* = \bar z \qquad z \rightsquigarrow \frac{1}{\bar z} \qquad z = e^{\frac{2\pi}{L}(t+ix)}, \quad \frac{1}{\bar z} = e^{\frac{2\pi}{L}(-t+ix)}$

$\bar z^{m-h} \ \bar z^{n-\bar h} \ \phi_{m,n}$

$(\bar z)^{n+\bar h} \phi_{m,n} \cdot \bar z^{-2h} z^{-2\bar h} = \sum_{m,n\in\mathbb{Z}} \bar z^{-m-h} z^{-n-\bar h} \phi_{-m,-n}$

$\bar z^{-m-h} \ z^{-n-\bar h} (\phi_{m,n})^\dagger$

$\boxed{\phi^\dagger_{m,n} = \phi_{-m,-n}}$

$z^{-n-2} L_n$ we have $\boxed{(L_n)^\dagger = L_{-n}}$

$$z^* = \bar{z} \qquad z \sim \frac{1}{\bar{z}} \qquad z = e^{\frac{2\pi}{L}(t+ix)}, \quad \frac{1}{\bar{z}} = \ldots$$

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} z^{-m-h}\, \bar{z}^{-n-\bar{h}}\, \phi_{m,n}$$

$$= \sum (\bar{z})^{m+h} (z)^{n+\bar{h}} \phi_{m,n} \cdot \bar{z}^{-2h} z^{-2\bar{h}} = \sum_{m,n\in\mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi_{-m,-n}$$

$$\boxed{\phi^\dagger_{m,n} = \phi_{-m,-n}}$$

$$(\ldots, \bar{z})^\dagger = \sum_{m,n} \bar{z}^{-m-h} z^{-n-\bar{h}} (\phi_{m,n})^\dagger$$

$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ we have $\boxed{(L_n)^\dagger = L_{-n}}$

What about OPE? How to rephrase them for operators?

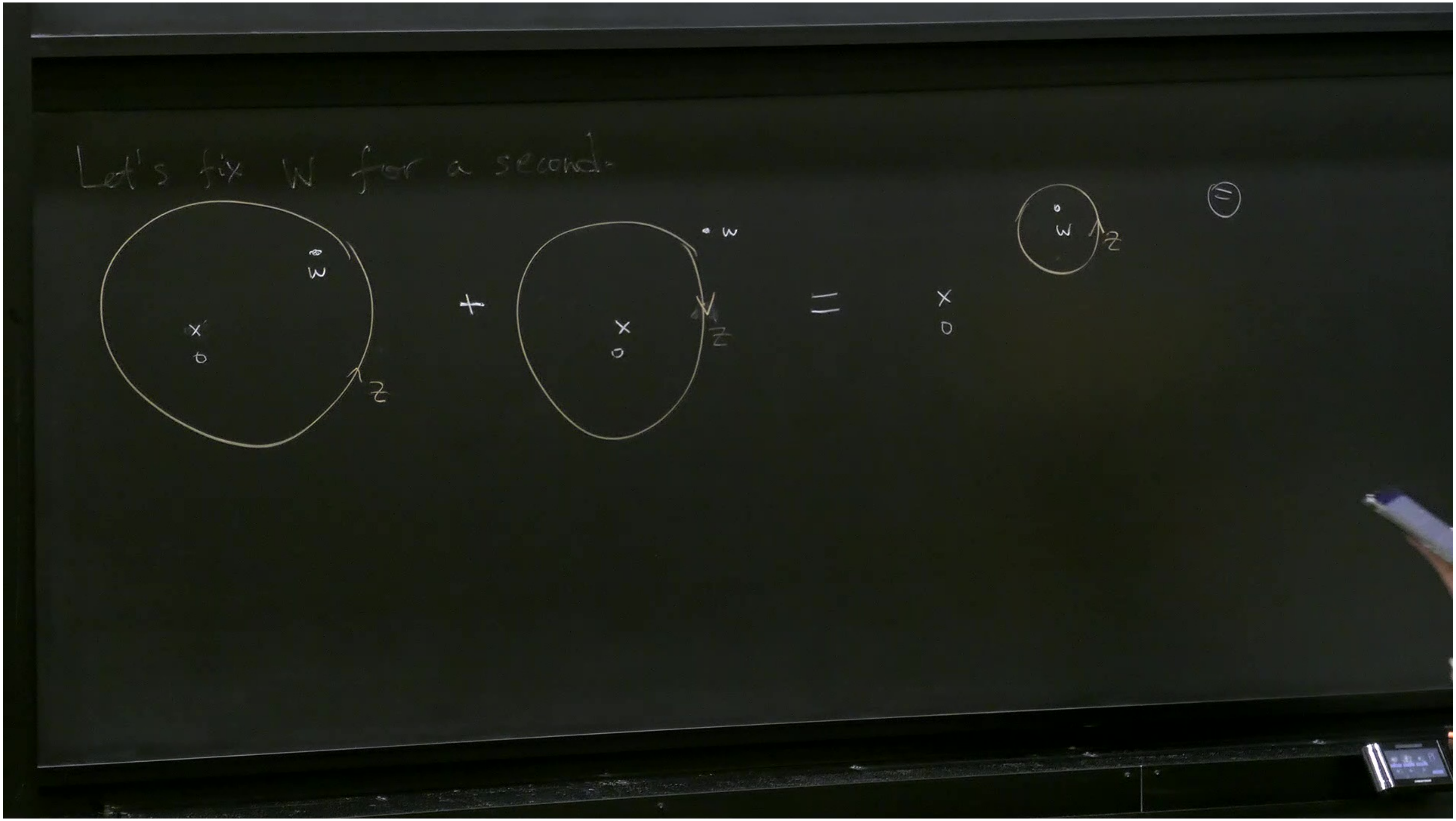
- We can extract modes from fields by $\phi_{m,n} = \oint_0 \frac{dz}{2\pi i} z^{m+h-1} \oint_0 \frac{d\bar{z}}{2\pi i}$

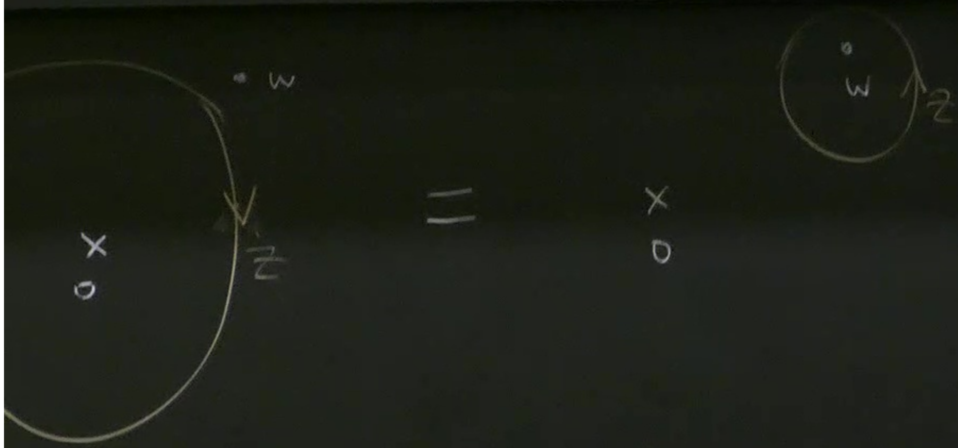
- We have to replace time ordering with radial ordering: $R\,\phi_{1}(z$

$$[L_n, L_m] \ni \underbrace{\oint_{|z|=R+\varepsilon} \frac{dz}{2\pi i} z^{n+1} T(z) \oint_{|w|=R-\varepsilon} \frac{dw}{2\pi i} w^{m+1} T(w)}_{L_n \cdot L_m} - \underbrace{\oint_{|w|=R+\varepsilon} \frac{dw}{2\pi i} w^{m+1} T(w) \oint_{|z|=R-\varepsilon} \frac{dz}{2\pi i} z^{n+1} T(z)}_{L_m \cdot L_n}$$

?

$$\ldots \bar{h}-1 \quad \oint_0 \frac{d\bar{z}}{2\pi i}\, \bar{z}^{\,n+\bar{h}-1}\, \phi(z,\bar{z}) \rightsquigarrow \oint \frac{dz}{2\pi i}\, z^n = \delta_{n,-1}$$


Let's fix W for a second.
w
z
+
w
z
=
w
z



$$= \oint_{|w|=R} \frac{dw}{2\pi i}\, w^{m} \oint_{w} \frac{dz}{2\pi i}\, z \ \ T(z)\,T(w)$$

$$= \oint_{|w|=R} \frac{dw}{2\pi i}\, w^{m+1} \left(\frac{c}{2} \cdot \frac{1}{3!} (n+1)n(n-1)\, w^{n-2} + \right.$$

$$= \delta_{m+n,0}\, \frac{c}{12} (n^2-1)n + 2(n+1) L_{n+m} -$$

$$\Rightarrow \boxed{[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^2-1)n\, \delta_{m+n,0}} \quad \text{- Virasoro}$$

$$\overset{\circ}{w} \quad z$$

$$= \oint_{|w|=R} \frac{}{2\pi i} w \quad \oint_{w} \frac{}{2\pi i} z \quad T(z)T(w) - \oint_{|w|=R} \frac{}{2\pi i} \quad \oint_{w}$$

$$= \quad \times \; 0$$

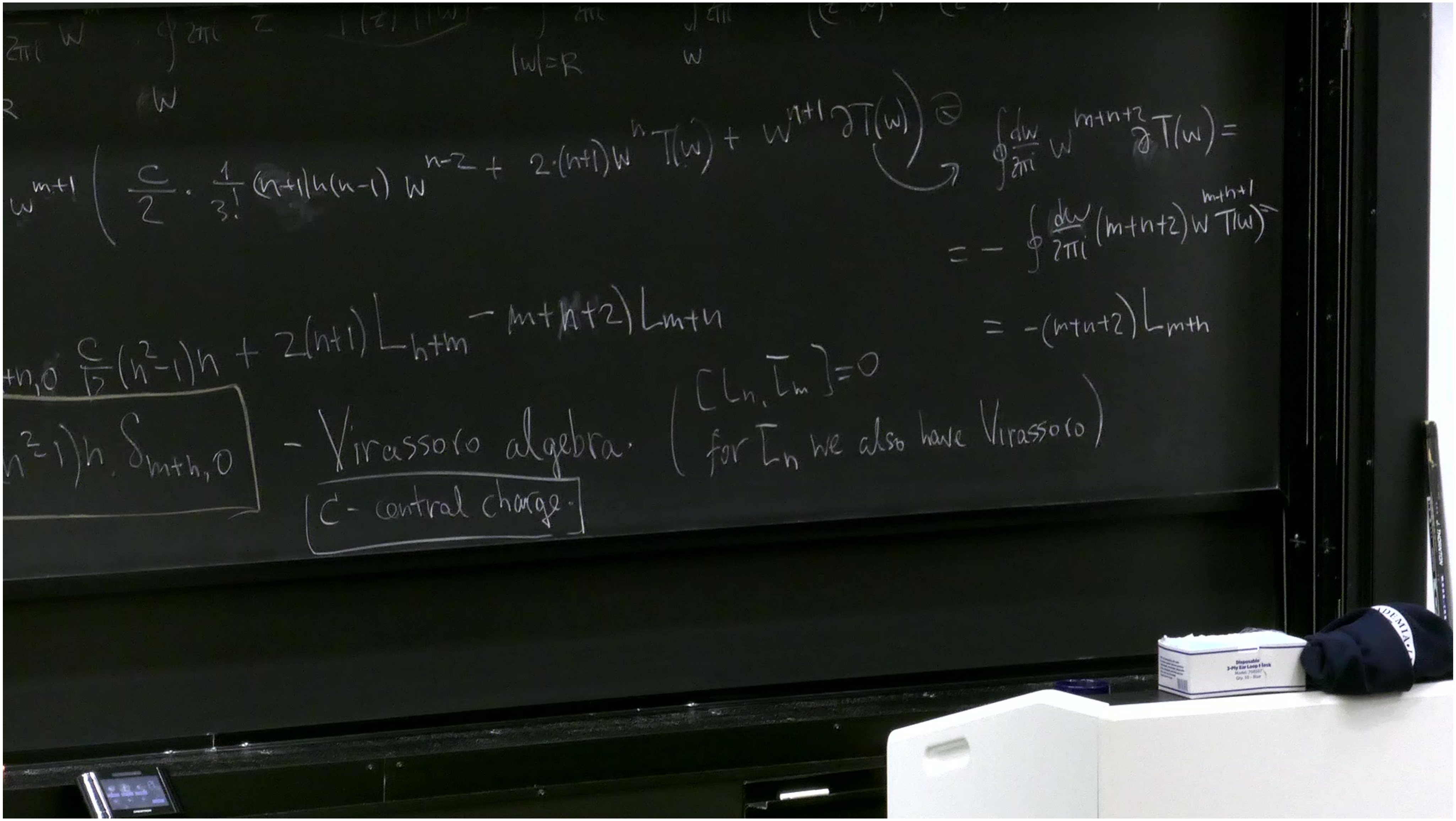
$$= \oint_{|w|=R} \frac{dw}{2\pi i} w^{m+1} \left(\frac{c}{2} \cdot \frac{1}{3!} (n+1)n(n-1)\, w^{n-2} + 2\cdot(n+1) w^{n} T(w) \right.$$

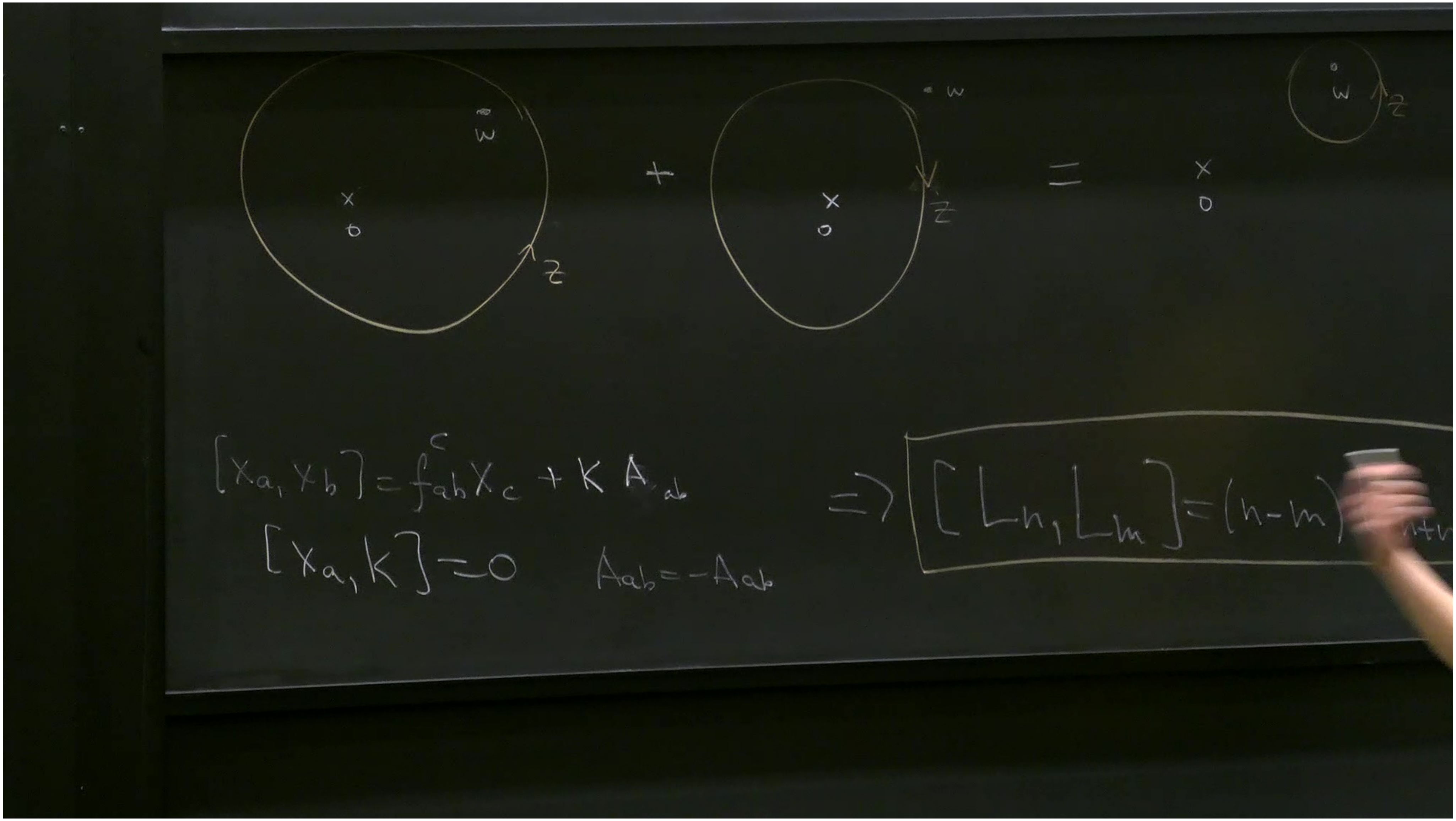
$$= \delta_{m+n,0} \frac{c}{12}(n^2-1)n + 2(n+1) L_{n+m} - (m+n+2) L_{m+n}$$

$$\boxed{[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^2-1)n\, \delta_{m+n,0}}$$

– Virasoro algebra.

$\boxed{c \text{ – central charge.}}$


$|w|=R$
$w^{m+1}\left(\frac{c}{2}\cdot\frac{1}{3!}(n+1)n(n-1)\,w^{n-2}+2\cdot(n+1)w^{n}T(w)+w^{n+1}\partial T(w)\right)$
$\oint\frac{dw}{2\pi i}w^{m+n+2}\partial T(w)=$
$=-\oint\frac{dw}{2\pi i}(m+n+2)w^{m+n+1}T(w)$
$=-(m+n+2)L_{m+n}$
$+n,0\ \frac{c}{12}(n^2-1)n+2(n+1)L_{n+m}-(m+n+2)L_{m+n}$
$n^2-1)n\,\delta_{m+n,0}$
- Virassoro algebra.
$[L_n,\bar{L}_m]=0$
for $\bar{L}_n$ we also have Virassoro
c - central charge.


w
z
x
0
+
w
z
x
0
=
w
z
x
0
[X_a, X_b] = f_ab^c X_c + K A_ab
[X_a, K] = 0
A_ab = -A_ab
=> [L_n, L_m] = (n-m)

$|w|=R$ $\qquad w$

$\frac{1}{3!}(n+1)n(n-1)\, w^{n-2} + 2\cdot(n+1) w^{n} T(w) + w^{n+1}\partial T(w)$

$$\oint \frac{dw}{2\pi i} w^{m+n+2}\partial T(w) =$$

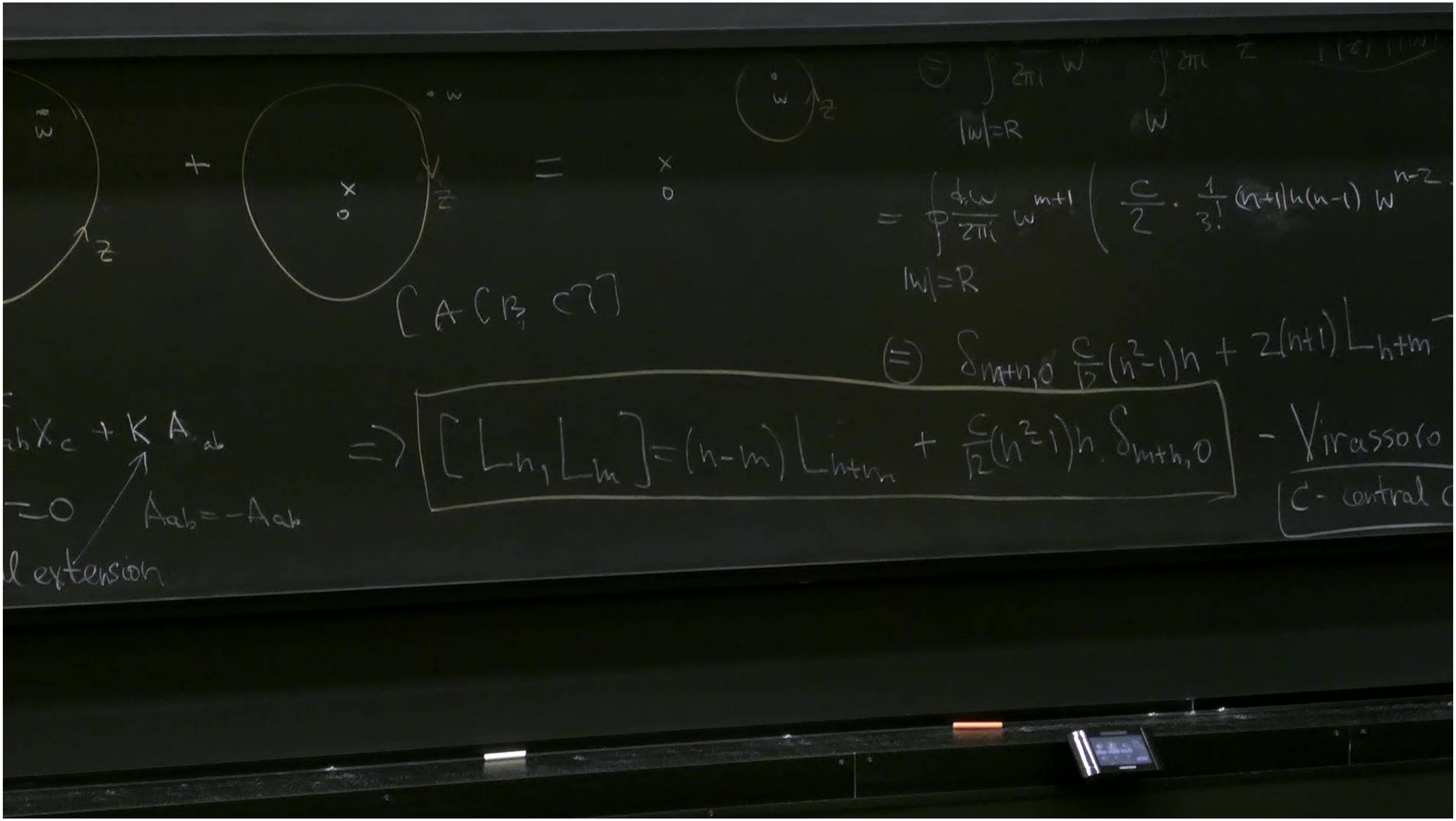
$$= -\oint \frac{dw}{2\pi i}(m+n+2) w^{m+n+1} T(w) =$$

$$= -(m+n+2) L_{m+n}$$

$+ 2(n+1) L_{n+m} - (m+n+2) L_{m+n}$

- Virasoro algebra. ($[L_n, \bar{L}_m] = 0$ for $\bar{L}_n$ we also have Virasoro)

$\boxed{c - \text{central charge.}}$


[A [B, C]]
|w|=R
$= \oint_{|w|=R} \frac{dw}{2\pi i} w^{m+1} \left(\frac{c}{2} \cdot \frac{1}{3!}(n+1)n(n-1)\, w^{n-2} \right.$
|w|=R
$= \delta_{m+n,0} \frac{c}{12}(n^2-1)n + 2(n+1)L_{n+m}$
$X_c + K A_{ab}$
$=0$
$A_{ab} = -A_{ab}$
l extension
$\Rightarrow \boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^2-1)n\, \delta_{m+n,0}}$
- Virasoro
c - central c