Title: QFT III Lecture

Speakers: Mykola Semenyakin

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Lecture 7 Symmetries of 2d CFT

$$\frac{\partial (x')^\lambda}{\partial x^\mu} \frac{\partial (x')^\sigma}{\partial x^\nu} \eta_{\lambda\sigma} = e^{2\omega} \eta_{\mu\nu}$$

$d > 2$ $\quad SO(2,d)$

$\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow SO(1,d+1)$

$$(x')^\mu = x^\mu + \epsilon^\mu(x) \Rightarrow \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\omega \eta_{\mu\nu}$$

$$\Downarrow$$

$$\partial_\mu \epsilon^\mu = 2\omega$$

$\mu = \nu = 1 : \cancel{2} \partial_1 \epsilon_1 = \cancel{\partial_1 \epsilon_1} + \partial_2 \epsilon_2$

$$\mu = 1, \nu = 2 : \begin{cases} \partial_1 \epsilon_1 - \partial_2 \epsilon_2 = 0 \\ \partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0 \end{cases}$$

$z = x^1 + i x^2$

$\epsilon = \epsilon^1 + i \epsilon^2$

$$\Longleftrightarrow \boxed{\frac{\partial \epsilon}{\partial \bar{z}} = 0, \quad \epsilon = \epsilon(z)}$$

CFT

$\epsilon = \epsilon(z)$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right)$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$$

$$\frac{\partial}{\partial x^1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z}$$

$$\frac{\partial}{\partial x^2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar z}\right)$$

$$(dx^1)^2 + (dx^2)^2 = \frac{1}{2}\, dz\, d\bar z$$

$$dx^0 \wedge dx^1 = \frac{i}{2}\, dz \wedge d\bar z \qquad \eta_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \qquad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}$$

Remark Most of conformal transform. in 2d are not invertible

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \qquad I: z \mapsto \frac{1}{z}$$

$0 \mapsto \infty$

$\infty \mapsto 0$

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc = 1$$

Sketch

$f(z)$ - some analytic $\Rightarrow f = \dfrac{P(z)}{Q(z)}$

$P(z_1) = P(z_2) = 0 \Rightarrow \deg P = 1$

$\deg Q = 1$

Global conf. transform: $= SL(2,\mathbb{C}) \simeq SO(1,3)$

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc=1$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Exercise: Composition $f \circ g$

$\Updownarrow$

Multiplication of corresponding matrices.

Remark: We will treat $z, \bar{z}$ – as independent, $z^* = \bar{z}$ – real plane.

$(1,3)$

Generators of algebra of conf transform

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z}, \quad \bar{\ell}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$$

$$[\ell_n, \ell_m] = (n-m)\,\ell_{n+m}$$

Witt algebra

$$[\ell_n, \bar{\ell}_m] = 0$$

$$[\bar{\ell}_n, \bar{\ell}_m] = (n-m)\,\bar{\ell}_{n+m}$$

Witt $\oplus$ Witt.

Witt $\supset sl_2 =$

$z^* = \bar{z}$ - real plane.

form ...)

algebra

$$\text{Witt} \supseteq \mathfrak{sl}_2 = \langle l_{-1}, l_0, l_1 \rangle$$

$[l_{-1}, l_0] = -l_{-1}$ $\qquad l_0 = -h$

$[l_1, l_0] = l_1$ $\qquad l_1 = f$

$[l_{-1}, l_1] = -2 l_0$ $\qquad l_1 = e$

$l_{-1} = -\frac{\partial}{\partial z}$ - translations

$l_0 = -z \frac{\partial}{\partial z}$ - dilatations / rotations.

$SO(1,3)$

Generators of algebra of conf. transform:

$$l_n = -z^{n+1}\frac{\partial}{\partial z}, \quad \bar{l}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$$

$$[l_n, l_m] = (n-m)\, l_{n+m}$$

$$[l_n, \bar{l}_m] = 0$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\,\bar{l}_{n+m}$$

Witt algebra

Witt $\oplus$ Witt.

Preserve real plane:

$l_n + \bar{l}_n, \; i(l_n - \bar{l}_n)$

$z^* = \bar{z}$ - real plane.

Witt $\supset sl_2 = \ldots$

$$[l_{-1}, l_0] = -\ldots$$

$$[l_1, l_0] = l\ldots$$

$$[l_{-1}, l_1] = -\ldots$$

$l_{-1} = -\frac{\partial}{\partial z}$ - tra…

$l_0 = -z\,\partial/\partial z$ - dil…

Transformations of fields

In 2d: rotations $SO(2) \simeq U(1)$

$$z \mapsto e^{i\varphi} z$$
$$\bar{z} \mapsto e^{-i\varphi} \bar{z}$$

Label of representation spin $s \in \frac{1}{2}\mathbb{Z}$ (or $\mathbb{C}$ in general
anyons
parafermions.)

$$\phi(e^{i\varphi} z, e^{-i\varphi} \bar{z}) = e^{-i\varphi s} \phi(z, \bar{z})$$

Dilatations: $\mathbb{R}_{>0}$: $z \mapsto e^{\sigma} z$, $\bar{z} \mapsto e^{\sigma} \bar{z}$

Label of representation scaling dimension: $\Delta \in \mathbb{R}$

$$\phi(e^{\sigma} z, e^{\sigma} \bar{z}) = e^{-\sigma \Delta} \phi(z, \bar{z})$$

al)

<u>Def</u> <u>Qusi</u>-primary fields: those, which transform as

$$\phi(w(z),\bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}),$$

where $W=w(z)$ — <u>global</u> conf. transform.

$h=\frac{1}{2}(\Delta+s)$
$\bar{h}=\frac{1}{2}(\Delta-s)$ — conformal weights

<u>Def</u> Primary fields. those which transforms:

$$\phi(w(z),\bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z})$$

under all conf. transf.

Infinitesimally: $w = z+\epsilon(z)$

$$\delta_{\epsilon,\bar{\epsilon}}\phi \equiv \tilde{\phi}(z)-\phi(z) = -(h\partial_z\epsilon+\epsilon\partial_z\phi) - (\bar{h}\partial_{\bar{z}}\bar{\epsilon}+\bar{\epsilon}\partial_{\bar{z}}\phi)$$

<u>Rem</u> There are fields which quasi-prim., but not primary e.g. T

<u>Rem</u> $\frac{dw}{dz} = \left|\frac{dw}{dz}\right| e^{i\arg\left(\frac{dw}{dz}\right)}$

local scaling factor

local rotation

s of fields

$z) \simeq U(1)$ $\quad z \mapsto e^{i\varphi} z$, $\bar{z} \mapsto e^{-i\varphi} \bar{z}$

pin $s \in \frac{1}{2}\mathbb{Z}$ (or $\mathbb{C}$ in general: anyons, parafermions.)

$= e^{-i\varphi s} \phi(z, \bar{z})$

$z \mapsto e^{\sigma} z, \ \bar{z} \mapsto e^{\sigma} \bar{z}$

scaling dimension: $\Delta \in \mathbb{R}$

$) = e^{-\sigma\Delta} \phi(z, \bar{z})$

Def Qusi-primary fields: those, which tran

$$\phi(w(z), \bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}$$

Def Primary fields. those which tran

$$\phi(w(z), \bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right.$$

Infinitesimally: $w = z + \epsilon(z)$ $\qquad \delta_{\epsilon, \bar{\epsilon}} \phi \equiv$

Rem There are fields which quasi-prim. but not

Rem $\frac{dw}{dz} = \left|\frac{dw}{dz}\right| e^{i \arg\left(\frac{dw}{dz}\right)}$ local scaling fact, local rotat

Noether theorem

When we have global symmetry.

$$\tilde{\phi}(\tilde{x}) = \mathcal{F}(\phi(x))$$
$$\tilde{x} = \tilde{x}(x)$$

$\longrightarrow$

$$\delta_\omega \phi = \tilde{\phi}(x) - \phi(x) = -i\omega G\phi(x)$$
$$iG\phi = \frac{\delta x^\mu}{\delta\omega}\partial_\mu\phi - \frac{\delta\mathcal{F}}{\delta\omega}$$

Symmetry means: $S[\tilde{\phi}(x)] = S[\phi(x)]$

$$S[\phi(x) - i\omega G\phi] - S[\phi(x)] \simeq O(\omega^2)$$

Assume that $\omega = \omega(x)$: $S[\phi(x) - i\omega G\phi] - S[\phi(x)] \simeq -\int d^dx\, j^\mu \partial_\mu \omega + O(\omega^2)$

(in case if $\mathcal{L}$ contains ϕ, $\partial_\mu \phi$)

On the other hand: the least action principle: under any variation of fields

$$S[\phi + \delta\phi] - S[\phi] \simeq O(\delta\phi^2)$$

on-shell

$\Downarrow$

On-shell $-\int d^dx\, j^\mu \partial_\mu \omega(x) = 0 \quad \forall\, \omega(x)$

$$\int d^dx\, \partial_\mu j^\mu\, \omega(x) \Leftrightarrow \boxed{\partial_\mu j^\mu = 0}$$

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \omega} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta F}{\delta \phi}$$

Noether theorem

When we have global symmetry.

$$\tilde{\phi}(\tilde{x}) = \mathcal{F}(\phi(x)) \qquad \tilde{x} = \tilde{x}(x) \longrightarrow \delta_\omega \phi = \tilde{\phi}(x) - \phi(x) = -i\omega G\phi(x)$$

$$iG\phi = \frac{\delta x^\mu}{\delta\omega}\partial_\mu\phi - \frac{\delta \mathcal{F}}{\delta\omega}$$

Symmetry means: $S[\tilde{\phi}(x)] = S[\phi(x)]$

$$S[\phi(x) - i\omega G\phi] - S[\phi(x)] \simeq \mathcal{O}(\omega^2)$$

$\mathfrak{g} = n_+ \oplus \mathfrak{h} \oplus n_-$

sl_n: upper triang

lower...

diagonal. $[\mathfrak{h}, \mathfrak{h}] = 0$

$[n_+, n_+] \subset n_+$

with $n_+ = \langle l_n \rangle \; n > 0$

$n_- = \langle l_n \rangle \; n < 0$

$\mathfrak{h} = \langle l_0 \rangle$

Assume that ω

On the other hand

$\Downarrow$

On-shell $-\int d^d x$

$\parallel$

$\int d^d x$

$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right.$