

Title: Quantum Field Theory for Cosmology - Lecture 20240328

Speakers: Achim Kempf

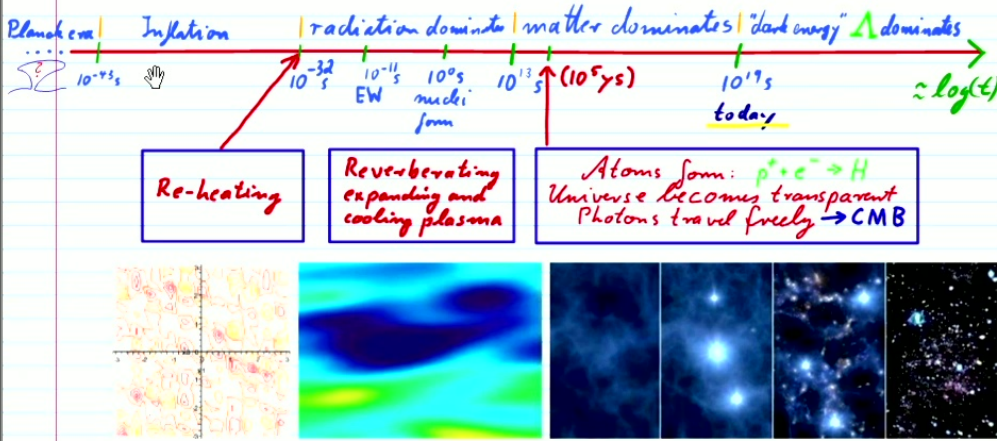
Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: March 28, 2024 - 4:00 PM

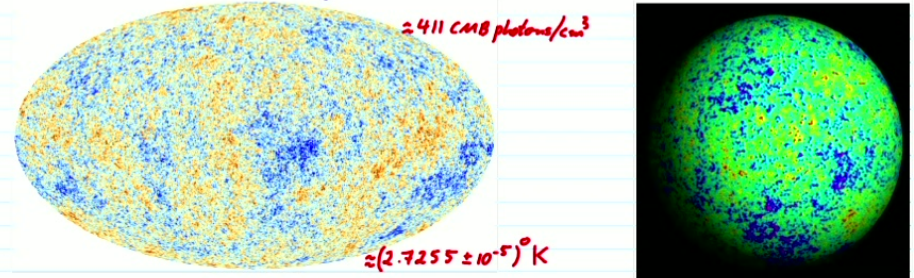
URL: <https://pirsa.org/24030015>

QFT for Cosmology, Achim Kempf, Lecture 22

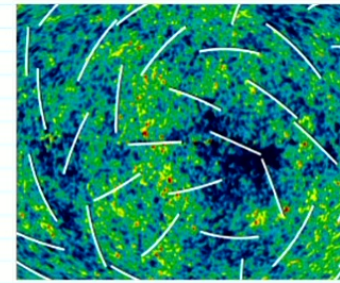
Time line of standard model of cosmology:



Actual observations of the CMB:



Zoom-in, with polarization: (avg polarization  $\approx 10^{-6}$ )



Recall:

$$\phi(x, \eta) = \phi_0(\eta) + \varphi(x, \eta) \quad \text{with } |\varphi(x, \eta)| \ll |\phi_0(\eta)|$$

$$g_{\mu\nu}(x, \eta) = a(\eta)^2 \eta_{\mu\nu} + \gamma_{\mu\nu}(x, \eta) \quad \text{with } |\gamma_{\mu\nu}(x, \eta)| \ll 1$$

$$ds^2 = a^2(\eta) (d\eta^2 - \sum_{i=1}^3 (dx^i)^2) + \underbrace{ds_s^2}_{\text{scalar}} + \underbrace{ds_v^2}_{\text{vector}} + \underbrace{ds_t^2}_{\text{tensor}}$$

$$ds_s^2 = a^2(\eta) \left[ 2\Xi(x, \eta) d\eta^2 - 2 \sum_{i,j=1}^3 \frac{\gamma_{ij}}{a^2} B(x, \eta) dx^i dx^j - \sum_{i,j=1}^3 (2\Phi(x, \eta) \delta_{ij} - 2\gamma_{ij}(x, \eta)) dx^i dx^j \right]$$

The surviving gauge-invariant degrees of freedom are:

- The purely tensorial part of the metric:  $h_{ij}(x, \eta)$
- A combination of a scalar part of the metric,  $\Psi(x, \eta)$ , and  $\varphi(x, \eta)$ :

$$\tau(x, \eta) := -\frac{a_i}{a_i} (\phi_0(\eta))'^{-1} \varphi(x, \eta) - \Psi(x, \eta)$$

They possess these actions:

$$ds_1^2 = a^2(\gamma) \left[ 2\mathbb{E}(x, \gamma) d\gamma^2 - 2 \sum_{i,j=1}^3 \frac{\partial}{\partial x^i} B(x, \gamma) dx^i d\gamma \right. \\ \left. - \sum_{i,j=1}^3 \left( 2\Phi(x, \gamma) \delta_{ij} - 2 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} E(x, \gamma) \right) dx^i dx^j \right]$$

$$ds_2^2 = a^2(\gamma) \left[ 2 \sum_{i,j=1}^3 V_i(x, \gamma) dx^i d\gamma \right. \\ \left. - \sum_{i,j=1}^3 \left( \frac{\partial}{\partial x^i} W_j(x, \gamma) + \frac{\partial}{\partial x^j} W_i(x, \gamma) \right) dx^i dx^j \right]$$

$$ds_3^2 = a^2(\gamma) \sum_{i,j=1}^3 h_{ij}(x, \gamma) dx^i dx^j$$

They possess these actions:

$$S_T = \frac{i}{64\pi G} \sum_{i,j=1}^3 \int a^2(\gamma) \frac{\partial}{\partial x^i} (h^i_j(x, \gamma)) \frac{\partial}{\partial x^j} (h^i_j(x, \gamma)) \gamma^{\mu\nu} d^4x$$

$$S_S = \frac{1}{2} \int z^2(\gamma) \left( \frac{\partial}{\partial x^i} r(x, \gamma) \right) \left( \frac{\partial}{\partial x^i} r(x, \gamma) \right) \gamma^{\mu\nu} d^4x \text{ with } z(\gamma) := \frac{a^2(\gamma)}{a^2(\gamma)} \dot{\phi}_s(\gamma)$$

To quantize without a friction term, change variable:

$$u(x, \gamma) := -z(\gamma) r(x, \gamma)$$

↙ constant factors

$$p_{ij}(x, \gamma) := \frac{i}{\sqrt{32\pi G}} a(\gamma) h_{ij}(x, \gamma)$$

Further, separate polarization matrices:

$$p_{ij}(k, \gamma) := \sum_{\lambda=1,2} v_{k,\lambda}(\gamma) \epsilon_{ij}(k, \lambda)$$

→ Equations of motion:

$$\hat{v}_{\alpha,2}''(\gamma) + \left( k^2 - \frac{a''}{a} \right) \hat{v}_{\alpha,2}(\gamma) = 0$$

$$\hat{u}_k''(\gamma) + \left( k^2 - \frac{z''(\gamma)}{z(\gamma)} \right) \hat{u}_k(\gamma) = 0$$

### Quantum fluctuations

As before, this reduces to solving the eqns of motion for the mode functions, which are complex number-valued, say  $\hat{u}_k(\gamma)$ ,  $\hat{v}_{\alpha,2}(\gamma)$ :

$$\hat{u}_k''(\gamma) + \left( k^2 - \frac{z''(\gamma)}{z(\gamma)} \right) \hat{u}_k(\gamma) = 0$$

$$\hat{v}_{\alpha,2}''(\gamma) + \left( k^2 - \frac{a''}{a} \right) \hat{v}_{\alpha,2}(\gamma) = 0$$

along with the Wronskian conditions.

#### Initial conditions?

At early times:

- \* The  $k^2$  term dominates
- ⇒ can choose Minkowski-like init. cond.

We say we choose the "Bunch Davies vacuum".



Further, separate polarization matrices:

$$P_{ij}(k, \gamma) := \sum_{\lambda=1,2} v_{k,\lambda}(\gamma) \epsilon_{ij}(k, \lambda)$$

Equations of motion:

$$\hat{v}_{k,2}''(\gamma) + \left(k^2 - \frac{a''}{a}\right) \hat{v}_{k,2}(\gamma) = 0$$

$$\hat{u}_k''(\gamma) + \left(k^2 - \frac{z''(\gamma)}{z(\gamma)}\right) \hat{u}_k(\gamma) = 0$$

$$\hat{u}_k''(\gamma) + \left(k^2 - \frac{z''(\gamma)}{z(\gamma)}\right) \hat{u}_k(\gamma) = 0$$

$$\hat{v}_{k,2}''(\gamma) + \left(k^2 - \frac{a''}{a}\right) \hat{v}_{k,2}(\gamma) = 0$$

along with the Wronskian conditions.

Initial conditions?

We say we choose the "Bunch Davies vacuum".

At early times:

- \* The  $k^2$  term dominates
- ⇒ Can choose Minkowski-like init. cond.

The mode fctns at late times?

At late times:

- \* The mode  $k$  crossed the Hubble horizon:
- \* The terms  $\frac{z''}{z}$  and  $\frac{a''}{a}$  dominate.
- \* The harmonic oscillator is inverted
- \* Instead of 2 oscillatory basis sol's we now expect one growing and one decaying basis solution.
- \* Soon after horizon crossing the mode function consists of essentially only the growing solution.

Which is the growing solution at late times?

Eqs of motion after horizon crossing:

$$\hat{u}_k''(\gamma) + \left(\gamma^2 - \frac{z''(\gamma)}{z(\gamma)}\right) \hat{u}_k(\gamma) = 0, \text{ i.e., } \frac{\hat{u}_k(\gamma)''}{\hat{u}_k(\gamma)} = \frac{z''(\gamma)''}{z(\gamma)}$$

$$\hat{v}_{k,2}''(\gamma) + \left(\gamma^2 - \frac{a''}{a}\right) \hat{v}_{k,2}(\gamma) = 0, \text{ i.e., } \frac{\hat{v}_{k,2}(\gamma)''}{\hat{v}_{k,2}(\gamma)} = \frac{a''(\gamma)''}{a(\gamma)}$$

⇒ Growing solution must behave as:

$$\hat{u}_k(\gamma) \sim z(\gamma) \text{ at late } \gamma$$

$$\hat{v}_{k,2}(\gamma) \sim a(\gamma) \text{ at late } \gamma$$

⇒ The mode fctns  $\tilde{r}_k(\gamma) = -\frac{\hat{u}_k(\gamma)}{z(\gamma)}$  and  $\tilde{h}_{r,s}(\gamma) = 12\pi G \frac{\hat{v}_{k,2}(\gamma) \epsilon_{ij}(k,2)}{a(\gamma)}$  become constant at late  $\gamma$ , i.e., after the mode  $k$  crosses the horizon!

- \* The harmonic oscillator is inverted
- \* Instead of 2 oscillatory basis sol's we now expect one growing and one decaying basis solution.
- \* Soon after horizon crossing the mode function consists of essentially only the growing solution.

$$v_{i,1}(\eta) + \left(\gamma - \frac{\epsilon}{2}\right)v_{i,2}(\eta) = 0, \text{ i.e., } \frac{v_{i,1}(\eta)}{v_{i,2}(\eta)} = \frac{a(\eta)^{2\epsilon}}{a(\eta)}$$

⇒ Growing solution must behave as:

$$\tilde{u}_i(\eta) \sim z(\eta) \text{ at late } \eta$$

$$\tilde{v}_{i,2}(\eta) \sim a(\eta) \text{ at late } \eta$$

⇒ The mode fctns  $\tilde{r}_i(\eta) = -\frac{\tilde{u}_i(\eta)}{a(\eta)}$  and  $\tilde{h}_{i,j_0}(\eta) = 12\pi G \frac{\tilde{v}_{i,2}(\eta) \epsilon_{ij}(k,z)}{a(\eta)}$  become constant at late  $\eta$ , i.e., after the mode  $k$  crosses the horizon!

Check:  $\tilde{r}_i(\eta) = \frac{1}{z(\eta)} \tilde{u}_i(\eta) \sim \frac{z(\eta)}{z(\eta)}$  for late  $\eta$

$$\tilde{h}_{i,j_0}(\eta) = \frac{1}{a(\eta)} \tilde{p}_{i,j_0}(\eta) \sim \frac{1}{a(\eta)} \tilde{v}_{i,2}(\eta) \sim \frac{a(\eta)}{a(\eta)} \text{ for late } \eta$$

⇒ As expected, the magnitude of the mode  $k$ 's quantum fluctuations

$$\delta r_k = k^{3/2} |\tilde{r}_k| \quad \text{and} \quad \delta h_{i,j_0} = k^{3/2} |\tilde{h}_{i,j_0}|$$

stay constant at the value that they possess when the mode crosses the horizon, even as the mode's proper wavelength then continues to increase rapidly.

\* Goal now: Calculate the magnitude of the fluctuations at horizon crossing!

Realistic example: "Power law inflation"

□ We need an explicit potential  $V(\phi)$  in order to be able to find explicit  $a(\eta)$ ,  $\phi_s(\eta)$  for which to calculate then the fluctuation spectrum.

□ De Sitter is ruled out because:

- \*  $V(\phi)$ , and therefore the temporary "cosmological constant  $H \sim \sqrt{V(\phi)}$ " must slowly decrease (slow roll).
- \* In any case, our perturbation assumptions don't allow exact de Sitter, as  $\delta v_k$  would diverge, invalidating the assumption that it is small.

Constant factor

⇒ As expected, the magnitude of the mode  $k$ 's quantum fluctuations

$$\delta r_{\vec{k}} = \underbrace{z^{-1} k^{3/2}}_n |\tilde{r}_{\vec{k}}| \quad \text{and} \quad \delta h_{i,j,\vec{k}} = \underbrace{\alpha^{-1} k^{3/2}}_n |\tilde{h}_{i,j,\vec{k}}|$$

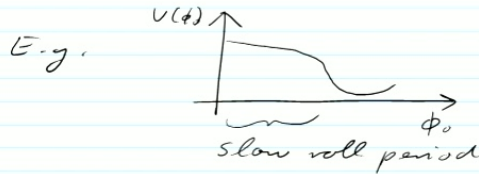
stay constant at the value that they possess when the mode crosses the horizon, even as the mode's proper wavelength then continues to increase rapidly.

\* Goal now: Calculate the magnitude of the fluctuations at horizon crossing!

### The "slow roll parameters"

Idea:

\* We do not know the exact slow roll potential:



\* For all values of  $\phi_0$  during the inflationary period we can parametrize the slope of the potential by its derivatives.

\* These are the so-called slow roll parameters: (Recall:  $H(\phi) \sim \sqrt{V(\phi)}$ )

□ De Sitter is ruled out because:

\*  $V(\phi)$ , and therefore the temporary "cosmological constant  $H \sim \sqrt{V(\phi)}$ " must slowly decrease (slow roll).

\* In any case, our perturbation assumptions don't allow exact de Sitter, as  $\delta r_{\vec{k}}$  would diverge, invalidating the assumption that it is small.

$$\epsilon(\phi) := \frac{1}{4\pi G} \left( \overset{\text{curvature factor}}{\frac{H'(\phi)}{H(\phi)}} \right)^2 \quad \left( = \frac{3}{2} \frac{\dot{\phi}^2}{V + \frac{1}{2} \dot{\phi}^2} \right)$$

$$\eta(\phi) := \frac{1}{4\pi G} \frac{H''(\phi)}{H(\phi)} \quad \left( = \epsilon - \frac{\epsilon'}{\sqrt{16\pi G \epsilon}} \right)$$

$$\xi(\phi) := \frac{1}{4\pi G} \sqrt{\frac{H'(\phi)H''(\phi)}{H^3(\phi)}}$$

etc...

□ The simplest solvable case:

\* The simplest case is that of

$$\epsilon(\phi) = c \quad \text{where } c \text{ is a constant.}$$



$\phi_0$   
slow roll period

\* For all values of  $\phi$ , during the inflationary period we can parametrize the slope of the potential by its derivatives.

\* These are the so-called slow roll parameters: (Recall:  $H(\phi) \sim \sqrt{V(\phi)}$ )

\* In this case:

$$c = \epsilon(\phi) := \frac{1}{4\pi G} \left( \frac{H'(\phi)}{H(\phi)} \right)^2$$

Thus,

$$H(\phi) \sim e^{\sqrt{4\pi G c} \phi}$$

and the potential is of the form:

$$V(\phi) = e^{s\phi}$$

\* Exercise: What is the value of  $s$ ?

\* Then, one also finds:

$$c = \epsilon = \eta = \xi = \dots$$

$$\xi(\phi) := \frac{1}{4\pi G} \sqrt{\frac{H'(\phi)H''(\phi)}{H^4(\phi)}}$$

etc...

□ The simplest solvable case:

\* The simplest case is that of

$$\epsilon(\phi) = c \quad \text{where } c \text{ is a constant.}$$

\* The expansion rate is polynomial:

Exercise:

Show that:

$$a(t) = a_0 t^{1/c} \quad (t \text{ is proper time})$$

Exercise:

Show that, in terms of the conformal time  $\eta$ :

$$a(\eta) = \frac{-1}{\eta H} \frac{1}{1-\epsilon}$$

Note: Still  $\eta$  is always negative and  $t \rightarrow \infty$  means  $\eta \rightarrow 0$ .

and the potential is of the form:

$$V(\phi) = e^{s\phi}$$

\* Exercise: What is the value of  $s$ ?

\* Then, one also finds:

$$c = \epsilon = \gamma = \xi = \dots$$

Exercise:

Show that, in terms of the conformal time  $\eta$ :

$$a(\eta) = \frac{-1}{\eta H} \frac{1}{1-\epsilon}$$

Note: Still  $\eta$  is always negative and  $t \rightarrow \infty$  means  $\eta \rightarrow 0$ .

The mode equations:

□ Scalar: We can now calculate  $z(\eta) = \frac{a^2(\eta)}{a'(\eta)} \phi'(\eta)$  and therefore also the mode equation's term  $z''/z$  explicitly, to obtain

$$\ddot{u}_s(\eta) + \left( k^2 - \frac{(\nu^2 - 1/4)}{\eta^2} \right) \tilde{u}_s(\eta) = 0$$

where:  $\nu := \frac{3}{2} + \frac{c}{1-c}$

\* Solution for Bunch Davies initial conditions:

$$\tilde{u}_s(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\frac{\pi}{2}} (-\eta)^{\nu+1/2} H_\nu^{(1)}(-k\eta)$$

↑ Hankel fun of 1st kind of order  $\nu$ .

\* Behavior after horizon crossing:

$$\tilde{u}_s(\eta) \rightarrow e^{i(\nu-1/2)\frac{\pi}{2}} 2^{\nu-1/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{-\nu+1/2}$$

\* Thus, the magnitude of intrinsic curvature fluctuations after horizon crossing becomes:

$$\delta\tau_s(\eta > \eta_{hor}) = G 2^{\nu-1/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (\nu-1/2)^{1/2-\nu} \frac{H^2}{|H'|} \Big|_{at k=k_H}$$

↑  
horizon crossing

Exercise: verify

Notice: Measurement of  $\delta\tau_s$  in CMB can only tell us  $\frac{H^2}{H'}$  (at horizon crossing) but not  $H$  or  $H'$  individually!



Now, for the intrinsic curvature (the Mukhanov variable), we found:

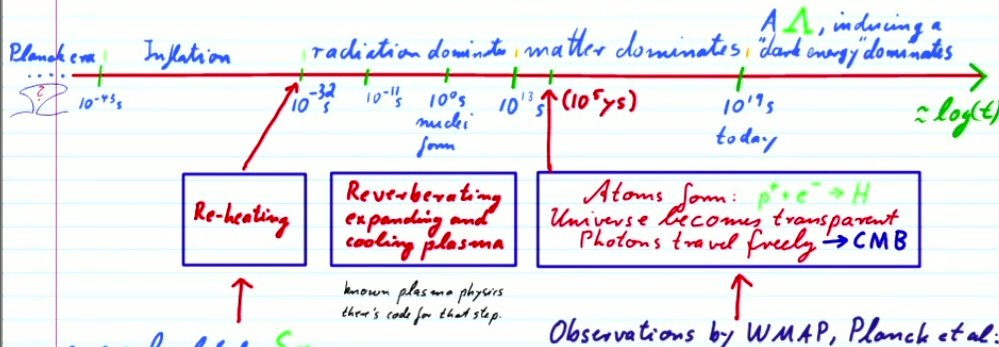
$$\delta r_k \sim H^2 / |H'|$$

Recall:  $r_k$  is the slicing-independent combination of the scalar part of  $\delta g_{\mu\nu}$  and  $\phi$ .

The slower the roll ( $|H'|$  small) the wider away from another fluctuate gauge equivalent and inequivalent slicings:



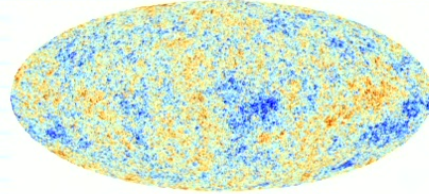
### Recall the timeline:



we calculated:  $\delta r_k$  at horizon crossing, then unchanged till end of inflation: re-heating.

known plasma physics there's code for that step.

Observations by WMAP, Planck et al.:



- $\delta r_k$  is predicted to have seeded oscillations in the hot plasma after re-heating. The plasma decohered the quantum fluctuations of the intrinsic curvature  $v$ .
- Standard plasma physics allows one to calculate the propagation and dispersion for the  $\approx 10^{11}s$  until hydrogen formed.
- The temperature fluctuation spectrum in the CMB is from gravitational blue and redshifts due to those curvature fluctuations.
- Theory matches experiment closely, while fixing cosmological parameters, including indications that  $\epsilon \neq 0$ , namely that  $\delta r_k \neq \text{const}$ .

$$\ddot{\tilde{u}}_k(\eta) + \left( k^2 - \frac{(\nu^2 - 1/4)}{\eta^2} \right) \tilde{u}_k(\eta) = 0$$

where:  $\nu := \frac{3}{2} + \frac{c}{1-c}$

\* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\frac{\pi}{2}} (-\eta)^{\nu+1/2} H_{\nu}^{(1)}(-k\eta)$$

↑ Hankel fun of 1st kind of order  $\nu$ .

$$\delta\tau_k(\eta > \eta_{hor}(\nu)) = G 2^{\nu-1/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (\nu - \frac{1}{2})^{1/2-\nu} \frac{H^2}{|H'|} \Big|_{\text{at } k=k_H}$$

↑ horizon crossing

Exercise: verify

Notice: Measurement of  $\delta\tau_k$  in CMB can only tell us  $\frac{H^2}{H'}$  (at horizon crossing) but not  $H$  or  $H'$  individually!

Intuition?

Earlier, for a K.G. field  $\phi$  in a fixed background FRW universe, we found:

$$\delta\phi_k \sim H$$

Now, for the intrinsic curvature (the Mukhanov variable), we found:

$$\delta\tau_k \sim \frac{H^2}{|H'|}$$

Recall:  $\tau_k$  is the slicing-independent combination of the scalar part of

$$\delta g_{\mu\nu} \text{ and } \phi.$$

The slower the roll ( $|H'|$  small) the wider away from another fluctuate gauge equivalent and inequivalent slicings:

Analogous to: A river in a plain meanders the more widely, the flatter the plain is.





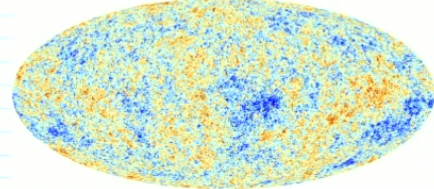
Re-heating

Reverberating expanding and cooling plasma

Atoms form:  $p^+ e^- \rightarrow H$   
 Universe becomes transparent  
 Photons travel freely  $\rightarrow$  CMB

known plasma physics there's code for that step.

Observations by WMAP, Planck et al.

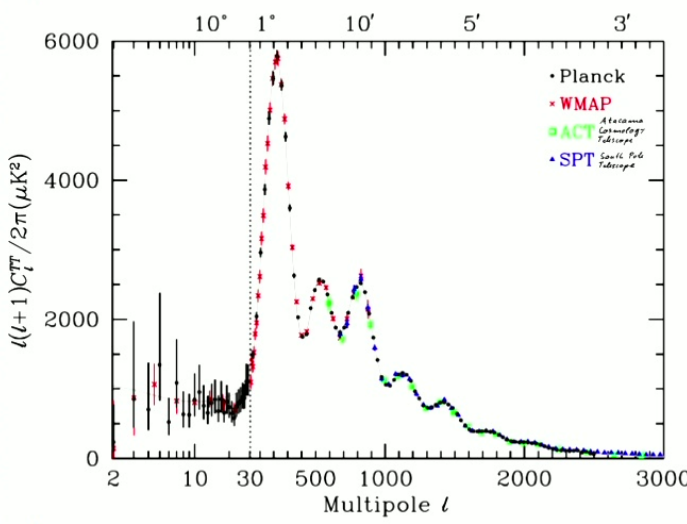


we calculated:  $\delta r_k$   
 at horizon crossing,  
 then unchanged till end of inflation: re-heating.

standard plasma physics allows one to calculate the propagation and dispersion for the  $\approx 10^5$  yrs until hydrogen formed.

- The temperature fluctuation spectrum in the CMB is from gravitational blue and redshifts due to these curvature fluctuations.
- Theory matches experiment closely, while fixing cosmological parameters, including indications that  $\epsilon \neq 0$ , namely that  $\delta r_k \neq \text{const}$ :

Best fit today:



- $K = 0$
- $\Lambda \approx 0.7 \rho_{critical}$
- $\rho_{matter} \approx 0.3 \rho_{critical}$
- $\rho_{dark matter} \approx 0.9 \rho_{matter}$
- $\rho_{visible matter} \approx 0.1 \rho_{matter}$
- $\rho_{CMB} \approx 5 \cdot 10^{-5} \rho_{critical}$
- $v_{peculiar} \approx 370 \text{ km/s}$  of earth

Tensor modes:  $\ddot{v}_{i,j} + \left(k^2 - \frac{a''}{a}\right) \check{v}_{i,j} = 0$

we obtain for the term  $a''/a$ :

$$\frac{a''}{a} = 2a^2 H^2 (1 - \epsilon/2)$$

which comes out to be (verify):

$$\frac{a''}{a} = \frac{1}{\eta^2} \left(v^2 - \frac{1}{4}\right) \quad \text{recall: } v = \frac{3}{2} - \frac{c}{1-c}$$

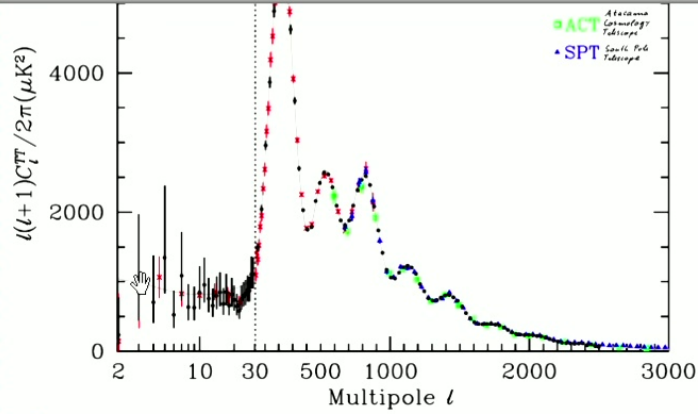
$\Rightarrow$  The mode eqn is again solved by the Hankel functions.

$\Rightarrow$  The tensor fluctuation spectrum:

$$\delta h_{i,j} = \frac{2}{\sqrt{4\pi}} 2^{\nu-1/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (2\nu-1/2)^{\nu-1/2} \sqrt{V} H \Big|_{k=aH}$$

horizon crossing  $\downarrow$





$\Omega \approx 0.7 \rho_{critical}$   
 $\rho_{matter} \approx 0.3 \rho_{critical}$   
 $\rho_{dark matter} \approx 0.9 \rho_{matter}$   
 $\rho_{visible matter} \approx 0.1 \rho_{matter}$   
 $\rho_{CMB} \approx 5 \cdot 10^{-5} \rho_{critical}$   
 $v_{peculiar} \approx 370 \text{ km/s of earth}$

which comes out to be (verify):

$$\frac{a''}{a} = \frac{1}{4} \left( v^2 - \frac{1}{4} \right) \quad \text{recall: } v = \frac{3}{2} - \frac{c}{1-c}$$

⇒ The mode eqn is again solved by the Hankel function.

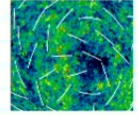
⇒ The tensor fluctuation spectrum:

$$\delta h_{ij} = \frac{2}{V_H^2} 2^{v-1/2} \frac{\Gamma(v)}{\Gamma(3/2)} (v-1/2)^{v-1/2} \sqrt{V_H} H \Big|_{k=aH}$$

horizon crossing.  
↓

$\delta h_{ij}$  should have left curl ("B") polarization in the CMB

Experiments show polarization in the CMB:



□ But most is gradient ("E") polarization that originated in  $\delta v_n$  or in foreground.

□ So far,  $h_{ij}$ -originated B-polarization cannot be distinguished from foreground.

□ Observation of  $h_{ij}$  polarization:

- \* Would show quantised gravitational waves!
  - \* Would determine the scale of  $H$ , and therefore of  $H^2$ !
  - \* This would tell the slope of the spectra
- ⇒ Nontrivial consistency conditions to check inflation.