Title: Quantum Field Theory for Cosmology - Lecture 20240321

Speakers: Achim Kempf

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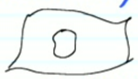
QFT for Cosmology, Achim Kempf, Lecture 20

How could an initial temporary strong inflation have been caused?

$\boxed{V(\phi) \text{ temporarily very large}}$

- Consider a universe like ours.
  Everywhere, at all times, all fields quantum fluctuate.

- As a rare fluke, the field ϕ quantum fluctuates in a patch a few Planck lengths in size

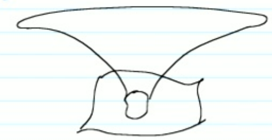


to a ϕ value that makes $V(\phi)$ close to the Planck scale.

(Assume homogeneity in that patch, so that the $\partial_i \phi$ are small)

- In this patch, $V(\phi)$ is dominant and impacting $a(t)$ like a large Λ would.

$\Rightarrow$ Before the fluctuation can "snap back", general relativity will quasi-exponentially inflate this patch (potentially, e.g., by 10^5 orders of magnitude).

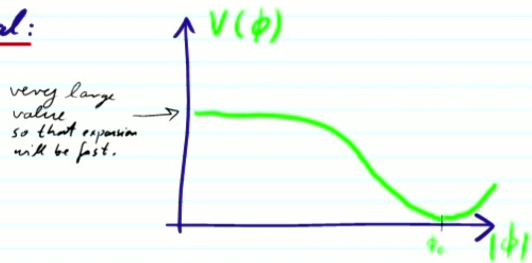


$\Rightarrow$ The mother universe spawns a daughter universe!

- $V(\phi)$ in the patch starts out high but will dynamically fall eventually to low value $\rightarrow$ Inflation ends.
- The energy in $V(\phi)$ turns into hot matter.

- The daughter universe can spawn new universes and so on...

Example potential:

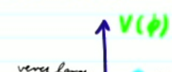


* **Concretely:**

The Klein Gordon equation reads:

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\underbrace{\dot{\phi}}_{\text{friction term}} - \frac{dV}{d\phi}$$

This is like the equation of motion of a ball rolling down a hill, with friction:



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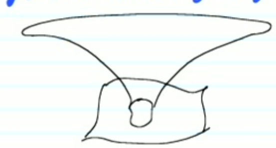
- Consider a universe like ours.
  Everywhere, at all times, all fields quantum fluctuate.

- As a rare fluke, the field ϕ quantum fluctuates in a patch a few Planck lengths in size

to a ϕ value that makes $V(\phi)$ close to the Planck scale.

(Assume homogeneity in that patch, so that the $\partial_i \phi$ are small)

(potentially, e.g., by $10^{?}$ orders of magnitude).



⇒ The mother universe spawns a daughter universe!

- $V(\phi)$ in the patch starts out high but will dynamically fall eventually to low value → Inflation ends.
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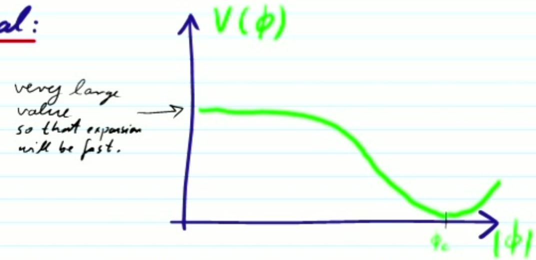

- Then, inflation starts when, in a patch, ϕ is very small, even though it is energetically expensive (a rare quantum fluctuation)
- Then, after ϕ starts out at $\phi = 0$ and large $V(\phi)$, it will slowly evolve towards ϕ_c while the universe inflates, thus flattens, and the matter dilutes.
- Once $\phi = \phi_c$ is reached, $V(\phi) = 0$, and inflation has ended.

* **Concretely:**

The Klein Gordon equation reads:

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\underbrace{\dot{\phi}}_{\text{friction term}} - \frac{dV}{d\phi}$$

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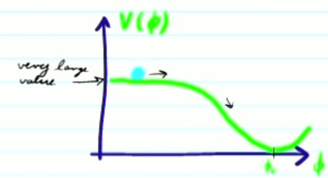

$\left(-\frac{dV}{d\phi}\right)$ acts to pull ϕ down the potential hill.

$\left(-3\frac{\dot{a}}{a}\dot{\phi}\right)$ acts as a "friction" term.

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will be fast.

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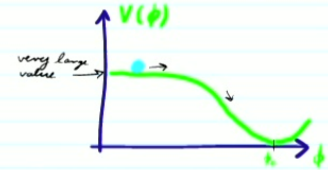
rolling down a hill, with friction:



$\left(-\frac{dV}{d\phi}\right)$ acts to pull ϕ down the potential hill.

$\left(-3\frac{\dot a}{a}\dot\phi\right)$ acts as a "friction" term.

* Definition:

If the initial value of $V(\phi)$ is very large and if the initial slope is very flat, i.e., if the ball for a period rolls slowly, with approximately constant $V(\phi)$, we call this a period of "Slow Roll Inflation".

* Observation:

During the slow roll period, we have, in particular, that $V(\phi)$ dominates over $\frac{1}{2}\dot\phi^2$.

$$\Rightarrow \quad w(t) = \frac{\frac{1}{2}\dot\phi^2 - V(\phi)}{\frac{1}{2}\dot\phi^2 + V(\phi)} \approx -1 \quad (\text{temporarily})$$

* But, do we also get temporary exponential inflation?

Indeed, the 0,0 component of the Einstein equation

$$\left(\frac{\dot a}{a}\right)^2 - \frac{1}{3}\Lambda = \frac{8\pi G}{3} T^0_0$$

is during the slow roll period:

$$\left(\frac{\dot a}{a}\right)^2 \approx \frac{8\pi G}{3} V(\phi)$$

whose solution during slow roll is

$$a(t) \approx a_0\, e^{t\sqrt{\frac{8\pi G}{3} V(\phi)}}$$

(ϕ and $V(\phi)$ change slowly over time in slow roll.)

* Thus, during slow roll, we have effectively a slowly varying Hubble parameter!

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(ϕ and $V(\phi)$ change slowly over time in slow roll.)

* Thus, during slow roll, we have effectively a slowly varying Hubble parameter!

Definition:

The function $H(t) := \frac{\dot{a}(t)}{a(t)}$ is called the Hubble parameter function.

* In the case $a(t) = e^{Ht}$ we recover $H = H(t)$.

* In the case of slow roll inflation, we have

$$H(t) = \sqrt{\frac{8\pi G}{3} V(\phi(t))}$$

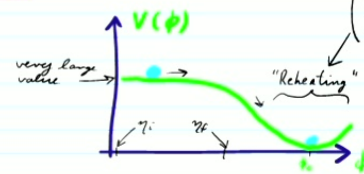
Remark:

□ As $V(t)$ decreases, also $H(t)$ decreases.

$\Rightarrow$ inflation predicts that $\delta\phi_k$ decreases for later and later horizon crossing modes, i.e., for smaller and smaller wavelength modes. The WMAP satellite's CMB data show evidence for this!

The end of inflation:

(Note that this is a misnomer. We do not necessarily assume the existence of a prior hot phase)



* In the period called "Re-heating", the energy of ϕ is transferred into the mode oscillators of the usual (low mass) particles, i.e., the inflaton particles decay and thereby create a high energetic, i.e. hot, plasma of literally all sorts of particles.

* From thereon, the usual big bang cosmology is assumed to have followed.

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Quantum fluctuations during cosmic inflation

e.g.

Strategy:

- We assume a suitable potential $V(\phi)$ and suitable initial conditions, as discussed before.

  $\Rightarrow$ Solutions $\phi(t)$ and $a(t)$ which exhibit slow roll inflation for a suitable finite time interval $[t_i, t_f]$, i.e., $[\eta_i, \eta_f]$.

- We consider the case of small inhomogeneities in the inflaton field:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \quad \text{with} \quad |\varphi(x,\eta)| \ll |\phi_0(\eta)|}$$

- This means that we must also consider small fluctuations in the metric, because:

  In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T^{(\phi)}_{\mu\nu}(\eta,\vec{x}) = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\tfrac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}m^2\phi^2\right]$$

$\Rightarrow$ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

inhomogeneities of $\phi(x)$ induce inhomogeneities of $g_{\mu\nu}$

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⇒ Solutions $\phi_0(t)$ and $a(t)$ which exhibit slow roll inflation for a suitable finite time interval $[t_i, t_f]$, i.e., $[\eta_i, \eta_f]$.

- We consider the case of small inhomogeneities in the inflaton field:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varepsilon(x,\eta) \quad \text{with } |\varepsilon(x,\eta)| \ll |\phi_0(\eta)|}$$

⇝ Consider also small inhomogeneities in the metric, i.e., in the spacetime

$$\boxed{g_{\mu\nu}(x,\eta) = a(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta) \quad \text{with } |\gamma_{\mu\nu}(x,\eta)| \ll 1}$$

- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\phi}(x)$ but this is too hard, inconsistent so far.

- In lowest order perturbation theory we first find the classical solutions $g_{0\,\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

- Then, we quantize only the $\hat{\varepsilon}(x,\eta)$ and $\hat{\gamma}_{\mu\nu}(x,\eta)$.

to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T^{\text{infl}}_{\mu\nu}(\eta,\vec{x}) = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\tfrac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2} m^2\phi^2\right]$$

⇒ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \tfrac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

inhomogenities of $\phi(x)$ induce inhomogenities of $g_{\mu\nu}(x)$:

- Why does this approximation help?

  * For fields $\hat{\varepsilon}$, $\hat{\gamma}_{\mu\nu}$ that are "small" the equations of motions are effectively linear in $\hat{\varepsilon}, \hat{\gamma}_{\mu\nu}$.

  * This is because we can assume that in their equations of motion all terms that are quadratic or of higher power are negligible.

  * This means that the quantum fields $\hat{\varepsilon}$ and $\hat{\gamma}_{\mu\nu}$ have no potential terms, nor any mass terms.

  ⇒ We will obtain a free, i.e., noninteracting quantum field theory whose nontriviality only stems from the time-varying parameters $\phi_0(\eta)$, $a_0(\eta)$.

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- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\varphi}(x)$ but this is too hard, inconsistent so far.
- In lowest order perturbation theory we first find the classical solutions $g_{\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\varphi_0(\eta)$ that are completely homogeneous and isotropic.
- Then, we quantize only the $\hat{\varphi}(x,\eta)$ and $\hat{\gamma}_{\mu\nu}(x,\eta)$.

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$\Rightarrow$ We will obtain a free, i.e., noninteracting quantum field theory whose nontriviality only stems from the time-varying parameters $\varphi_0(\eta)$, $a_0(\eta)$.

- **Intuition:** We should expect two more sources of nontriviality:

1) Interdependence of φ and $\gamma_{\mu\nu}$ inhomogeneities:

* Much of the inhomogeneities of $\hat{\gamma}_{\mu\nu}(x,\eta)$ will be induced by the inhomogeneities of the inflaton, $\hat{\varphi}(x,\eta)$.

* Vice versa: we can also read the Einstein eqn from left to right $\Rightarrow$ these gravity inhomogeneities induce the inflaton's inhomogeneities.

* Thus, the inflaton's inhomogeneities' dynamics cannot be separated from that of the metric.

Recall:

Gravity is a force with some similarity to electromagnetism:

- Some electromagnetic fields only exist because there are charges or currents.

Similarly: Some part of the metric will depend on φ, while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of φ. $\rightarrow$

- **But:** also, some electromagnetic fields are self-sustaining, i.e., they exist independently, with their own dynamics.

Exercise: show this $\rightarrow$

- **Namely:** $\vec{E}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Any vector field $\vec{E}(x)$ can be decomposed into:

$$\vec{E}(x) = \vec{E}_s(x) + \vec{E}_v(x)$$

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* Thus, the inflaton's inhomogeneities' dynamics cannot be separated from that of the metric.

2) Some of $\gamma_{\mu\nu}(x,\eta)$ is independent of ε!

Recall:

Gravity is a force with some similarity to electromagnetism:

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Similarly: Some part of the metric will depend on ϕ, while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of ϕ. $\}\rightarrow$

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"gradient part" or "scalar part" — "curl part" or "vector part"

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induced by the inhomogeneities of the inflaton, $\varphi(x,y)$.

* Vice versa: we can also read the Einstein eqn from left to right ⇒ these gravity inhomogeneities induce the inflaton's inhomogeneities.

* Thus, the inflaton's inhomogeneities' dynamics cannot be separated from that of the metric.

2) Some of $g_{\mu\nu}(x,y)$ is independent of φ!

Similarly: Some part of the metric will depend on φ, while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of φ. →

□ But: also, some electromagnetic fields are self-sustaining, i.e., they exist independently, with their own dynamics.

Exercise: show this → □ Namely: $\vec{E}:\mathbb{R}^3\to\mathbb{R}^3$

Any vector field $\vec{E}(x)$ can be decomposed into:

$$\vec{E}(x) = \vec{E}_s(x) + \vec{E}_v(x)$$

"gradient part" or "scalar part" — "curl part" or "vector part"

Here, $\vec{E}_s$ and $\vec{E}_v$ derive from a scalar function Λ and a vector field $\vec{A}$ respectively:

$$\vec{E}_s = \vec{\nabla}\Lambda \quad\text{and}\quad \vec{E}_v = \vec{\nabla}\times\vec{A}$$

They obey: $\vec{\nabla}\times\vec{E}_s = \vec{0}$ and $\vec{\nabla}\cdot\vec{E}_v = 0$ (A)

Exercise for physics students: verify → □ According to the Maxwell equations, the scalar part, e.g., of the electric field, $\vec{E}$, is caused by (or causes) the electric charge density

$$\vec{\nabla}\cdot\vec{E}_s = \rho$$

[* An unusual but mathematically equivalent viewpoint.
* E.g. D-branes in string theory are charges that are defined from this viewpoint.

while the vector part is charge independent

$$\vec{\nabla}\times\vec{E}_v = -\frac{\partial\vec{B}}{\partial t}$$

and similarly for the magnetic field $\vec{B}(x)$.

Thus, $\vec{E}$ and $\vec{B}$ fields can sustain each other, which makes possible nonzero electromagnetic fields (namely traveling waves) even when there are no charges. ←

□ Similarly, some curvature exists only where there is energy-momentum.

□ But, also, some curvature is self-sustaining, with dynamics, e.g., gravitational waves.

⇒ We should expect that $g_{\mu\nu}(x,y)$ contains:

* some curvature that is induced by (or induces) the inflaton inhomogeneities.

* some curvature inhomogeneities that are self-sustaining, i.e., that possess their own dynamics - and therefore also their own quantum fluctuations.

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Exercise for physics students: verify →

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and similarly for the magnetic field $\vec{B}(v)$.

Thus, $\vec{E}$ and $\vec{B}$ fields can sustain each other, which makes possible nonzero electromagnetic fields (namely traveling waves) even where there are no charges. ←

→ We should expect that $\gamma_{\mu\nu}(x,\eta)$ contains:

* some curvature that is induced by (or induces) the inflaton inhomogeneities.
* some curvature inhomogeneities that are self-sustaining, i.e., that possess their own dynamics - and therefore also their own quantum fluctuations.

How to separate these inhomogeneities of $\gamma_{\mu\nu}(x,\eta)$?

Similar to vector fields $\vec{E}: \mathbb{R}^3 \to \mathbb{R}^3$ we have for the tensor field γ:

- The perturbations $\gamma_{\mu\nu}$ of the metric tensor can be decomposed into three types:

  a) The part of $\gamma_{\mu\nu}$ which can be derived from scalar functions.

  b) The part of $\gamma_{\mu\nu}$ which can be derived from vector fields.

  c) The part of $\gamma_{\mu\nu}$ which is purely tensor.

Decomposition of $g_{\mu\nu}(x,\eta)$, with respect to its spatial structure:

- One usually writes the "line element" ds^2, i.e., the infinitesimal proper distance (squared) from x to $x+dx$ as

$$ds^2 = g_{\mu\nu}(x,\eta)\, dx^\mu dx^\nu \quad \text{with} \quad dx^\mu = (d\eta, dx^1, dx^2, dx^3)$$

- Then, the decomposition takes the form:

$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3} (dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

- Here, the spatially "scalar" part of the inhomogeneities reads

$$ds_S^2 = a^2(\eta)\Big[2\Phi(x,\eta)\, d\eta^2 - 2\sum_{i=1}^{3} \frac{\partial}{\partial x^i} B(x,\eta)\, dx^i d\eta - \sum_{i,j=1}^{3}\Big(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} E(x,\eta)\Big) dx^i dx^j\Big]$$

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where Φ, Ψ, B and E are scalar functions.

- The spatially "vector" part of the metric reads:

$$ds_V^2 = a^2(\eta)\Big[2\sum_{i=1}^{3} V_i(x,\eta)\, dx^i d\eta - \sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x^j} W_i(x,\eta) + \frac{\partial}{\partial x^i} W_j(x,\eta)\right) dx^i dx^j\Big]$$

where V_i and W_i are 3-vector fields.

- The spatially "tensor" part of the metric is the remainder, i.e., is what cannot be derived from a scalar or vector field:

$$ds_T^2 = a^2(\eta) \sum_{i,j=1}^{3} h_{ij}(x,\eta)\, dx^i dx^j$$

Here, h_{ij} is a spatial tensor field.

- **Remark** regarding the fields V^i, W^i and h_{ij}:

  Analogous to the equations **(A)** above in electromagnetism,

  $$\vec\partial \times \vec E_S = \vec 0 \quad \text{and} \quad \vec\partial \cdot \vec E_V = 0,$$

  we now have:

  * V_i, W_j obey:

  $$\vec\partial \cdot \vec V = 0, \quad \vec\partial \cdot \vec W = 0 \qquad \left(\text{i.e. } \sum_{i=1}^{3} \frac{\partial}{\partial x^i} V^i = 0 \text{ etc.}\right)$$

  * h_{ij} obeys:

  $$h_{ij} = h_{ji}, \quad \sum_{i=1}^{3} h^i{}_i = 0, \quad \sum_{i=1}^{3} \frac{\partial}{\partial x_i} h_{ij} = 0$$

  Remark: This implies that h_{ij} describes "Weyl curvature" which is known to describe gravitational waves

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$$-\sum_{i,j=1}\left(\frac{\partial}{\partial x^j} W_i(x,\eta) + \frac{\partial}{\partial x^i} W_j(x,\eta)\right) dx^i dx^j \Big]$$

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$$ds_T^2 = a^2(\eta) \sum_{i,j=1}^{3} h_{ij}(x,\eta)\, dx^i dx^j$$

Here, h_{ij} is a spatial tensor field.

* V_i, W_j obey:

$$\vec{\nabla}\cdot\vec{V} = 0, \quad \vec{\nabla}\cdot\vec{W} = 0 \qquad \left(\text{i.e. } \sum_{i=1}^{3} \frac{\partial}{\partial x^i} V^i = 0 \text{ etc.}\right)$$

* h_{ij} obeys:

$$h_{ij} = h_{ji}, \quad \sum_{i=1}^{3} h^i{}_i = 0, \quad \sum_{i=1}^{3} \frac{\partial}{\partial x_i} h_{ij} = 0$$

Remark: This implies that h_{ij} describes "Weyl curvature" which is known to describe gravitational waves

Recall:

- We decompose the inflaton field $\phi(x,\eta)$:

$$\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta)$$

where:

* $\phi_0(\eta)$ is assumed large and is treated classically.

* $\varphi(x,\eta) =: \delta\phi(x,\eta)$ describes a field of small inhomogeneities and is to be quantized: $\hat{\varphi}(x,\eta)$

- We decompose the metric $g_{\mu\nu}(x,\eta)$:

$$g_{\mu\nu}(x,\eta) = a_0^2(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta)$$

treated classically | assumed small, to be quantized

- Here, $\gamma_{\mu\nu}(x,\eta)$ can be decomposed into scalar, vector and tensor-type inhomogeneities, using functions

$$E, B, \Psi, \Phi, V_i, W_i, h_{ij}.$$

namely: $ds^2 = g_{\mu\nu}(x,\eta)\, dx^\mu dx^\nu$

$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

Where:

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vector and tensor-type inhomogeneities, using functions $E, B, \Psi, \Phi, V_i, W_i, h_{ij}$.

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$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_s^2}_{\text{scalar}} + \underbrace{ds_v^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

$$ds_s^2 = a^2(\eta)\Big[2\Phi(x,\eta)\, d\eta^2 - 2\sum_{i=1} \frac{\partial}{\partial x^i} B(x,\eta)\, dx^i d\eta$$
$$- \sum_{i,j=1}^{3}\left(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} E(x,\eta)\right) dx^i dx^j\Big]$$

$$ds_v^2 = a^2(\eta)\Big[2\sum_{i=1}^{3} V_i(x,\eta)\, dx^i d\eta$$
$$- \sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x^j} W_i(x,\eta) + \frac{\partial}{\partial x^i} W_j(x,\eta)\right) dx^i dx^j\Big]$$

$$ds_T^2 = a^2(\eta)\sum_{i,j=1}^{3} h_{ij}(x,\eta)\, dx^i dx^j$$

We insert the approximation

$$\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta)$$

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta)$$

with φ, γ assumed small, into the action:

$$S = \frac{-1}{16\pi G}\int R\sqrt{|g|}\, d^4x$$

$$+ \frac{1}{2}\int (\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\sqrt{|g|}\, d^4x$$

$$+ \text{neglected (other fields)}$$

One obtains many terms with $\Phi, \Psi, B, E, V, W, h$!

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$$ds_V^2 = a^2(\eta)\Big[2\sum_{i=1}^{3} V_i(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\Big(\frac{\partial}{\partial x^j}W_i(x,\eta) + \frac{\partial}{\partial x^i}W_j(x,\eta)\Big)dx^i dx^j\Big]$$

$$ds_T^2 = a^2(\eta)\sum_{i,j=1}^{3} h_{ij}(x,\eta)\;dx^i dx^j$$

..., ψ assumed small, into the action:

$$S = \frac{-1}{16\pi G}\int R\sqrt{|g|}\,d^4x$$

$$+\frac{1}{2}\int(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\sqrt{|g|}\,d^4x$$

$$+ \text{neglected (other fields)}$$

One obtains many terms with $\Phi, \Psi, B, E, V, W, h$!

- These terms can be simplified! Why?

  Now that space is curved, there is no longer a preferred foliation of spacetime into spacelike hypersurfaces!

  $\Rightarrow$ No preferred choice for the coordinate system.

  (e.g., no preferred conformal time & space cds)

- But the choice of cds will affect the functions above, i.e. they are in part coordinate system dependent.

  $\Rightarrow$ We may choose our $\overbrace{\text{spacelike hypersurfaces}}^{\text{and thus our notion of equal time}}$ so that these functions $\Phi, \Psi, E, B, V, W, h$ vanish or simplify.

It took on the order of 10 years to clarify this "gauge" question!

- For detailed references, see e.g.:

  * A. Riotto, hep-ph/0210162 (relatively compact)
  * R. H. Brandenberger et al, Physics Reports 215, 203 (1992) (long)

- Result:

  * For small inhomogeneities (1st order perturbation) nearly all inhomogeneities can be eliminated by suitable coordinate choice.
  * Except, there are two fields, which are coordinate system, i.e., "gauge" independent. Namely:

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(e.g., no preferred conformal time & space cds)

- But the choice of cds will affect the functions above, i.e. they are in part coordinate system dependent.

and thus our notion of equal time

⇒ We may choose our spacelike hypersurfaces so that these functions $\Phi, \Psi, C, B, v, w, h$ vanish or simplify.

It took on the order of 10 years to clarify this "gauge" question!

I) A spatial tensor field:

This is $h_{ij}(x,\eta)$ itself. It represents $T_{\mu\nu}$-independent, so-called Weyl curvature, namely gravitational waves. $h_{ij}(x,\eta)$ measures how much space is locally distorted against itself in different directions.

II) A spatially scalar field, r, made of φ and $\gamma_{\mu\nu}$'s scalar part:

Due to the Einstein eqn,

$$\delta\phi(x,\eta) = \varphi(x,t)$$

combines with the scalar part of the metric inhomogeneities

$$\Psi(x,\eta),$$

to yield one dynamical entity, namely:

- **Result:**

  * For small inhomogeneities (1st order perturbation) nearly all inhomogeneities can be eliminated by suitable coordinate choice.

  * Except, there are two fields, which are coordinate system, i.e., "gauge" independent. Namely:

recall: $\phi_0(\tau)$ = classical homogeneous inflaton field.

$$r(x,\eta) := -\frac{a_i'}{a_i}\left(\phi_0(\eta)'\right)^{-1}\varphi(x,\eta) - \Psi(x,\eta)$$

from inflaton

from "scalar" part of the metric

Physically, what is $r(x,\eta)$?

* First term: In $\frac{a'}{a}\frac{1}{\phi_0'}\varphi$, the $\varphi(x,\eta)$ is the scalar field's fluctuation.

* Second term: $\Psi(x,\eta)$ is the (scalar) metric's fluctuation.