

Title: Quantum Field Theory for Cosmology - Lecture 20240314

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QFT for Cosmology, Achim Kempf, Lecture 18

Time evolution and the fluctuation spectrum:

Recall:

- We assume the system is in the state $|0\rangle$ which is the vacuum at η_0 .
- ☞ \Rightarrow The system is always in the state $|0\rangle$ (Heisenberg picture).
- We solve the QFT with $\hat{X}_k(\eta) = a(\eta) \hat{\phi}_k(\eta)$ and the ansatz

$$\hat{X}_k(\eta) = \frac{1}{\sqrt{2}} (v_k^*(\eta) a_k + v_k(\eta) a_{-k}^\dagger)$$
 where for convenience we choose the mode functions $\{v_k(\eta)\}_k$ so that $a_k |0\rangle = 0$.
- The technical challenge will be:
 - Identify $|0\rangle$, i.e., identify the initial conditions for the v_k at η_0 .
 - Solve the K.G. eqn for the $v_k(\eta)$.

Benefit:

- State $|0\rangle$ known
 - Operators $\hat{\phi}_k(\eta)$ known $\forall \eta > \eta_0$.
- \Rightarrow We can calculate all predictions for all times, even, e.g., at times of nonadiabaticity or inverted potential!

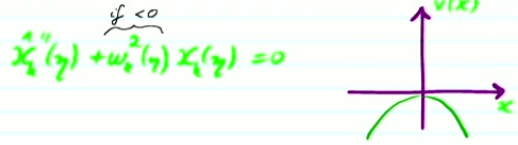
In particular, we can calculate for all $\eta > \eta_0$:

$$\delta \phi_k(\eta) = k^{3/2} \left| \frac{v_k(\eta)}{a(\eta)} \right|$$

We observe: The dynamics of $|v_k(\eta)|$ crucially affects $\delta \phi_k(\eta)$.

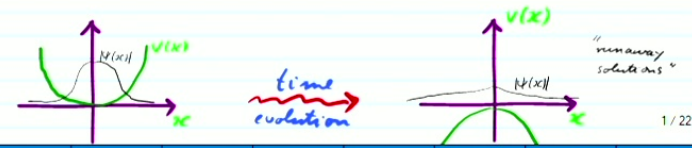
Q: In which circumstance does $v_k(\eta)$ grow most?

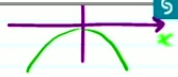
Answer: The most efficient mechanism to enlarge v_k occurs when the mode is nonadiabatically evolving in the sense that the mode oscillator is inverted:



In such a time period the v_k (and a_k)

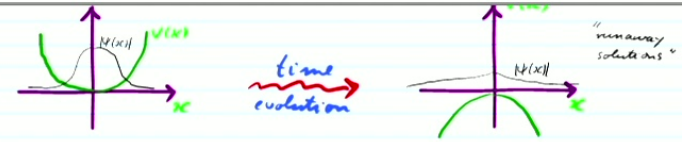
* Instead, there will be one exponentially decaying and one exponentially growing solution. Inverting a harmonic oscillator is an efficient way to increase $\Delta \phi$:





In such a time period, the Klein Gordon equation's solutions are not oscillatory because $\omega_k(\eta)$ is imaginary:

$$\omega_k(\eta) = \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}} \quad \left\{ \begin{array}{l} \text{This term may be large} \\ \text{enough to make the} \\ \text{discriminant negative.} \end{array} \right.$$



Caution:

Notice that this argument applies to χ but $\phi = \frac{1}{a} \chi$. Thus, ϕ 's growth is slower than that of χ . Recall: The equation of motion of ϕ has a friction-type term.

Before we calculate the fluctuation amplification explicitly:

Relationship of fluctuation amplification to particle creation

Assume that at a later time, η_1 , the evolution is adiabatic for mode k (i.e. its ω_k changes slowly).

\Rightarrow We can identify $|vac_{\eta_1}\rangle$:

Using the adiabatic vacuum identification criterion, we find the mode function \tilde{v}_k for which:

$$|\tilde{0}\rangle = |vac_{\eta_1}\rangle$$

Case 1: The evolution of mode k was adiabatic from η_0 to η_1 .

* Therefore:

$$v_k = \tilde{v}_k \quad \text{and} \quad |0\rangle = |\tilde{0}\rangle$$

* Therefore:

The state of the system, $|\Omega\rangle = |0\rangle$, is still the vacuum state at time η_1 :

$$|\Omega\rangle = |\tilde{0}\rangle$$

* There is no particle creation.

* But since $v_k = \frac{1}{\sqrt{2\omega_k(\eta_1)}} e^{i \int_{\eta_0}^{\eta_1} \omega_k(\eta') d\eta'}$, in general:

$$|v_k(\eta_1)| \neq |v_k(\eta_0)| \quad (\text{namely } |v_k(\eta)| = \omega_k(\eta)^{-\frac{1}{2}})$$

\Rightarrow the fluctuations, which depend on $|v_k(\eta)|$ can be affected even if there is no particle creation?

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□ Case 1: The evolution of mode k was adiabatic from η_0 to η_1 .

* Therefore:

□ Case 2: The evolution was not always adiabatic between η_0 and η_1 .

* Then, $v_k \neq \tilde{v}_k$

* But since both are in the same 2 dimensional solution space to the K.G. equation, there exist α_k, β_k :

$$v_k(\eta) = \alpha_k \tilde{v}_k(\eta) + \beta_k \tilde{v}_k^*(\eta)$$

* Substitute in the fluctuations equation:

$$\begin{aligned} \delta\phi_k(\eta)^2 &= \alpha^{-2}(\eta) k^3 |v_k(\eta)|^2 \\ &= \alpha^{-2}(\eta) k^3 |\alpha_k \tilde{v}_k(\eta) + \beta_k \tilde{v}_k^*(\eta)|^2 \end{aligned}$$

Recall: When particle concept applies, $|\beta_k|$ yields nonadiabatic particle production

$$|\Omega\rangle = |\tilde{0}\rangle$$

* There is no particle creation.

* But since $v_k = \frac{1}{\sqrt{\omega_k(\eta)}} e^{i\int_{\eta_0}^{\eta} \omega_k(\eta') d\eta'}$, in general:

$$|v_k(\eta_1)| \neq |v_k(\eta_0)| \quad (\text{namely } |v_k(\eta)| = \omega_k(\eta)^{-1/2})$$

⇒ the fluctuations, which depend on $|v_k(\eta)|$ can be affected even if there is no particle creation?

* For clarity, assume that the non-adiabatic period is over by η_1 .

* Also, assume that spacetime is again Minkowski around η_1 . (Thus, we focus on nonadiabatic effects only)

* In this case:

$$\tilde{v}_k(\eta) = \frac{1}{\sqrt{\omega_k(\eta)}} e^{i\omega_k(\eta_1)\eta} \quad \text{for all } \eta \approx \eta_1$$

$$\Rightarrow \delta\phi_k^2(\eta) = \alpha^{-2}(\eta) \frac{k^3}{\omega_k(\eta)} (|\alpha_k|^2 + |\beta_k|^2 - 2 \operatorname{Re}(\alpha_k \beta_k^* e^{2i\omega_k(\eta_1)\eta}))$$

Over a long enough time period this term averages 0.

* We use: $|\alpha_k|^2 - |\beta_k|^2 = 1$ (W)

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$$\Rightarrow \delta\phi_k^2(\eta) = \alpha^{-2}(\eta) \frac{k^3}{\sqrt{k^2 + m^2}} (1 + 2|\beta_k|^2) \quad (\mathcal{F})$$

This term is the same, whether or not the expansion of equation is done adiabatic.

This term is only non zero if the evolution was non-adiabatic.

* Notice: $|\beta_k|$ and $|\alpha_k|$ can both become very large. This is consistent with the Wronskian condition, (W).

* Particle production:

Recall that the expected number of created particles is also given by $|\beta_k|^2$:

$$\begin{aligned} \bar{N}_k(\eta) &= \langle \Omega | \hat{N}_k | \Omega \rangle = \dots \\ &= |\beta_k|^2 \end{aligned}$$

$\hat{N}_k = \tilde{a}_k^\dagger \tilde{a}_k$

* Remark:

But (F) holds even if $\omega_k^2 \ll 0$ at η_* ! We know what we mean by field fluctuations even when we do not have a concept of vacuum and particles.

\Rightarrow 'Quantum fields more fundamental than quantum particles.'

□ Fluctuations in proper coordinates as opposed to comoving coordinates

* We have: $d = a(\eta) L$, $p = \frac{1}{a(\eta)} k$

proper length comoving length proper momentum comoving momentum

* Therefore, $\delta\phi_k^2(\eta) = \alpha^{-2}(\eta) \frac{k^3}{\sqrt{k^2 + m^2}} (1 + 2|\beta_k|^2)$ becomes:

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* Particle production:

Recall that the expected number of created particles is also given by $|\beta_\pm|^2$:

$$\bar{N}_\pm(\eta) = \langle \Omega | \hat{N}_\pm | \Omega \rangle = \dots = |\beta_\pm|^2$$

$\hat{N}_\pm = \hat{a}_\pm^\dagger \hat{a}_\pm$

$$\delta\phi_p^2(\eta) = a^{-2} \frac{a^3 p^3}{|a^2 p^2 + \omega^2|} (1 + 2|\beta_{\pm p}|^2)$$

$$= \frac{p^3}{\sqrt{p^2 + \omega^2}} (1 + 2|\beta_{\pm p}|^2)$$

Same as Minkowski

* Note: The nonadiabatic term depends on p .

* Note: This was the case when we end in a Minkowski space. We see that in this case we have the original Minkowski spectrum if the evolution from η_0 to η_1 was adiabatic.

* Note: Also in thermodynamics, "adiabatic" processes are reversible processes.

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Application to specific cosmological models

The standard model of cosmology holds that the very early universe underwent a short period of almost exponential expansion, "inflation".

→ Begin by studying QFT in de Sitter spacetime:

The deSitter FRW spacetime can be defined through

$$a(t) := e^{Ht} \text{ for all } t \in \mathbb{R}$$

Notes: * t is the time on a comoving observer's worldsheet
* large $H \Leftrightarrow$ large acceleration

Here: $H > 0$ is a constant, the "Hubble constant".

□ Exercise: Read Mukhanov's comments on de Sitter space.

* **Note:** The nonadiabatic term depends on p .

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begin by studying $d+1$ in de Sitter spacetime:

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Notes: * t is the time on a comoving observer's worldline
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□ **Exercise:** Read Mukhanov's comments on de Sitter space.

The de Sitter horizon

Proposition: (in particle picture) (Note: large $H \Leftrightarrow$ small horizon d_H)
Objects (or any observers) who are further apart than a proper distance of $d_H = 1/H$ can never meet, and cannot communicate.

Proof: * Consider an observer in a galaxy A . Let us choose the origin of the comoving coordinate system(s) to be where this observer sits.
* Now suppose that, at some arbitrary time, t_s , this observer sends a radio signal towards another galaxy, B .
* The signal travels in a small time Δt the small comoving distance Δx :

$$\frac{a(t) \Delta x}{\Delta t} = c = 1$$

↑ speed of light

$$\Rightarrow \frac{dx}{dt} = a^{-1}(t) \text{ i.e.: } \frac{dx}{dt} = e^{-Ht}$$

$$\Rightarrow x(t) = -\frac{1}{H} e^{-Ht} + C$$

Fix the integration constant C so that $x(t_s) = 0 \Rightarrow C = \frac{1}{H} e^{-Ht_s}$

$$\Rightarrow x(t) = -\frac{1}{H} e^{-Ht} + \frac{e^{-Ht_s}}{H}$$

\Rightarrow As $t \rightarrow \infty$ we have $x(t) \rightarrow \frac{e^{-Ht_s}}{H}$. Terminal comoving distance traveled.

Thus, can reach galaxy B if comoving distance is at most $d_c = \frac{e^{-Ht_s}}{H}$.

Q: Proper distance d_p of such B from A at t_s ?

$$\mathbf{A:} \quad d_p = a(t) d_c \Rightarrow d_p = e^{+Ht_s} \frac{e^{-Ht_s}}{H} = \frac{1}{H}$$

Recall: The proper distance traveled is:
 $d(t) = a(t) x(t)$
Clearly: $d(t) \rightarrow \infty$ as $t \rightarrow \infty$

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(Recall: The proper distance traveled is:
 $d(t) = a(t) x(t)$
 (Check: $d(t) \rightarrow 0$ as $t \rightarrow 0$)

Fix the integration constant C so that $x(t_s) = 0 \Rightarrow C = \frac{1}{H} e^{-Ht_s}$

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Q: Proper distance d_p of such B from A at t_s ?

A: $d_p = a(t) d_c \Rightarrow d_p = e^{+Ht_s} \frac{e^{-Ht_s}}{H} = \frac{1}{H}$

Recall: This holds for arbitrary t_s .

\Rightarrow A signal sent by A at any time t_s can only ever reach B if at the time of sending, t_s , the proper distance between A and B is at most $\frac{1}{H}$.

\Rightarrow Any two observers further apart than a proper distance of $1/H$ cannot communicate!

Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper wavelength obeys $\lambda \ll \frac{1}{H}$ but stop oscillating and possess instead an imaginary frequency when their proper wavelength has grown beyond, i.e., when $\lambda \gg \frac{1}{H}$, assuming that their mass is small: $m \ll H$.

Interpretation: In the case where a de Sitter exponential expansion lasts forever, between any objects of proper distance $> 1/H$, space is being created faster than what can be crossed when travelling with the speed of light.

(Remark: Notice that the proper size of the de Sitter horizon is constant in time.)

Proof: 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

□ Recall: $\eta(t) := \int \frac{1}{a(t')} dt'$

here: $\eta(t) = \int e^{-Ht'} dt' = -\frac{1}{H} e^{-Ht} + C$

The choices of the integration constant C merely mean different fixed shifts in the time coordinate η relative to the time coordinate t .

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Remark: Notice that the proper size of the de Sitter horizon is constant in time.

Notice:

□ As $t \rightarrow -\infty$ we have $\eta \rightarrow -\infty$.

□ But as $t \rightarrow +\infty$ we have $\eta \rightarrow 0$.

□ Choose $C = 0$:

⇒

$$\eta(t) = -\frac{1}{H} \frac{1}{a(t)}$$

$$a(t) = -\frac{1}{H\eta(t)}$$

i.e.:

$$a(\eta) = -\frac{1}{H\eta}$$

then proper distance has grown beyond, i.e., $> 1/H$, assuming that their mass is small: $m \ll H$.

Proof: 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

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The choices of the integration constant C merely mean different fixed shifts in the time coordinate η relative to the time coordinate t .

2) Introduce $\hat{\chi}_\nu(\eta) := a(\eta) \hat{\phi}_\nu(\eta)$:

□ We have: $\hat{\chi}_\nu(\eta) = -\frac{1}{H\eta} \hat{\phi}_\nu(\eta)$

□ $\hat{\chi}_\nu$ obeys this Klein Gordon equation

$$\hat{\chi}_\nu''(\eta) + \omega_\nu^2(\eta) \hat{\chi}_\nu(\eta) = 0$$

with:

$$\omega_\nu^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

□ Exercise: Show that in the de Sitter case this yields:

$$\omega_\nu^2(\eta) = k^2 + \frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2}$$

⇒

$$\eta(t) = -\frac{1}{H} \frac{1}{a(t)}$$

$$a(t) = -\frac{1}{H\eta(t)}$$

i.e.:

$$a(\eta) = -\frac{1}{H\eta}$$

with:

$$\omega_b^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

□ Exercise: Show that in the de Sitter case this yields:

$$\omega_b^2(\eta) = k^2 + \frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2}$$

3.) Check for imaginary frequencies.

We are assuming $m \ll H$. ⇒ $\frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2} < 0$

□ Thus, in $\omega_b^2(\eta) = k^2 + \underbrace{\frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2}}_{\text{we have: } < 0}$

□ Therefore: For each mode k there comes a time when ω_b^2 becomes negative!

The case relevant in cosmology: $m = 0$ (we'll assume this)

⇒ The time when a mode k crosses the horizon is given by:

$$\eta_{\text{hor}}(k) \approx -\frac{\sqrt{2}}{k}$$

4.) Conclusion:

□ A mode oscillates as long as:

$$|\eta| \gg \frac{1}{k} \quad \text{i.e., while } |\eta|k \gg 1 \quad \textcircled{a}$$

Recall: $\eta \in (-\infty, 0)$
i.e. $|\eta| \gg \frac{1}{k}$ means
early times.

(Used that η and 1 are of same order of magnitude)

□ A mode has imaginary frequency from when

$$|\eta| \ll \frac{1}{k} \quad \text{i.e., from when } |\eta|k \ll 1 \quad \textcircled{b}$$

This is late times, i.e.
when $\eta \approx 0$.

Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = \frac{2\pi}{L}$ we obtain:

$$|\eta|k = \frac{1}{Ha} \frac{2\pi}{L}$$

↑ comoving wavelength

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Therefore: For each mode k there comes a time when u_k^2 becomes negative!

The case relevant in cosmology: $m=0$ (we'll assume this)

⇒ The time when a mode k crosses the horizon is given by:

$$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k}$$

A mode has imaginary frequency from when

$$|\gamma| \ll \frac{1}{k} \quad \text{i.e., from when } |\gamma|k \ll 1 \quad \textcircled{b}$$

This is late times, i.e. when $\gamma \approx 0$.

Re-expressed in terms of proper wavelength?

Noting $|\gamma| = \frac{1}{Ha}$ and multiplying it with $k = \frac{2\pi}{L}$ we obtain:

$$|\gamma|k = \frac{1}{Ha} \frac{2\pi}{L}$$

← comoving wavelength

Transforming to the proper wavelength, $\lambda = a(\eta)L$, we obtain:

$$|\gamma|k = \frac{2\pi}{H\lambda} \quad \left(\begin{array}{l} \text{Thus, the proper wavelength, } \lambda, \text{ of a fixed} \\ \text{comoving mode, } k, \text{ obeys:} \\ \lambda(\eta) = \frac{2\pi}{Hk|\gamma|} \end{array} \right)$$

Thus, finally, the two cases, \textcircled{a} and \textcircled{b} become:

A mode oscillates as long as: $|\gamma|k \gg 1$

i.e., as long as $\frac{2\pi}{H\lambda} \gg 1$ i.e.: $\lambda \ll \frac{1}{H}$ \textcircled{a}

A mode has imaginary frequency from when:

$$\frac{2\pi}{H\lambda} = |\gamma|k \ll 1, \text{ i.e., from when } \lambda \gg \frac{1}{H} \quad \textcircled{b}$$

This is what we had set out to show.