Title: Quantum Field Theory for Cosmology - Lecture 20240312

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: March 12, 2024 - 4:00 PM

URL: https://pirsa.org/24030010

QFT for Cosmology, Achim Kempf, Lecture 17

From the particle picture to the wave picture

So far: Spacetime dynamics can produce particles.

When? When mode oscillators $\omega_k(t)$ changes nonadiabatically fast: $\omega_k(t)'/\omega_k(t)^2 \gg 1$.

In cosmology? No: particles mostly produced conventionally from inflation potential at the end of inflation.

Now: Spacetime dynamics can enhance quantum field fluctuations!

When? When $\omega_k(\eta)$ becomes imaginary.

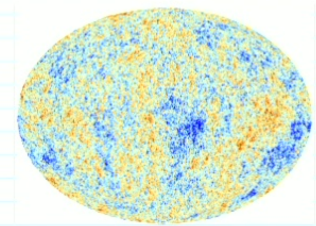
Recall: $\omega_k^2(\eta) := k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$

Expectation value:
$$\bar{\phi}(x,\eta_0) = \langle f|\hat{\phi}(x,\eta_0)|f\rangle = f(x)\langle f|f\rangle = f(x)$$

Variance:
$$\Delta\phi^2(x,\eta_0) = \langle f|\left(\hat{\phi}(x,\eta_0) - \bar{\phi}(x,\eta_0)\right)^2|f\rangle = \langle f|(f(x) - f(x))^2|f\rangle = 0$$
i.e. no fluctuations.

Quantum field fluctuations

Do the amplitudes $\hat{\phi}(x,t)$ of a quantum field necessarily quantum fluctuate?



- Consider a real-valued function $f: \mathbb{R}^3 \to \mathbb{R}$ and a time η_0.
- Then, we define the state $|f\rangle$ as the joint eigenstate of all operators $\hat{\phi}(x,\eta_0)$ with eigenvalues $f(x)$:

$$\hat{\phi}(x,\eta_0)|f\rangle = f(x)|f\rangle$$

But, can such states $|f\rangle$ occur in practice?

No: The reason is that for these states:

$$\langle f|\hat{H}^{fpo}(\eta_0)|f\rangle = \infty$$ Exercise: Show this.

Hint: For these states, $\Delta\phi = 0$, and so $\Delta\pi^{fpo} = \infty$

But $\hat{H}^{fpo}$ contains a term $\hat{\pi}^2$...

$\Longrightarrow$ × Even the state $|f\rangle$ with $f(x)=0$ ...

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

Variance:
$$\Delta\hat\phi^2(x,y) = \langle f \vert (\hat\phi(x,y) - \bar\phi(x,y))^2 \vert f \rangle$$
$$= \langle f \vert (f(x) - f(x))^2 \vert f \rangle$$
$$= 0 \qquad \text{i.e. no fluctuations.}$$

$\Longrightarrow$ There are no quantum fluctuations of the quantum field $\hat\phi$ if the system is in such a state $\vert f \rangle$:

Hint: For these states, $\Delta\phi = 0$, and so $\Delta\pi^{(x)} = \infty$

But $\hat H^{(x)}$ contains a term $\hat\pi^2$...

$\Longrightarrow$ * Even the state $\vert f \rangle$ with $f(x)=0$ for all x has ∞ energy and is, therefore, not accessible.

* Thus, all finite energy states do possess quantum fluctuations

Exercise:

What is the analogue of this observation in the case of the harmonic oscillator in quantum mechanics?

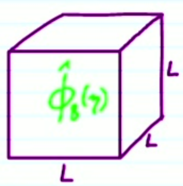
How to calculate the amount of quantum field fluctuations?

- It is not realistic to measure all operators $\hat\phi(x)$ individually.
- Realistically, we could at most hope to measure an average of the values of $\hat\phi$ over regions $B \subset \mathbb{R}^3$ of not too small volume $V = L \times L \times L$:

$$\hat\phi_B(y) := \int_{\mathbb{R}^3} \hat\phi(x,y)\, W(x)\, d^3x$$

with "window function" W

$$W(x) = \begin{cases} \approx 0 & \text{for all } x \notin B \\ \approx V^{-1} & \text{for all } x \in B \end{cases}$$



(we'll also allow B to be spherical)

Water analog:

A ship of size B averages the sea over the region B.

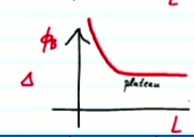
Observe: Each ship rocks mostly due to the waves of wavelength of about the scale of B.

Example:

In QFT:

- Calculate the typical amplitude of quantum fluctuations as a function of their spatial size:

- Calculate how this relationship is affected by cosmic expansion:



3/23

3

File Edit View Insert Actions Tools Help

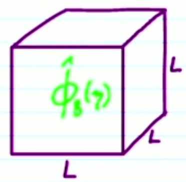
Sharing Desktop | Stop Sharing

values of ϕ over regions $B \subset \mathbb{R}^3$ of not too small volume V ...

$$\hat{\phi}_B(\eta) := \int_{\mathbb{R}^3} \hat{\phi}(x,\eta)\; W(x)\, d^3x$$

with "window function" W

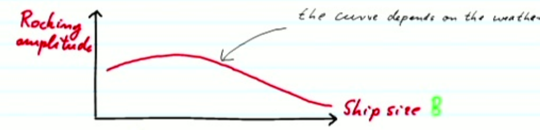
$$W(x) = \begin{cases} \approx 0 & \text{for all } x \notin B \\ \approx V^{-1} & \text{for all } x \in B \end{cases}$$



(we'll also allow B to be spherical)

about the scale of B.

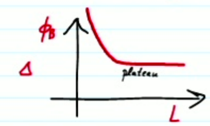
Example:



In QFT:

- Calculate the typical amplitude of quantum fluctuations as a function of their spatial size:

- Calculate how this relationship is affected by cosmic expansion:



Quantum field fluctuations in FRW spacetime

- Choose conformal time η and comoving coordinates x.
- Choose a region B of size $L \times L \times L$.

  Note: In proper coordinates, this is a box of increasing size:

  $$a(\eta)L \times a(\eta)L \times a(\eta)L$$

- Assume that at η_0 the system's state, $|\Omega\rangle$, is the vacuum state:

  $$|\Omega\rangle = |vac_{\eta_0}\rangle$$

- We choose the mode functions $v_k(\eta)$ so that $|vac_{\eta_0}\rangle = |0\rangle$ with:

  $$\hat{x}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)a_k + v_k a_{-k}^+\right) \quad \text{and} \quad a_k|0\rangle = 0$$

⇝ The expectation value of the region-averaged field at a time $\eta \geq \eta_0$:

$$\bar{\phi}_B(\eta) = \langle\Omega|\,\hat{\phi}_B(\eta)\,|\Omega\rangle$$

$$= \langle vac_{\eta_0}|\,\hat{\phi}_B(\eta)\,|vac_{\eta_0}\rangle$$

$$= \langle 0|\int_{\mathbb{R}^3}\hat{\phi}(x,\eta)\, W(x)\, d^3x\,|0\rangle$$

$$= \langle 0|\frac{1}{a(\eta)}\int_{\mathbb{R}^3}\hat{x}(x,\eta)\, W(x)\, d^3x\,|0\rangle$$

$$= \frac{1}{a(\eta)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle 0|\frac{1}{\sqrt{2}}\left(v_k^*(\eta)\cancel{a_k} + v_k(\eta)\cancel{a_{-k}^+}\right)W(x)|0\rangle\, e^{ixk}(2\pi)^{-\frac{3}{2}}d^3k\, d^3x$$

$$= 0$$

⇒ The average amplitude of $\hat{\phi}_B$ vanishes in the vacuum state.

5 / 23

5

Note: In proper coordinates, this is a box of increasing size:

$$a(\eta)L \times a(\eta)L \times a(\eta)L$$

- Assume that at η_0 the system's state, $|\Omega\rangle$, is the vacuum state:

$$|\Omega\rangle = |vac_{\eta_0}\rangle$$

- We choose the mode functions $v_k(\eta)$ so that $|vac_{\eta_0}\rangle = |0\rangle$ with:

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k\, a_{-k}^+\right) \quad \text{and} \quad a_k|0\rangle = 0$$

$$= \langle 0| \int_{\mathbb{R}^3} \hat{\phi}(x,\eta)\, W(x)\, d^3x\, |0\rangle$$

$$= \langle 0| \frac{1}{a(\eta)} \int_{\mathbb{R}^3} \hat{\chi}(x,\eta)\, W(x)\, d^3x\, |0\rangle$$

$$= \frac{1}{a(\eta)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle 0| \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\cancel{a_k} + v_k(\eta)\cancel{a_{-k}^+}\right) W(x)|0\rangle\, e^{ixk} (2\pi)^{-\frac{3}{2}} d^3k\, d^3x$$

$$= 0$$

$\Rightarrow$ The average amplitude of $\hat{\phi}_B$ vanishes in the vacuum state.

The quantum fluctuations

While $\overline{\phi}_B(\eta)$ vanishes, measurement outcomes for $\hat{\phi}_B(\eta)$ are not fully predictable because subject to fluctuations around zero with this standard deviation:

$$\Delta\phi_B^2(\eta) = \langle\Omega|\left(\hat{\phi}_B(\eta) - \cancel{\overline{\phi}_B(\eta)}\right)^2|\Omega\rangle$$

$$= \langle 0|\, \hat{\phi}_B(\eta)^2\, |0\rangle$$

$$= \frac{1}{a(\eta)^2} \langle 0| \left(\int_{\mathbb{R}^3} \hat{\chi}(x,\eta)\, W(x)\, d^3x\right)^2 |0\rangle$$

$=$ ... Exercise: fill in the steps

$$= \frac{1}{2a(\eta)^2}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} |v_k(\eta)|^2\, |\widetilde{W}(k)|^2\, d^3k$$

Fourier transform of the window function $W(x)$.

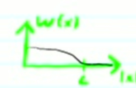
Assume now for simplicity that B is spherical with radius L. Then use spherical coordinates:

$$= \frac{1}{2a(\eta)^2}\frac{1}{(2\pi)^3}\int_0^\infty k^2\, 4\pi\, |v_k(\eta)|^2\, |\widetilde{W}(k)|^2\, dk$$

$k = \sqrt{k_1^2 + k_2^2 + k_3^2}$

Notice the dimension dependence of the integral's measure!

Approximation: Consider that:



typical scale is

$$\Delta \phi_B^2(\eta) = \langle \Omega | (\hat{\phi}_B(\eta) - \cancel{\bar{\phi}_B(\eta)})^2 | \Omega \rangle$$

$$= \langle 0 | \hat{\phi}_B(\eta)^2 | 0 \rangle$$

$$= \frac{1}{a(\eta)^2} \langle 0 | \left(\left(\int_{\mathbb{R}^3} \hat{\chi}(x,\eta)\, W(x)\, d^3x \right)^2 | 0 \rangle$$

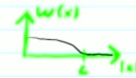
$=$ ... Exercise: fill in the steps

with radius L. Then use spherical coordinates:

$$= \frac{1}{2\, a(\eta)^2} \frac{1}{(2\pi)^3} \int_0^\infty k^2\, 4\pi\, |v_k(\eta)|^2 |\tilde{W}(k)|^2 dk$$

$k = \sqrt{k_1^2 + k_2^2 + k_3^2}$

Notice the dimension dependence of the integral's measure!

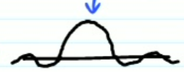
Approximation: Consider that: 

typical scale is L

(using Fourier) $\Rightarrow$ We can assume that, roughly:

$$\tilde{W}(k) \approx 0 \text{ for } |k| > \frac{2\pi}{L}$$

Example: If $W(x) =$ 

then $\tilde{W}(k) =$ 

and we approximate that $\tilde{W}(k) \approx$ 

$$\Longrightarrow \Delta \phi_B^2(\eta) \approx \frac{1}{4\pi^2 a(\eta)^2} \int_0^{2\pi/L} k^2 |v_k(\eta)|^2 dk$$

In the integral, the values of $|v_k(\eta)|^2$ for small k are suppressed by k^2.

$\Rightarrow$ Can approximately replace $|v_k(\eta)|$ by its value at $k = 2\pi/L$:

$$\Delta \phi_B^2(\eta) \approx \frac{1}{4\pi^2 a(\eta)^2} \int_0^{2\pi/L} k^2 |v_{\frac{2\pi}{L}}(\eta)|^2 dk$$

$$\Delta \phi_B^2(\eta) \approx \frac{1}{4\pi^2} \frac{(2\pi)^3}{3L^3} \left| \frac{v_{\frac{2\pi}{L}}(\eta)}{a(\eta)} \right|^2$$

$$= \frac{2\pi}{3L^3} |w_{\frac{2\pi}{L}}(\eta)|^2$$

Here, w is the mode function of $\hat{\phi}$, because $\frac{\hat{\chi}(\eta)}{a(\eta)} = \hat{\phi}(\eta)$.

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

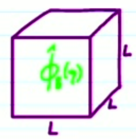
$$\Longrightarrow \Delta\phi_B^2(\eta) \approx \frac{1}{4\pi^2 a(\eta)^2}\int_0 k^2 |v_k(\eta)|^2\, dk$$

Conclusion:

Assume that at a time η_0 the vacuum state was the state $|0\rangle$ corresponding to $\{v_k\}$.

Then, the typical amplitude of fluctuations of size L at an arbitrary time η is:

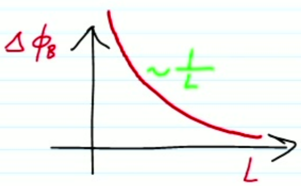
$$\Delta\phi_B^2(\eta) \approx \frac{2\pi}{3L^3}\left|\frac{v_{\frac{2\pi}{L}}(\eta)}{a(\eta)}\right|^2 = \frac{2\pi}{3L^3}\left|W_{\frac{2\pi}{L}}(\eta)\right|^2$$



Special case: Minkowski space (massless field)

- $v_k(\eta) = w_k(\eta)$
- $\eta = t$
- $|v_k(t)|^2 = \frac{1}{\omega_k(t)} = \frac{1}{|k|} = \frac{L}{2\pi}$

$$\Rightarrow \Delta\phi_B^2 \approx \frac{1}{3L^2} \quad \Rightarrow \Delta\phi_B \approx \frac{1}{\sqrt{3}}\frac{1}{L}$$



$$\underbrace{\overline{n(t)\,n(t+L)}}_{\text{"Auto-correlator"}} = \int f(t) f(t+L)\, dt$$

(suitably regularize if non-normalizable, e.g. $\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} f(t) f(t+L)\, dt$)

$$= \iiint \tilde f(\omega)\tilde f(\omega') e^{i\omega t} e^{i\omega'(t+L)}\, dt\, d\omega\, d\omega'$$

$$= \iiint \underbrace{e^{it(\omega+\omega')} dt}_{=(2\pi)^{-1}\delta(\omega+\omega')} \tilde f(\omega)\tilde f(\omega') e^{i\omega' L}\, d\omega\, d\omega'$$

Here, w is the mode function of $\hat\phi$, because $\frac{\hat\chi(\eta)}{a(\eta)} = \hat\phi(\eta)$.

How to describe field quantum fluctuations using correlators

A primer on classical fluctuations:

- Assume $n(t)$ is a Ω-bandlimited gaussian white (all frequencies occur to same amount) noise signal, i.e., a random signal with gaussian distributed amplitudes, filtered to leave only frequencies in the interval $[-\Omega, \Omega]$.
- Then, for an ensemble of such noise signals, one can show:

$$\overline{n(t)} = 0 \qquad \forall t$$

"2-point correlator": $$\overline{n(t)\,n(t+L)} = c\,\frac{\sin(\Omega L)}{\Omega L} \qquad \forall t$$

How can we see this?

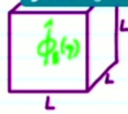
This noise is ergodic, i.e. we could instead average over all t:

The 2-point correlator in QFT:

$$\langle 0|\hat\phi(\vec x,\eta)\,\hat\phi(\vec x+\vec L,\eta)|0\rangle = \frac{1}{a(\eta)^2}\langle 0|\hat\chi(\vec x,\eta)\,\hat\chi(\vec x+\vec L,\eta)|0\rangle$$

Exercise: use mode expansion of $\hat\chi$ and spherical coordinates to derive

$$\Delta\phi_B^2(\eta) \approx \frac{2\pi}{3L^3}\left|\frac{V_{\frac{2\pi}{L}}(\eta)}{a(\eta)}\right|^2 = \frac{2\pi}{3L^3}\left|W_{\frac{2\pi}{L}}(\eta)\right|^2$$



Special case: Minkowski space (massless field)

- $V_k(\eta) = w_k(\eta)$
- $\eta = t$
- $|v_k(t)|^2 = \frac{1}{\omega_k(t)} = \frac{1}{|k|} = \frac{L}{2\pi}$

$\Rightarrow \Delta\phi_B^2 \approx \frac{1}{3L^2} \quad \Rightarrow \Delta\phi_B \approx \frac{1}{\sqrt{3}}\frac{1}{L}$

to leave only frequencies in the interval $[-\Omega, \Omega]$.

Then, for an ensemble of such noise signals, one can show:

$\overline{n(t)} = 0 \quad \forall t$

"2-point correlator": $\overline{n(t)\,n(t+L)} = c\,\frac{\sin(\Omega L)}{\Omega L} \quad \forall t$

How can we see this?

This noise is ergodic, i.e. we could instead average over all t:

$$\underbrace{\overline{n(t)\,n(t+L)}}_{\text{"Auto-correlator"}} = \int f(t)\,f(t+L)\,dt$$

(suitably regularize if non-normalizable, e.g. $\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} f(t)\,f(t+L)\,dt$)

$$= \iiint \tilde f(\omega)\,\tilde f(\omega')\,e^{i\omega t}\,e^{i\omega'(t+L)}\,dt\,d\omega\,d\omega'$$

$$= \iiint \underbrace{e^{it(\omega+\omega')}\,dt}_{=(2\pi)^{-1}\delta(\omega+\omega')}\,\tilde f(\omega)\,\tilde f(\omega')\,e^{i\omega' L}\,d\omega\,d\omega'$$

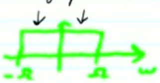
$$= \frac{1}{2\pi}\int \tilde f(\omega)\,\tilde f(-\omega)\,e^{i\omega L}\,d\omega$$

$$= \frac{1}{2\pi}\int |\tilde f(\omega)|^2\,e^{i\omega L}\,d\omega$$

"Spectral power function"

$\Rightarrow$ Autocorrelation and power spectrum are a Fourier pair!

Recall: flatness of spectrum means noise is "white"

For white bandlimited noise: $|\tilde f(\omega)|^2 =$ 

Exercise: Show that its Fourier transform is indeed $\sin(\Omega L)/\Omega L$.

The 2-point correlator in QFT:

$$\langle 0|\hat\phi(\vec x,\eta)\,\hat\phi(\vec x+\vec L,\eta)|0\rangle = \frac{1}{a(\eta)^2}\langle 0|\hat\chi(\vec x,\eta)\,\hat\chi(\vec x+\vec L,\eta)|0\rangle$$

Exercise: use mode expansion of $\hat\chi$ and spherical coordinates to derive:

$$= \frac{1}{a(\eta)^2}\int_0^\infty \frac{k^2\,dk}{4\pi^2}\,\frac{\sin(kL)}{kL}\,|v_k(\eta)|^2 \quad \text{with } k = |\vec k|,\ L = |\vec L|.$$

Notice dimension dependence of the integral's measure!

Observe: $\frac{\sin(kL)}{kL} \approx \begin{cases} 1 & \text{for } k < 1/L \\ 0 & \text{for } k > 1/L \end{cases}$

Observe: $k^2\,\frac{\sin(kL)}{kL}$ has largest amplitude around $k \approx \frac{2\pi}{L}$

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$$\underbrace{\overline{m(t)\,m(t+L)}}_{\text{"Auto-correlator"}} = \int f(t) f(t+L)\,dt$$

(suitably regularize if non-normalizable, e.g. $\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} f(t)\, f(t+L)\,dt$)

$$= \iiint \tilde f(\omega)\tilde f(\omega')\, e^{i\omega t} e^{i\omega'(t+L)}\, dt\, d\omega\, d\omega'$$

$$= \iiint \underbrace{e^{it(\omega+\omega')}\,dt}_{=(2\pi)^{-1}\delta(\omega+\omega')}\, \tilde f(\omega)\tilde f(\omega')\, e^{i\omega' L}\, d\omega\, d\omega'$$

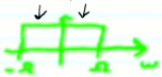
$$= \frac{1}{2\pi}\int \tilde f(\omega)\,\tilde f(-\omega)\, e^{i\omega L}\, d\omega$$

$$= \frac{1}{2\pi}\int |\tilde f(\omega)|^2\, e^{i\omega L}\, d\omega$$

"Spectral power function"

$\Rightarrow$ ▫ Autocorrelation and power spectrum are a Fourier pair!

Recall: flatness of spectrum means noise is "white"

▫ For white bandlimited noise: $|\tilde f(\omega)|^2 =$ 

Exercise: Show that its Fourier transform is indeed $\sin(\Omega L)/\Omega L$.

The 2-point correlator in QFT:

$$\langle 0|\hat\phi(\vec x,\eta)\,\hat\phi(\vec x+\vec L,\eta)|0\rangle = \frac{1}{a(\eta)^2}\langle 0|\hat\chi(\vec x,\eta)\,\hat\chi(\vec x+\vec L,\eta)|0\rangle$$

Exercise: use mode expansion of $\hat\chi$ and spherical coordinates to derive:

$$= \frac{1}{a(\eta)^2}\int_0^\infty \frac{k^2\,dk}{4\pi^2}\,\frac{\sin(kL)}{kL}\,|v_k(\eta)|^2 \quad \text{with } k=|\vec k|,\ L=|\vec L|.$$

Notice dimension dependence of the integral's measure!

Observe: $\frac{\sin(kL)}{kL} \approx \begin{cases} 1 & \text{for } k<1/L \\ 0 & \text{for } k>1/L \end{cases}$

Observe: $k^3\,\frac{\sin(kL)}{kL}$ has largest amplitude around $k\approx\frac{2\pi}{L}$

$\Rightarrow$ Estimate:

$$\langle 0|\hat\phi(\vec x,\eta)\,\hat\phi(\vec x+\vec L,\eta)|0\rangle \approx a(\eta)^{-2}\int_0^{2\pi/L}\frac{k^2}{4\pi^2}\,|v_{\frac{2\pi}{L}}(\eta)|^2$$

$$\approx a(\eta)^{-2}\,\frac{k^3}{12\pi^2}\,|v_k(\eta)|^2\Big|_{k=\frac{2\pi}{L}}$$

Special case: Minkowski space

We notice: The variance in a box scales like the correlator! Both are good measures of the fluctuations.

Definition: We define the so-called Fluctuation Spectrum at time η as a function of k:

$$\boxed{\delta\phi_k(\eta) := a(\eta)^{-1} k^{3/2}\,|v_k(\eta)|}$$

$k = \frac{2\pi}{L}$

Special case: Minkowski space with massive field:

15 / 23

15

File Edit View Insert Actions Tools Help

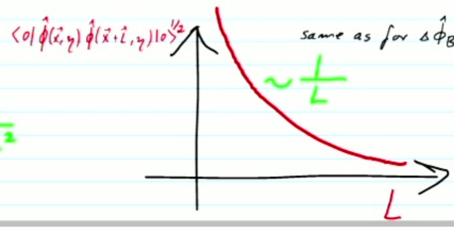
Sharing Desktop ■ Stop Sharing

$$\approx a(\eta)^{-2} \frac{k^3}{12\pi^2} |v_k(\eta)|^2 \Big|_{k=\frac{2\pi}{L}}$$

Special case: Minkowski space

Mode function: $|v_k|^2 = \frac{1}{|k|}$

$$\Rightarrow \langle 0|\hat\phi(\vec x,\eta)\,\hat\phi(\vec x+\vec\ell,\eta)|0\rangle \approx \frac{1}{3}\frac{1}{L^2}$$



$$\boxed{\delta\phi_k(\eta) := a(\eta)^{-1} k^{3/2} |v_k(\eta)|} \qquad k = \frac{2\pi}{L}$$

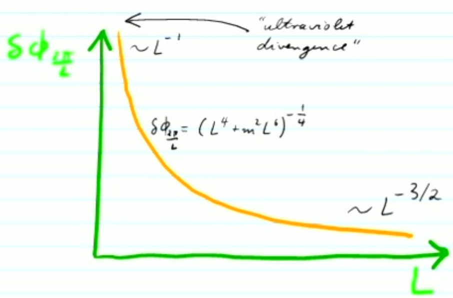
Special case Minkowski space with massive field:

- Scale factor: $a(\eta) = 1$ for all η
- Mode functions: $v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i\eta\omega_k}$ with $\omega_k = \sqrt{k^2+m^2}$

$\Rightarrow$ The fluctuation spectrum is: (recall: $k = \frac{2\pi}{L}$)

$$\delta\phi_k = \frac{k^{3/2}}{(m^2+k^2)^{1/4}} = \begin{cases} k & \text{for } k\to\infty \\ \frac{k^{3/2}}{\sqrt{m}} & \text{for } k\to 0 \end{cases}$$

and as a function of L it is: $\delta\phi_{\frac{2\pi}{L}} = \frac{(2\pi)^{3/2} L^{-3/2}}{\left(\frac{4\pi^2}{L^2}+m^2\right)^{1/4}}$



Notice: The two different scaling behaviors are not clearly visible in this plot.

Recall "Log-log plots":

$$\boxed{x := \ln(k), \quad y = \ln(\delta\phi_k)}$$

Here: $$\ln\delta\phi_k = \ln\left(\frac{k^{3/2}}{(m^2+k^2)^{1/4}}\right) = \begin{cases} \ln k & \text{for } k\to\infty \\ \underbrace{\ln\left(\frac{k^{3/2}}{\sqrt{m}}\right)}_{= -\frac{1}{2}\ln(m)+\frac{3}{2}\ln k} & \text{for } k\to 0 \end{cases}$$

Thus: $$y \approx \begin{cases} x & \text{for } x\to\infty \\ -\frac{1}{2}\ln(m)+\frac{3}{2}x & \text{for } x\to-\infty \end{cases}$$

17 / 23

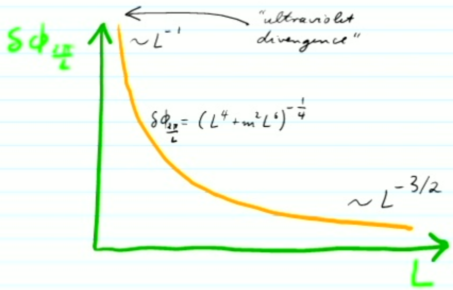
17

File Edit View Insert Actions Tools Help

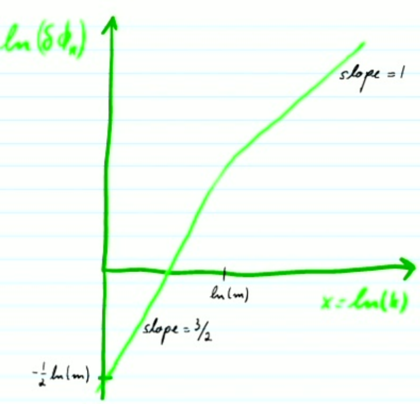
Sharing Desktop ■ Stop Sharing

$$\delta\phi_k = \frac{k^{3/2}}{(m^2+k^2)^{1/4}} = \begin{cases} k & \text{for } k\to\infty \\ \frac{k^{3/2}}{\sqrt{m}} & \text{for } k\to 0 \end{cases}$$

and as a function of L it is: $\delta\phi_{\frac{2\pi}{L}} = \dfrac{(2\pi)^{3/4} L^{-3/2}}{\left(\frac{4\pi^2}{L^2}+m^2\right)^{1/4}}$



Plot:



□ We notice that, in Minkowski space, large scale

Recall "Log-log plots":

$$\boxed{x := \ln(k), \quad y = \ln(\delta\phi_k)}$$

Here: $$\ln\delta\phi_k = \ln\left(\frac{k^{3/2}}{(m^2+k^2)^{1/4}}\right) = \begin{cases} \ln k & \text{for } k\to\infty \\ \underbrace{\ln\left(\frac{k^{3/2}}{\sqrt{m}}\right)}_{= -\frac{1}{2}\ln(m) + \frac{3}{2}\ln k} & \text{for } k\to 0 \end{cases}$$

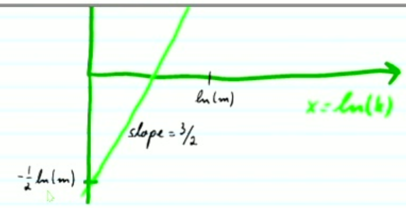
Thus: $$y \approx \begin{cases} x & \text{for } x\to\infty \\ -\frac{1}{2}\ln(m) + \frac{3}{2}x & \text{for } x\to -\infty \end{cases}$$

□ **Regarding the Infrared (IR):**

* The mass m does not matter at very short distances, i.e., in the ultraviolet.
* But, for large L the mass term does help to suppress the quantum fluctuations. ($\delta\phi \sim L^{-3/2}$ vs. $\sim L^{-1}$)
* Generally, in phenomena of QFT, the mass of particles tends to play a role only in the infrared, but not in the ultraviolet, which is why in studies of UV phenomena the ma...

19 / 23

19



☐ We notice that, in Minkowski space, large scale (i.e., large L, small k) fluctuations are strongly suppressed, especially if mass $m \neq 0$.

* But, for large L the mass term does help to suppress the quantum fluctuations. ($\delta\phi \sim L^{-3/2}$ vs. $\sim L^{-1}$)

* Generally, in phenomena of QFT, the mass of particles tends to play a rôle only in the infrared, but not in the ultraviolet, which is why in studies of UV phenomena the mass can often be neglected (e.g. for "perturbative power counting" - a method you encounter in renormalization).

☐ Regarding the Ultraviolet (UV)

* We see that QFT predicts divergingly large quantum fluctuations to be found in measurements that resolve smaller and smaller regions.

* For small enough regions B the fluctuations $\Delta\phi_B$ in ϕ_B would lead to fluctuations in the Klein Gordon energy momentum tensor that are large enough to cause black holes.

$\Rightarrow$ At this scale, $\approx 10^{-35}m$, the Planck scale, the notion of distance is assumed to break down.

(Accelerators can probe only distances down to about $10^{-19}m$)