

Title: Quantum Field Theory for Cosmology - Lecture 20240305

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QFT for Cosmology, Achim Kempf, Lecture 15

Solving the quantized K.G. eqn. on FRW spacetimes

Recall:

1.) We obtain the solution $\hat{\phi}(u, t)$ through the ansatz

$$\hat{\phi}(u, t) = \sum_k u_k(x, t) a_k + u_k^*(x, t) a_k^\dagger \quad (*)$$

* if we use operators a_k obeying $[a_k, a_{k'}^\dagger] = \delta^3(k - k')$ and* if we find classical solutions $\{u_k(x, t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\nabla_{\mu\nu} g^{\mu\sigma} \sum_k (u_k(x, t) \partial_\sigma u_k^*(u, t) - u_k^*(x, t) \partial_\sigma u_k(u, t)) = i \delta^3(\vec{x} - \vec{x}') \quad (G)$$

Application to FRW spacetime□ For convenience (namely, to avoid a "friction"-type term) we aim to solve not for $\hat{\phi}(u, t)$ directly, but instead for:

$$\hat{\chi}(\gamma, x) := a(\gamma) \hat{\phi}(\gamma, x)$$

□ In terms of $\hat{\chi}(\gamma, x)$ the quantum K.G. eqn. reads:2.) Then, we can use the $\{a_k\}$ to build a convenient basis in the Hilbert space:□ Namely: $|0\rangle$ is the vector obeying $a_k |0\rangle = 0$

□ The other basis vectors are:

$$a_k^\dagger |0\rangle, \dots, \frac{1}{\sqrt{n!}} (a_k^\dagger)^n |0\rangle, \dots, a_k^\dagger a_{k'}^\dagger |0\rangle, \dots$$

$$\frac{1}{\sqrt{n!}} \dots \frac{1}{\sqrt{m!}} (a_{k_1}^\dagger)^n \dots (a_{k_m}^\dagger)^m |0\rangle, \dots \text{etc.}$$

3.) Choosing a different set of classical solutions $\{\tilde{u}_k(u, t)\}$ which obey (G) yields the same $\hat{\phi}(u, t)$, namely

$$\hat{\phi}(u, t) = \sum_k \tilde{u}_k(x, t) \tilde{a}_k + \tilde{u}_k^*(x, t) \tilde{a}_k^\dagger$$

but the basis of vectors $|0\rangle, \tilde{a}_k^\dagger |0\rangle, \tilde{a}_k^\dagger \tilde{a}_{k'}^\dagger |0\rangle, \dots$ is a different basis. Recall: Stone von Neumann theorem.□ Observation: The derivatives $\frac{\partial}{\partial x^i}$ become multiplication operators ik_i under spatial Fourier transform.

□ Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

□ Define: $\hat{\chi}_k(\gamma) := \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}(\gamma, x) e^{-ikx} d^3x$

$$\hat{\phi}(x,t) = \sum_k u_k(x,t) a_k + u_k^*(x,t) a_k^\dagger \quad (*)$$

* if we use operators a_k obeying $[a_k, a_{k'}^\dagger] = \delta^3(k-k')$ and

* if we find classical solutions $\{u_k(x,t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\nabla^2 u_k - \sum_j (u_k(x,t) \partial_j^2 u_j^*(x,t) - u_j^*(x,t) \partial_j^2 u_k(x,t)) = i \delta^3(\vec{x}-\vec{x}') \quad (G)$$

$\sqrt{2\pi} \dots \sqrt{2\pi} (u_{k_1}) \dots (u_{k_n}) |0\rangle, \dots$

3.) Choosing a different set of classical solutions $\{\tilde{u}_k(x,t)\}$ which obey (G) yields the same $\hat{\phi}(x,t)$, namely

$$\hat{\phi}(x,t) = \sum_k \tilde{u}_k(x,t) \tilde{a}_k + \tilde{u}_k^*(x,t) \tilde{a}_k^\dagger$$

but the basis of vectors $|0\rangle, \tilde{a}_k^\dagger |0\rangle, \tilde{a}_k^\dagger \tilde{a}_{k'}^\dagger |0\rangle, \dots$ is a different basis. Recall: Stone von Neumann theorem.

Application to FRW spacetime

□ For convenience (namely, to avoid a "function"-type term) we aim to solve not for $\hat{\phi}(x,t)$ directly, but instead for:

$$\hat{\chi}(\eta, x) := a(\eta) \hat{\phi}(\eta, x)$$

□ In terms of $\hat{\chi}(\eta, x)$ the quantum K.G. eqn. reads:

$$\hat{\chi}''(\eta, x) - \Delta \hat{\chi}(\eta, x) + (m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}) \hat{\chi}(\eta, x) = 0$$

□ Note: This is a partial differential equation because both time and space derivatives occur.

□ Observation: The derivatives $\frac{\partial}{\partial x_i}$ become multiplication operators ik_i under spatial Fourier transform.

□ Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

□ Define: $\hat{\chi}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}(\eta, x) e^{-ikx} d^3x$

i.e.: $\hat{\chi}(\eta, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}_k(\eta) e^{ikx} d^3k$

□ Analogously:

$$\hat{\pi}_k^{(x)}(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\pi}^{(x)}(\eta, x) e^{-ikx} d^3x$$

□ In terms of $\hat{\chi}(\gamma, x)$ the quantum K.G. eqn. reads:

$$\hat{\chi}''(\gamma, x) - \Delta \hat{\chi}(\gamma, x) + (m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)}) \hat{\chi}(\gamma, x) = 0$$

□ **Note:** This is a partial differential equation because both time and space derivatives occur.

□ Thus, in terms of $\hat{\chi}_k(\gamma)$, the K.G. eqn. reads:

$$\hat{\chi}_k''(\gamma) + \left(k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) \hat{\chi}_k(\gamma) = 0 \quad (\text{EoM})$$

⇒ for each comoving Fourier mode k the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{\chi}_k''(\gamma) + \omega_k^2(\gamma) \hat{\chi}_k(\gamma) = 0$$

$$\text{with: } \omega_k(\gamma) = \sqrt{k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)}}$$

$$\square \text{ Define: } \hat{\chi}_k(\gamma) := \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}(\gamma, x) e^{-ikx} d^3x$$

$$\text{i.e.: } \hat{\chi}(\gamma, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}_k(\gamma) e^{ikx} d^3k$$

□ Analogously:

$$\hat{\pi}_k^{(+)}(\gamma) := \int \frac{1}{(2\pi)^{3/2}} \hat{\pi}^{(+)}(\gamma, x) e^{-ikx} d^3x$$

In extreme cases:

The frequency $\omega_k(\gamma)$ may become imaginary, namely if $a''(\gamma)$ is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on k , i.e., some modes may have imaginary frequencies while others don't.

□ Exercise:

$$* \text{ Show that } \hat{\chi}_k^+(\gamma) = \hat{\chi}_{-k}^-(\gamma), \hat{\pi}_k^{(+)}(\gamma) = \hat{\pi}_{-k}^{(-)}(\gamma) \quad (\text{HC})$$

* Show that

$$[\hat{\chi}_k(\gamma), \hat{\pi}_{k'}(\gamma)] = i \delta^3(k+k') \quad (\text{CCR})$$

$$\text{i.e. } [\hat{\chi}_k(\gamma), \hat{\pi}_{k'}^+(\gamma)] = i \delta^3(k-k')$$

(Note: The equation depends on...)

discriminant also depends on k , i.e., some modes may have imaginary frequencies while others don't.

⇒ for each comoving Fourier mode k the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\eta) + \omega_k^2(\eta) \hat{x}_k(\eta) = 0$$

with: $\omega_k(\eta) := \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$



□ Exercise:

* Show that $\hat{x}_k^+(\eta) = \hat{x}_{-k}(\eta)$, $\hat{\pi}_k^{loc+}(\eta) = \hat{\pi}_{-k}^{loc}(\eta)$ (HC)

* Show that $[\hat{x}_k(\eta), \hat{\pi}_{k'}(\eta)] = i \delta^3(k+k')$ (CCR)

i.e. $[\hat{x}_k(\eta), \hat{\pi}_{k'}(\eta)] = i \delta^3(k-k')$

□ In order to solve EoM, HC, CCR for $\hat{x}_k(\eta)$, we make this ansatz:

↙ convenient later

$$\hat{x}_k(\eta) := \frac{1}{\sqrt{2}} (v_k^+(\eta) a_k + v_k(\eta) a_k^\dagger) \quad (A)$$

□ Exercise: Express the mode functions $u_k(\eta, x)$ of (*) in terms of the functions $v_k(\eta)$.

□ Proposition: The ansatz (A)

- 1.) solves the hermiticity condition (HC) by construction.
- 2.) solves the (EoM), if the $v_k(\eta)$ each solve (EoM) as (complex!) number-valued functions:

(Note: The equation depends only on $|k|$, not on the direction of k . Thus if $v_k(\eta)$ is a solution for one k then it's a solution for all k' with $|k|=|k'|$. ⇒ We can and will choose $v_k(\eta) = v_{|k|}(\eta)$)

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0 \quad (M)$$

3.) the commutation relations (CCR) if the v_k are chosen such that they also obey:

$$v_k'(\eta) v_{k'}^+(\eta) - v_k(\eta) v_{k'}^{+\prime}(\eta) = 2i \quad (W)$$

□ Exercise:

- a) Prove the proposition.
- b) Assume that $v_k(\eta)$ is any solution of (EoM). Show that if (W) holds at one time then it holds at all time.
(Note: The LHS of (W) is the "Wronskian" of the v_k)

□ **Exercise:** Express the mode functions $u_k(y, x)$ of $(*)$ in terms of the functions $v_k(y)$.

□ **Proposition:** The ansatz (A)

1) solves the hermiticity condition (HC) by construction.

2) solves the (EoM), if the $v_k(y)$ each solve (EoM) as (complex!) number-valued functions:

$$v_k'(y) v_k''(y) - v_k(y) v_k'''(y) = 2i \quad (W)$$

□ **Exercise:**

a) Prove the proposition.

b) Assume that $v_k(y)$ is any solution of (EoM). Show that if (W) holds at one time then it holds at all time.

(Note: The LHS of (W) is the "Wronskian" of the ODE)

Conclusion:

In order to obtain the solution $\hat{\phi}(y, x)$, we do:

- Find for each mode $k \in \mathbb{R}^3$ a solution $v_k(y)$ to (M), i.e., a solution to the classical harmonic oscillator with time-dependent frequency.
- Make sure $v_k(y)$ obeys (W), if need be by multiplying with a constant. (Recall exercise b))
- Build a basis in the Hilbert space:
 $a_k |0\rangle = 0$, $a_k^\dagger |0\rangle$, $a_k^\dagger a_k^\dagger |0\rangle$, etc...

Choice of mode solutions $\{v_k(y)\}$

□ For each choice, say $\{v_k(y)\}_{k \in \mathbb{R}^3}$ or $\{\tilde{v}_k(y)\}_{k \in \mathbb{R}^3}$ we obtain the same $\hat{\phi}(x, t)$ but the bases

$$|0\rangle, a_k^\dagger |0\rangle, a_k^\dagger a_k^\dagger |0\rangle, \dots$$

and

$$|\tilde{0}\rangle, \tilde{a}_k^\dagger |\tilde{0}\rangle, \tilde{a}_k^\dagger \tilde{a}_k^\dagger |\tilde{0}\rangle, \dots$$

will of course be different.

□ We will often find it convenient to use the basis $|0\rangle, a_k^\dagger |0\rangle, a_k^\dagger a_k^\dagger |0\rangle, \dots$ that comes with one set of mode functions $\{v_k(y)\}$ at one time (say initially) and then the basis $|\tilde{0}\rangle, \tilde{a}_k^\dagger |\tilde{0}\rangle, \tilde{a}_k^\dagger \tilde{a}_k^\dagger |\tilde{0}\rangle, \dots$ of some $\{\tilde{v}_k(y)\}$ later.

frequency.

- B) Make sure $v_k(\gamma)$ obeys (W), if need be by multiplying with a constant. (Recall exercise b))
- C) Build a basis in the Hilbert space:
 $a_k |0\rangle = 0$, $a_k^\dagger |0\rangle$, $a_k^\dagger a_k^\dagger |0\rangle$, etc...

$|0\rangle$, $a_k^\dagger |0\rangle$, $a_k^\dagger a_k^\dagger |0\rangle$, ...

will of course be different.

- We will often find it convenient to use the basis $|0\rangle$, $a_k^\dagger |0\rangle$, $a_k^\dagger a_k^\dagger |0\rangle$, ... that comes with one set of mode functions $\{v_k(\gamma)\}$ at one time (say initially) and then the basis $|\tilde{0}\rangle$, $a_k^\dagger |\tilde{0}\rangle$, $a_k^\dagger a_k^\dagger |\tilde{0}\rangle$, ... of some $\{\tilde{v}_k(\gamma)\}$ later.

Why?

- In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.
- But the observables evolve in time!
- ⇒ The meanings of all Hilbert space vectors change over time
- ⇒ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

- How many possible choices of

$$\{v_k(\gamma)\}_{k \in \mathbb{R}^3}, \{\tilde{v}_k(\gamma)\}_{k \in \mathbb{R}^3}, \{\tilde{v}_k(\gamma)\}_{k \in \mathbb{R}^3}, \dots$$

do exist?

- Let us consider each mode, k , separately:
- The solution space of (M), for fixed k ,

$$v_k''(\gamma) + (k^2 + m^2 a^2(\gamma) - \frac{a^2(\gamma)}{a^2(\gamma)}) v_k(\gamma) = 0$$

has of course 2 complex dimensions.

- If v_k is a complex-valued solution, then v_k and v_k^* form a basis in the solution space.
- Note: Every solution obeying (W) must be complex-valued. Why?

⇒ The meanings of all Hilbert space vectors change over time

⇒ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.



Every solution, \tilde{v}_k , is a linear combination of v_k, v_k^* , i.e., there must exist $\alpha, \beta \in \mathbb{C}$, so that:

$$\tilde{v}_k(\eta) = \alpha v_k(\eta) + \beta v_k^*(\eta)$$

□ The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions v_k, \tilde{v}_k , etc must also obey (W), i.e.:

$$v_k'(\eta) v_k^*(\eta) - v_k(\eta) v_k'^*(\eta) = 2i \quad (W)$$

(i.e. $\text{Im}(v'v^*) = 1$, which is only one real equation)

□ The solution space of (M), for fixed k ,

$$v_k''(\eta) + \left(k^2 - m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

□ If v_k is a complex-valued solution, then v_k and v_k^* form a basis in the solution space.

Note: Every solution obeying (W) must be complex-valued. Why?

□ Proposition:

Assume v_k obey (W). Then, \tilde{v}_k defined through

$$\tilde{v}_k(\eta) = \alpha v_k(\eta) + \beta v_k^*(\eta) \quad (B)$$

also obeys (W), iff the coefficients α, β obey:

$$|\alpha|^2 - |\beta|^2 = 1$$

□ Proof: Exercise

⇒ we easily obtain: $\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}} (v_k^*(\eta) \hat{a}_k + v_k(\eta) \hat{a}_k^\dagger)$
 $= \frac{1}{\sqrt{2}} (\tilde{v}_k^*(\eta) \tilde{a}_k + \tilde{v}_k(\eta) \tilde{a}_k^\dagger)$
 $= \dots$ } (P)

⇒ The meanings of all Hilbert space vectors change over time

⇒ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.



Every solution, \tilde{v}_k , is a linear combination of v_k, v_k^* , i.e., there must exist $\alpha, \beta \in \mathbb{C}$, so that:

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The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions v_k, \tilde{v}_k , etc must also obey (W), i.e.:

$$v_k'(\eta) v_k^*(\eta) - v_k(\eta) v_k'^*(\eta) = 2i \quad (W)$$

(i.e. $\text{Im}(v'v^*) = 1$, which is only one real equation)

□ The solution space of (M), for fixed k ,

$$v_k''(\eta) + \left(k^2 - m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

□ If v_k is a ^{Note: Every solution obeying (W) must be complex-valued. Why?} complex-valued solution, then v_k and v_k^* form a basis in the solution space.

□ Proposition:

Assume v_k obey (W). Then, \tilde{v}_k defined through

$$\tilde{v}_k(\eta) = \alpha v_k(\eta) + \beta v_k^*(\eta) \quad (B)$$

also obeys (W), iff the coefficients α, β obey:

$$|\alpha|^2 - |\beta|^2 = 1$$

□ Proof: Exercise

⇒ we easily obtain: $\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}} (v_k^*(\eta) \hat{a}_k + v_k(\eta) \hat{a}_k^\dagger)$
 $= \frac{1}{\sqrt{2}} (\tilde{v}_k^*(\eta) \tilde{a}_k + \tilde{v}_k(\eta) \tilde{a}_k^\dagger)$
 $= \dots$ } (P)

□ The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions v_k, \tilde{v}_k , etc must also obey (W), i.e.:

$$v_k(\gamma) v_k^*(\gamma) - v_k^*(\gamma) v_k(\gamma) = 2i \quad (W)$$

(i.e. $\text{Im}(v v^*) = 1$, which is only one real equation)

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

□ Proof: Exercise

$$\Rightarrow \text{we easily obtain: } \hat{c}_k(\gamma) = \frac{1}{\sqrt{2}} (v_k^*(\gamma) a_k + v_k(\gamma) a_k^*) \quad (P)$$

$$= \frac{1}{\sqrt{2}} (\tilde{v}_k^*(\gamma) \tilde{a}_k + \tilde{v}_k(\gamma) \tilde{a}_k^*)$$

$$= \dots$$

□ Terminology: Such a transformation from one choice $\{v_k\}, a_k$ and corresponding basis $|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^- |0\rangle, \dots$

$$|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^- |0\rangle, \dots$$

to some $\{\tilde{v}_k\}, |\tilde{0}\rangle, \tilde{a}_k$ and their basis

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^- |\tilde{0}\rangle, \dots$$

is called a "Bogolubov transformation".

Bogolubov transformations of Hilbert bases

□ How can we express the basis vectors

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \frac{1}{\sqrt{2!}} \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots, \tilde{a}_k^+ \tilde{a}_k^- |\tilde{0}\rangle, \dots$$

as linear combinations of the basis

$$|0\rangle, a_k^+ |0\rangle, \frac{1}{\sqrt{2!}} a_k^+ a_k^+ |0\rangle, \dots, a_k^+ a_k^- |0\rangle, \dots ?$$

Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.

(E.g. so that $|0\rangle$ is, at least at one time, the vacuum.)

* Find out when which choice of $\{v_k\}$ is convenient.

□ Proposition: Equations (B) & (P) yield: $a_k = \alpha_k \tilde{a}_k + \beta_k \tilde{a}_k^+$

Proof: Exercise.

□ Now we observe that $a_k |0\rangle = 0$ becomes:

$$(\alpha_k \tilde{a}_k + \beta_k \tilde{a}_k^+) |0\rangle = 0$$

(We will soon explore how to identify the vacuum... 13/19)

$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$
is called a "Bogoleubov transformation".

Strategy: We have two tasks now:



* Make Bogoleubov Hilbert basis transforms explicit.

(E.g. so that $|\tilde{0}\rangle$ is, at least at one time, the vacuum.)

* Find out when which choice of $\{v_k\}$ is convenient.

□ Try to solve for $|\tilde{0}\rangle$ using ansatz: $|\tilde{0}\rangle := \left(\prod_k f_k(\tilde{a}_k^+, \tilde{a}_k^-) \right) |\tilde{0}\rangle$

□ Proposition:

$$|\tilde{0}\rangle = \left[\prod_k \frac{1}{|d_k|^{1/2}} e^{-\frac{\beta_k}{2d_k} \tilde{a}_k^+ \tilde{a}_k^-} \right] |\tilde{0}\rangle \quad (T)$$

needed for normalization

□ Proof: Exercise.

Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \dots$

Interpretation of (T):

□ Assume, e.g., that $|\tilde{0}\rangle$ and $|\tilde{0}\rangle$ are those Hilbert space vectors which happen to be the vacuum state vectors at the times η_1 and η_2 respectively.

as linear combinations of the basis

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \frac{1}{2!} \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots ?$$

□ Proposition: Equations (B) & (P) yield: $a_k = d_k^+ \tilde{a}_k + \beta_k \tilde{a}_k^-$

Proof: Exercise.

□ Now we observe that $a_k |\tilde{0}\rangle = 0$ becomes:

$$(d_k^+ \tilde{a}_k + \beta_k \tilde{a}_k^-) |\tilde{0}\rangle = 0$$

(We will soon explore how to identify the vacuum state at any given time)

□ Assume at time η_1 the system's state $|\Omega\rangle$ is the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions $\{v_k\}$ so that:

$$|\Omega\rangle = |\tilde{0}\rangle$$

At a later time, η_2 , the system is still in this state but the vacuum is then some other vector $|\tilde{0}\rangle$, which obeys $\tilde{a}_k |\tilde{0}\rangle = 0$ with mode functions $\tilde{v}_k(t)$.

□ Thus, from (T) we see that the system's state, $|\Omega\rangle$, is at η_2 a state with many particles.

□ Note: the particles have been created in $k, -k$ pairs.

□ Intuitively: The expansion rips virtual particle + antiparticle pairs apart.