Title: Quantum Field Theory for Cosmology - Lecture 20240305

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: March 05, 2024 - 4:00 PM

URL: https://pirsa.org/24030008

File Edit View Insert Actions Tools Help

# QFT for Cosmology, Achim Kempf, Lecture 15

## Solving the quantized K.G. eqn. on FRW spacetimes

**Recall:**

**1.)** We obtain the solution $\hat\phi(x,t)$ through the ansatz

$$\hat\phi(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger \qquad (*)$$

* if we use operators $a_k$ obeying $[a_k, a_{k'}^\dagger] = \delta^3(k-k')$ and

* if we find classical solutions $\{u_k(x,t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\sqrt{g}\, g^{0\nu} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_k(x',t) \right) = i\, \delta^3(\vec{x} - \vec{x}') \qquad (C)$$

**2.)** Then, we can use the $\{a_k\}$ to build a convenient basis in the Hilbert space:

- Namely: $|0\rangle$ is the vector obeying $a_k|0\rangle = 0$
- The other basis's vectors are:

$$a_k^\dagger|0\rangle, \ldots \frac{1}{\sqrt{n!}}(a_k^\dagger)^n|0\rangle, \ldots a_k^\dagger a_{k'}^\dagger|0\rangle, \ldots$$

$$\frac{1}{\sqrt{n_1!}} \cdots \frac{1}{\sqrt{n_m!}} (a_{k_1}^\dagger)^{n_1} \ldots (a_{k_m}^\dagger)^{n_m} |0\rangle, \ldots \text{ etc.}$$

**3.)** Choosing a different set of classical solutions $\{\tilde u_k(x,t)\}$ which obey **(C)** yields the same $\hat\phi(x,t)$, namely

$$\hat\phi(x,t) = \sum_k \tilde u_k(x,t)\, \tilde a_k + \tilde u_k^*(x,t)\, \tilde a_k^\dagger$$

but the basis of vectors $|\tilde 0\rangle, \tilde a_k^\dagger|\tilde 0\rangle, \tilde a_k^\dagger \tilde a_{k'}^\dagger|\tilde 0\rangle \ldots$ is a different basis. Recall: Stone von Neumann theorem.

## Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for $\hat\phi(x,t)$ directly, but instead for:

$$\hat\chi(\eta,x) := a(\eta)\, \hat\phi(\eta,x)$$

- In terms of $\hat\chi(\eta,x)$ the quantum K.G. eqn. reads:

- **Observation:** The derivatives $\frac{\partial}{\partial x^i}$ become multiplication operators $ik_i$ under spatial Fourier transform.

- **Plan:** Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more managable set of ordinary DEs.

- **Define:** $\hat\chi_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat\chi(\eta,x)\, e^{-ikx}\, d^3x$

1/19

File Edit View Insert Actions Tools Help

$$\hat\phi(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^* \qquad (*)$$

* if we use operators $a_k$ obeying $[a_k, a_{k'}^+] = \delta^3(k-k')$ and
* if we find classical solutions $\{u_k(x,t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\sqrt{|g|}\, g^{00} \sum_k \left( u_k(x,t) \frac{\partial}{\partial t} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial t} u_k(x',t) \right) = i\, \delta^3(\vec x - \vec x') \qquad (C)$$

3.) Choosing a different set of classical solutions $\{\tilde u_k(x,t)\}$ which obey (C) yields the same $\hat\phi(x,t)$, namely

$$\hat\phi(x,t) = \sum_k \tilde u_k(x,t)\, \tilde a_k + \tilde u_k^*(x,t)\, \tilde a_k^*$$

but the basis of vectors $|\tilde 0\rangle$, $\tilde a_k^+|\tilde 0\rangle$, $\tilde a_k^+ \tilde a_{k'}^+ |\tilde 0\rangle \ldots$ is a different basis. Recall: Stone von Neumann theorem.

## Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for $\hat\phi(x,t)$ directly, but instead for:

$$\hat\chi(\eta,x) := a(\eta)\, \hat\phi(\eta,x)$$

- In terms of $\hat\chi(\eta,x)$ the quantum K.G. eqn. reads:

$$\hat\chi''(\eta,x) - \Delta\hat\chi(\eta,x) + \left(m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat\chi(\eta,x) = 0$$

- Note: This is a partial differential equation because both time and space derivatives occur.

- Observation: The derivatives $\frac{\partial}{\partial x^i}$ become multiplication operators $i k_i$ under spatial Fourier transform.

- Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

- Define: $\hat\chi_k(\eta) := \int \frac{1}{(2\pi)^{3/2}}\, \hat\chi(\eta,x)\, e^{-ikx}\, d^3x$

  i.e.: $\hat\chi(\eta,x) = \int \frac{1}{(2\pi)^{3/2}}\, \hat\chi_k(\eta)\, e^{ikx}\, d^3k$

- Analogously:

$$\hat\pi_k^{(\chi)}(\eta) := \int \frac{1}{(2\pi)^{3/2}}\, \hat\pi^{(\chi)}(\eta,x)\, e^{-ikx}\, d^3x$$

File Edit View Insert Actions Tools Help

- In terms of $\hat{\chi}(\eta,x)$ the quantum K.G. eqn. reads:

$$\hat{\chi}''(\eta,x) - \Delta\hat{\chi}(\eta,x) + \left(m^2a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat{\chi}(\eta,x) = 0$$

- Note: This is a partial differential equation because both time and space derivatives occur.

- Define: $\hat{\chi}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}(\eta,x) e^{-ikx} d^3x$

i.e.: $\hat{\chi}(\eta,x) = \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}_k(\eta) e^{ikx} d^3k$

- Analogously:

$$\hat{\pi}_k^{(\chi)}(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\pi}^{(\chi)}(\eta,x) e^{-ikx} d^3x$$

- Thus, in terms of $\hat{\chi}_k(\eta)$, the K.G. eqn. reads:

$$\boxed{\hat{\chi}_k''(\eta) + \left(k^2 + m^2a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat{\chi}_k(\eta) = 0} \quad (EoM)$$

$\Rightarrow$ for each comoving Fourier mode $k$ the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta)\hat{\chi}(\eta) = 0$$

with: $\omega_k(\eta) := \sqrt{k^2 + m^2a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$

In extreme cases:

The frequency $\omega_k(\eta)$ may become imaginary, namely if $a''(\eta)$ is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on $k$, i.e., some modes may have imaginary frequencies while others don't.

- Exercise:

* Show that $\hat{\chi}_k^{+}(\eta) = \hat{\chi}_{-k}(\eta)$, $\hat{\pi}_k^{(\chi)+}(\eta) = \hat{\pi}_{-k}^{(\chi)}(\eta)$ (HC)
* Show that

$$[\hat{\chi}_k(\eta), \hat{\pi}_{k'}(\eta)] = i\,\delta^3(k+k') \quad (CCR)$$

i.e. $[\hat{\chi}_k(\eta), \hat{\pi}_{k'}^{+}(\eta)] = i\,\delta^3(k-k')$

3/19

$\Rightarrow$ for each **comoving** Fourier mode $k$ the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\eta) + \omega_k^2(\eta)\,\hat{x}(\eta) = 0$$

$$\text{with:} \quad \omega_k(\eta) := \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$$

discriminant also depends on $k$, i.e., some modes may have imaginary frequencies while others don't.

Exercise:

* Show that $\hat{x}_k^{+}(\eta) = \hat{x}_{-k}(\eta)$, $\hat{\pi}_k^{(x)+}(\eta) = \hat{\pi}_{-k}^{(x)}(\eta)$ **(HC)**

* Show that

$$[\hat{x}_k(\eta), \hat{\pi}_{k'}(\eta)] = i\,\delta^3(k+k') \qquad \textbf{(CCR)}$$

$$\text{i.e.} \quad [\hat{x}_k(\eta), \hat{\pi}_{k'}^{+}(\eta)] = i\,\delta^3(k-k')$$

In order to solve **EoM, HC, CCR** for $x_k(\eta)$, we we make this ansatz:

convenient later

$$\hat{x}_k(\eta) := \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\,a_k + v_k(\eta)\,a_{-k}^+\right) \qquad \textbf{(A)}$$

Exercise: Express the mode functions $u_k(\eta, x)$ of **(*)** in terms of the functions $v_k(\eta)$.

**Proposition**: The ansatz **(A)**

**1.)** solves the hermiticity condition **(HC)** by construction.

**2)** solves the **(EoM)**, if the $v_k(\eta)$ each solve **(EoM)** as (complex!) number-valued functions:

(Note: The equation depends only on $|k|$, not on the direction of $k$. Thus if $v_k(\eta)$ is a solution for one $k$ then it is solution for all $k'$ with $|k'| = |k|$. $\Rightarrow$ We can and will choose $v_k(\eta) = v_{|k|}(\eta)$)

$$\boxed{v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0 \qquad \textbf{(M)}}$$

**3.)** the commutation relations **(CCR)** if the $v_k$ are chosen such that they also obey:

$$\boxed{v_k'(\eta)\,v_k^*(\eta) - v_k(\eta)\,v_k^*(\eta)' = 2i \qquad \textbf{(W)}}$$

Exercise:

**a)** Prove the proposition.

**b.)** Assume that $v_k(\eta)$ is any solution of **(EoM)**. Show that if **(W)** holds at one time then it holds at all time.

(Note: The LHS of **(W)** is the "Wronskian" of the ODE)

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

□ Exercise: Express the mode functions $u_k(\eta,x)$ of (*) in terms of the functions $v_k(\eta)$.

□ Proposition: The ansatz (A)

1.) solves the hermiticity condition (HC) by construction.

2) solves the (EoM), if the $v_k(\eta)$ each solve (EoM) as (complex!) number-valued functions:

$$\boxed{v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad (W)}$$

□ Exercise:

a) Prove the proposition.

b.) Assume that $v_k(\eta)$ is any solution of (EoM). Show that if (W) holds at one time then it holds at all time.

(Note: The LHS of (W) is the "Wronskian" of the ODE)

Conclusion:

In order to obtain the solution $\hat\phi(\eta,x)$, we do:

A) Find for each mode $k \in \mathbb{R}^3$ a solution $v_k(\eta)$ to (M), i.e., a solution to the classical harmonic oscillator with time-dependent frequency.

B) Make sure $v_k(\eta)$ obeys (W), if need be by multiplying with a constant. (Recall exercise b))

C) Build a basis in the Hilbert space:

$a_k|0\rangle = 0, \quad a_k^+|0\rangle, \quad a_k^+ a_{k'}^+|0\rangle$, etc ...

Choice of mode solutions $\{v_k(\eta)\}$

□ For each choice, say $\{v_k(\eta)\}_{k\in\mathbb{R}^3}$ or $\{\tilde v_k(\eta)\}_{k\in\mathbb{R}^3}$ we obtain the same $\hat\phi(x,t)$ but the bases

$|0\rangle, a_k^+|0\rangle, a_k^+ a_{k'}^+|0\rangle, \ldots$

and

$|\tilde 0\rangle, \tilde a_k^+|\tilde 0\rangle, \tilde a_k^+ \tilde a_{k'}^+|\tilde 0\rangle, \ldots$

will of course be different.

□ We will often find it convenient to use the basis $|0\rangle, a_k^+|0\rangle, a_k^+ a_{k'}^+|0\rangle, \ldots$ that comes with one set of mode functions $\{v_k(\eta)\}$ at one time (say initially) and then the basis $|\tilde 0\rangle, \tilde a_k^+|\tilde 0\rangle, \tilde a_k^+ \tilde a_{k'}^+|\tilde 0\rangle, \ldots$ of some $\{\tilde v_k(\eta)\}$ later.

7/19

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

frequency.

**B)** Make sure $v_k(\eta)$ obeys **(W)**, if need be by multiplying with a constant. (Recall exercise 6))

**C)** Build a basis in the Hilbert space:

$a_k|0\rangle = 0, \quad a_k^+|0\rangle, \quad a_k^+ a_{k'}^+|0\rangle$, etc...

$|0\rangle, \; \tilde{a}_k|\tilde{0}\rangle, \; \tilde{a}_k \tilde{a}_{k'}|\tilde{0}\rangle, \ldots$

will of course be different.

- We will often find it convenient to use the basis $|0\rangle, \; a_k^+|0\rangle, \; a_k^+ a_{k'}^+|0\rangle, \ldots$ that comes with one set of mode functions $\{v_k(\eta)\}$ at one time (say initially) and then the basis $|\tilde{0}\rangle, \; \tilde{a}_k^+|\tilde{0}\rangle, \; \tilde{a}_k^+ \tilde{a}_{k'}^+|\tilde{0}\rangle, \ldots$ of some $\{\tilde{v}_k(\eta)\}$ later.

**Why?**

- In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.
- But the observables evolve in time!

$\Longrightarrow$ The meanings of all Hilbert space vectors change over time

$\rightsquigarrow$ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

- How many possible choices of

$$\{v_k(\eta)\}_{k\in\mathbb{R}^3}, \; \{\tilde{v}_k(\eta)\}_{k\in\mathbb{R}^3}, \; \{\tilde{\tilde{v}}_k(\eta)\}_{k\in\mathbb{R}^3}, \ldots$$

do exist?

- Let us consider each mode, $k$, separately:
- The solution space of **(M)**, for fixed $k$,

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

- If $v_k$ is a $\overbrace{\text{complex-valued}}^{\text{Note: Every solution obeying (W) must be complex-valued. Why?}}$ solution, then

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.

9

9 / 19

$\Longrightarrow$ The meanings of all Hilbert space vectors change over time

$\rightsquigarrow$ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

$\Longrightarrow$

> Every solution, $\tilde{v}_k$, is a linear combination of $v_k, v_k^*$, i.e., there must exist $\alpha, \beta \in \mathbb{C}$, so that:
>
> $$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta)$$

- The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions $v_k, \tilde{v}_k$, etc must also obey (W), i.e.:

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad \textbf{(W)}$$

(i.e. $\mathrm{Im}(v'v^*) = 1$, which is only one real equation)

- The solution space of (M), for fixed $k$,

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

- If $v_k$ is a $\underbrace{\text{complex-valued}}_{}$ solution, then

Note: Every solution obeying (W) must be complex-valued. Why?

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.

- <u>Proposition:</u>

Assume $v_k$ obeys (W). Then, $\tilde{v}_k$ defined through

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta) \qquad \text{(B)}$$

also obeys (W), iff the coefficients $\alpha_k, \beta_k$ obey:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

- Proof: Exercise

$\Rightarrow$ we easily obtain:

$$\left.\begin{aligned}\hat{x}_k(\eta) &= \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)\\ &= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right)\\ &= \ldots\end{aligned}\right\}\textbf{(P)}$$

File Edit View Insert Actions Tools Help

Sharing Desktop ▲ ■ Stop Sharing

$\Longrightarrow$ The meanings of all Hilbert space vectors change over time

$\rightsquigarrow$ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

$\Longrightarrow$

$$\boxed{\begin{array}{l}\text{Every solution, } \tilde{v}_k\text{, is a linear combination of } v_k, v_k^*\text{, i.e., there must exist } \alpha, \beta \in \mathbb{C}\text{, so that:}\\ \qquad \tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta)\end{array}}$$

- The actual dimensionality is 3 !

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions $v_k, \tilde{v}_k$, etc must also obey **(W)**, i.e.:

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad \textbf{(W)}$$

(i.e $\mathrm{Im}(v' v^*) = 1$, which is only one real equation)

- The solution space of **(M)**, for fixed $k$,

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

Note: Every solution obeying (W) must be complex-valued. Why?

- If $v_k$ is a complex-valued solution, then

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.

- Proposition:

Assume $v_k$ obeys **(W)**. Then, $\tilde{v}_k$ defined through

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta) \qquad (B)$$

also obeys **(W)**, iff the coefficients $\alpha_k, \beta_k$ obey:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

- Proof: Exercise

$\Rightarrow$ we easily obtain:

$$\left.\begin{aligned}\hat{x}_k(\eta) &= \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)\\ &= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right)\\ &= \dots\end{aligned}\right\}\textbf{(P)}$$

11 / 19

11

□ The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions $v_k, \tilde{v}_k$, etc must also obey (W), i.e.:

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \quad \textbf{(W)}$$

(i.e. $\mathrm{Im}(v' v^*) = 1$, which is only one real equation)

□ Terminology: Such a transformation from one choice $\{v_k\}$, $a_k$ and corresponding basis

$$|0\rangle,\ a_k^+|0\rangle,\ a_k^+ a_{k'}^+|0\rangle, \dots$$

to some $\{\tilde{v}_k\}$, $|\tilde{0}\rangle$, $\tilde{a}_k$ and their basis

$$|\tilde{0}\rangle,\ \tilde{a}_k^+|0\rangle,\ \tilde{a}_k^+ \tilde{a}_{k'}^+|\tilde{0}\rangle, \dots$$

is called a "Bogolubov transformation".

Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.

(E.g. so that $|0\rangle$ is, at least at one time, the vacuum.) * Find out when which choice of $\{v_k\}$ is convenient.

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

□ Proof: Exercise

⇒ we easily obtain:

$$\left.\begin{aligned}\hat{x}_k(\eta) &= \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)\\ &= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right)\\ &= \dots\end{aligned}\right\}\textbf{(P)}$$

## Bogolubov transformations of Hilbert bases

□ How can we express the basis vectors

$$|\tilde{0}\rangle,\ \tilde{a}_k^+|\tilde{0}\rangle,\ \frac{1}{\sqrt{2!}}\tilde{a}_k^{+2}|\tilde{0}\rangle, \dots,\ \tilde{a}_k^+ \tilde{a}_{k'}^+|\tilde{0}\rangle, \dots$$

as linear combinations of the basis

$$|0\rangle,\ a_k^+|0\rangle,\ \frac{1}{\sqrt{2!}} a_k^{+2}|0\rangle, \dots,\ a_k^+ a_{k'}^+|0\rangle, \dots\ ?$$

□ Proposition: Equations (B) & (P) yield: $\boxed{a_k = \alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+}$

Proof: Exercise.

□ Now we observe that $a_k|0\rangle = 0$ becomes:

$$\left(\alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+\right)|0\rangle = 0$$

(We will soon explore how to identify the vacuum...

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$|\tilde{0}\rangle,\ \tilde{a}_k^+|0\rangle,\ \tilde{a}_k^+\tilde{a}_{k'}^+|\tilde{0}\rangle, \dots$

is called a "Bogolubov transformation".

Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.

(E.g. so that $|0\rangle$ is, at least at one time, the vacuum.) * Find out when which choice of $\{v_k\}$ is convenient.

- Try to solve for $|0\rangle$ using ansatz: $|0\rangle := \left(\prod_k f_k(\tilde{a}_k^+, \tilde{a}_{-k}^+)\right)|\tilde{0}\rangle$

- Proposition:

$$|0\rangle = \left[\prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k^*}\tilde{a}_k^+\tilde{a}_{-k}^+}\right]|\tilde{0}\rangle \quad (T)$$

needed for normalization

- Proof: Exercise.
  Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \dots$

Interpretation of (T):

- Assume, e.g., that $|0\rangle$ and $|\tilde{0}\rangle$ are those Hilbert space vectors which happen to be the vacuum state vectors at the times $\eta_1$ and $\eta_2$ respectively.

as linear combinations of the basis

$|0\rangle,\ a_k^+|0\rangle,\ \frac{1}{\sqrt{2!}}a_k^{+2}|0\rangle, \dots,\ a_k^+a_{k'}^+|0\rangle, \dots$ ?

- Proposition: Equations (B) & (P) yield: $\boxed{a_k = \alpha_k^*\tilde{a}_k + \beta_k\tilde{a}_{-k}^+}$
  Proof: Exercise.

- Now we observe that $a_k|0\rangle = 0$ becomes:

$$(\alpha_k^*\tilde{a}_k + \beta_k\tilde{a}_{-k}^+)|0\rangle = 0$$

(We will soon explore how to identify the vacuum state at any given time)

- Assume at time $\eta_1$ the system's state $|\Omega\rangle$ is the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions $\{v_k\}$ so that:
$$|\Omega\rangle = |0\rangle$$
At a later time, $\eta_2$, the system is still in this state but the vacuum is then some other vector $|\tilde{0}\rangle$, which obeys $\tilde{a}_k|\tilde{0}\rangle = 0$ with mode functions $\tilde{v}_k(t)$.

- Thus, from (T) we see that the system's state, $|\Omega\rangle$, is at $\eta_2$ a state with many particles.

- Note: the particles have been created in $k, -k$ pairs.

- Intuitively: The expansion rips virtual particle + antiparticle pairs apart.

15/19

15