Title: Advanced General Relativity - 240327 (afternoon)

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Collection: Advanced General Relativity (PHYS7840)

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Kerr event horizon

Intrinsic coordinates $y^a = (v, \theta, \chi)$

Embedding relations: $x^\alpha = x^\alpha(y^a)$:

$$v = v$$
$$r = r_+$$
$$\theta = \theta$$
$$\psi = \Omega_H v + \chi$$

Tangent vectors:

$$k^\alpha = \frac{\partial x^\alpha}{\partial v} = (1, 0, 0, \Omega_H)$$

$$e^\alpha_\theta = \frac{\partial x^\alpha}{\partial \theta} = (0, 0, 1, 0)$$

$$e^\alpha_\chi = \frac{\partial x^\alpha}{\partial \chi} = (0, 0, 0, 1)$$

$$\Omega_H = \frac{a}{r_+^2 + a^2}$$

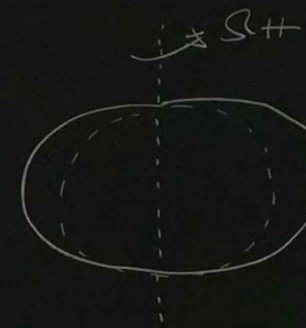
$$\kappa = \frac{r_+ - M}{r_+^2 + a^2}$$

$$r_+ = M + \sqrt{M^2 - a^2}$$

Induced metric: $\Omega_{AB} = g_{\alpha\beta}\, e^{\alpha}_{A}\, e^{\beta}_{B}$

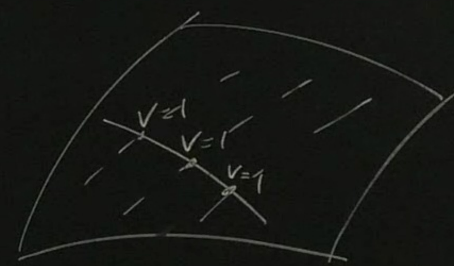
$$\begin{cases} \Omega_{\theta\theta} = r_+^2 + a^2\cos^2\theta \\ \Omega_{\chi\chi} = \dfrac{(r_+^2+a^2)^2}{r_+^2+a^2\cos^2\theta}\sin^2\theta \end{cases}$$

$$\sqrt{\Omega} = (r_+^2+a^2)\sin\theta$$



Cross-sectional area:

$$A = \int \sqrt{\Omega}\, d^2\theta = (r_+^2+a^2)\underbrace{\int \sin\theta\, d\theta\, d\varphi}_{4\pi} = 4\pi(r_+^2+a^2) = 8\pi M r_+$$



∂x $\quad r_+ = M + \sqrt{M^2 - a^2}$ $\quad = (r_+ + a)$ $\quad 4\pi$ $\quad = 8\pi$

$$\boxed{A = 8\pi M\left(M + \sqrt{M^2 - (J/M)^2}\right)}$$

$$A = A(M, J)$$

BH #1 : (M, J)

BH #2 ; $(M + \partial M, J + \partial J)$

Differential law

$$\partial A = \left(\frac{\partial A}{\partial M}\right)\partial M + \left(\frac{\partial A}{\partial J}\right)\partial J$$

$$\frac{\kappa}{8\pi}\partial A = \partial M - \Omega_H \partial J$$

$$\longrightarrow \boxed{\partial M = \frac{\kappa}{8\pi}\partial A + \Omega_H \partial J}$$

$$\sqrt{M^2 - (J/M)^2}$$

$$A = A(M, J)$$

BH #1 ; (M, J)

BH #2 ; $(M + \delta M, J + \delta J)$

$$\delta A = \left(\frac{\partial A}{\partial M}\right) \delta M + \left(\frac{\partial A}{\partial J}\right) \delta J$$

$$M - \Omega_H \, \delta J$$

$$\longrightarrow \boxed{\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \, \delta J}$$

$$\boxed{M(A, J)}$$

$$\boxed{\frac{\kappa}{8\pi} = \left(\frac{\partial M}{\partial A}\right)_J \qquad \Omega_H = \left(\frac{\partial M}{\partial J}\right)_A}$$

$$\boxed{\ldots + \sqrt{M^2 - (J/M)^2}\ )}$$

$$A = A(M, J)$$

BH #1 ; (M, J)

BH #2 ; $(M + \delta M, J + \delta J)$

$$\delta A = \left(\frac{\partial A}{\partial M}\right)\delta M + \left(\frac{\partial A}{\partial J}\right)\delta J$$

$$\delta M - \Omega_H\, \delta J$$

$$\longrightarrow \boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\, \delta J}$$

"$T\delta S$"

$$\boxed{M(A, J)}$$

$$\boxed{\frac{\kappa}{8\pi} = \left(\frac{\partial M}{\partial A}\right)_J \qquad \Omega_H = \left(\frac{\partial M}{\partial J}\right)_A}$$

Integral law

Euler thm on homogeneous functions :

$f(x^a)$: $f(\lambda x^a) = \lambda^K f(x$

$\Rightarrow \quad Kf = \sum_a x^a \frac{\partial f}{\partial x}$

$[M] = L$

$[A] = L^2$

$[J] = L^2$

$M = M(A, J)$

$M(\lambda A, \lambda J) = \lambda^{1/2} M(A, J) \rightarrow$ M is homo. func. of deg

Euler : $\frac{1}{2} M = A \frac{\partial M}{\partial A} + J \frac{\partial M}{\partial J}$

Integral law

Euler thm on homogeneous functions:

$f(x^a)$: $f(\lambda x^a) = \lambda^k f(x$

$\Rightarrow \quad kf = \sum_a x^a \frac{\partial f}{\partial x}$

$[M] = L$

$[A] = L^2$

$[J] = L^2$

$M = \cancel{M(A,J)}$

$M(\lambda A, \lambda J) = \lambda^{1/2} M(A,J) \longrightarrow$ M is homo. func. of deg

Euler: $\frac{1}{2}M = A\frac{\partial M}{\partial A} + J\frac{\partial M}{\partial J} = \frac{\kappa}{8\pi}A + \Omega$

$$\boxed{M = \frac{\kappa}{4\pi}A + 2\Omega}$$

mogeneous functions :

$$f(x^a) \; : \; f(\lambda x^a) = \lambda^k f(x^a)$$ homo func of degree k.

$$\Rightarrow \quad kf = \sum_a x^a \frac{\partial f}{\partial x^a}$$

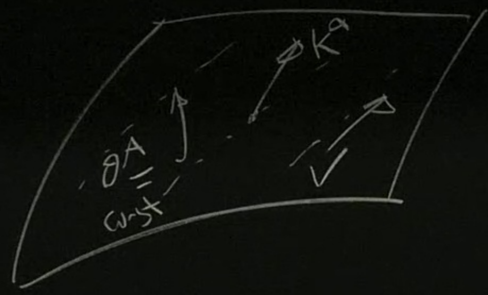
$A, J)$

$A, \lambda J) = \lambda^{1/2} M(A, J) \longrightarrow$ M is homo. func. of degree $\frac{1}{2}$.

$$\frac{1}{2} M = A \frac{\partial M}{\partial A} + J \frac{\partial M}{\partial J} = \frac{\kappa}{8\pi} A + \Omega_H J$$

$$\boxed{M = \frac{\kappa}{4\pi} A + 2\Omega_H J}$$ (Smarr's formula)

Null hypersurfaces



$$y^a = (v, \theta^A)$$

$$k^\alpha = \frac{\partial x^\alpha}{\partial v}, \quad e^\alpha_A = \frac{\partial x^\alpha}{\partial \theta^A}, \quad N_\alpha$$

$$y^\alpha = (v, \theta^A)$$

$$K^\alpha = \frac{\partial x^\alpha}{\partial v}, \quad e^\alpha_A = \frac{\partial x^\alpha}{\partial \theta^A}, \quad N_\alpha$$

$$K_\alpha K^\alpha = 0$$

$$K_\alpha e^\alpha_A = 0$$

$$N_\alpha N^\alpha = 0, \quad N_\alpha K^\alpha = -1, \quad N_\alpha e^\alpha_A = 0$$

$$\Omega_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$$

$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$$

$$K^\beta \nabla_\beta e^\alpha_A = \omega_A K^\alpha + B_A{}^B e^\alpha_A$$

$$B_{AB} = e^\alpha_A e^\beta_B \nabla_\alpha K_\beta = \frac{1}{2} \Theta \Omega_{AB} + \sigma_{AB}$$

$$= \partial_v \Omega_{AB}$$

$$\Theta = \frac{1}{\sqrt{\Omega}} \partial_v \sqrt{\Omega}$$

$$\frac{dA}{dv} = \int_S \Theta \, dS$$

$$dS = \sqrt{\Omega} \, d^2\theta$$

Integration:
$$d\Sigma_\alpha = -K_\alpha \, dv \, dS$$
$$dS_{\alpha\beta} = 2K_{[\alpha} N_{\beta]} \, dS$$

Gauss–Codazzi eqns:
$$R_{\mu\nu} K^\mu K^\nu = -\partial_v \Theta + \kappa \Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}$$
$$R_{\mu\nu} K^\mu e^\nu_A = \partial_v \omega_A - \partial_A \kappa$$

$\partial x \qquad r_+ = M + \sqrt{M^2 - a^2} \qquad = (r_+ + a) \qquad 4\pi \qquad = 8\pi M$

Integration: $\partial \Sigma_\alpha = - K_\alpha \, \partial v \, \partial S$

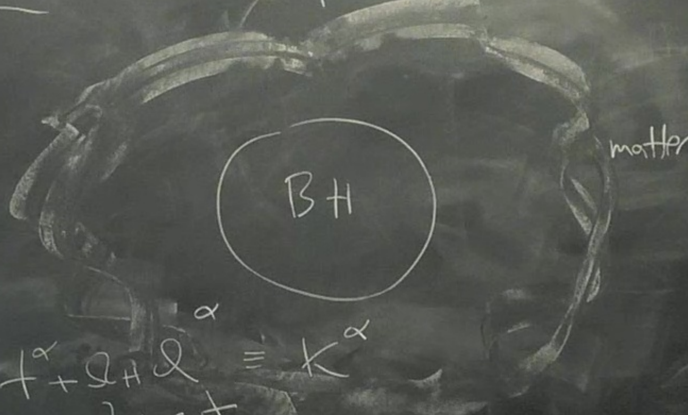
$\partial S_{\alpha\beta} = 2 K_{[\alpha} N_{\beta]} \, \partial S$

Gauss–Codazzi eqns:

$$\begin{cases} R_{\mu\nu} K^\mu K^\nu = -\partial_v \Theta + \kappa \Theta - \frac{1}{2} \Theta^2 - \sigma_{AB}\sigma^{AB} & \text{(Raychaudhuri)} \\ R_{\mu\nu} K^\mu e^\nu_A = \partial_v \omega_A - \partial_A \kappa - \frac{1}{2} \partial_A \Theta + D_B \sigma_A{}^B + \Theta \omega_A & \end{cases}$$

Stationary BH (nothing depends on time)

NOT KERR!



— Linear combination $t^\alpha + \Omega_H \varphi^\alpha \equiv K^\alpha$ (Ω_H const)

is null on the EH.

Hawking (1972)

— must be either static

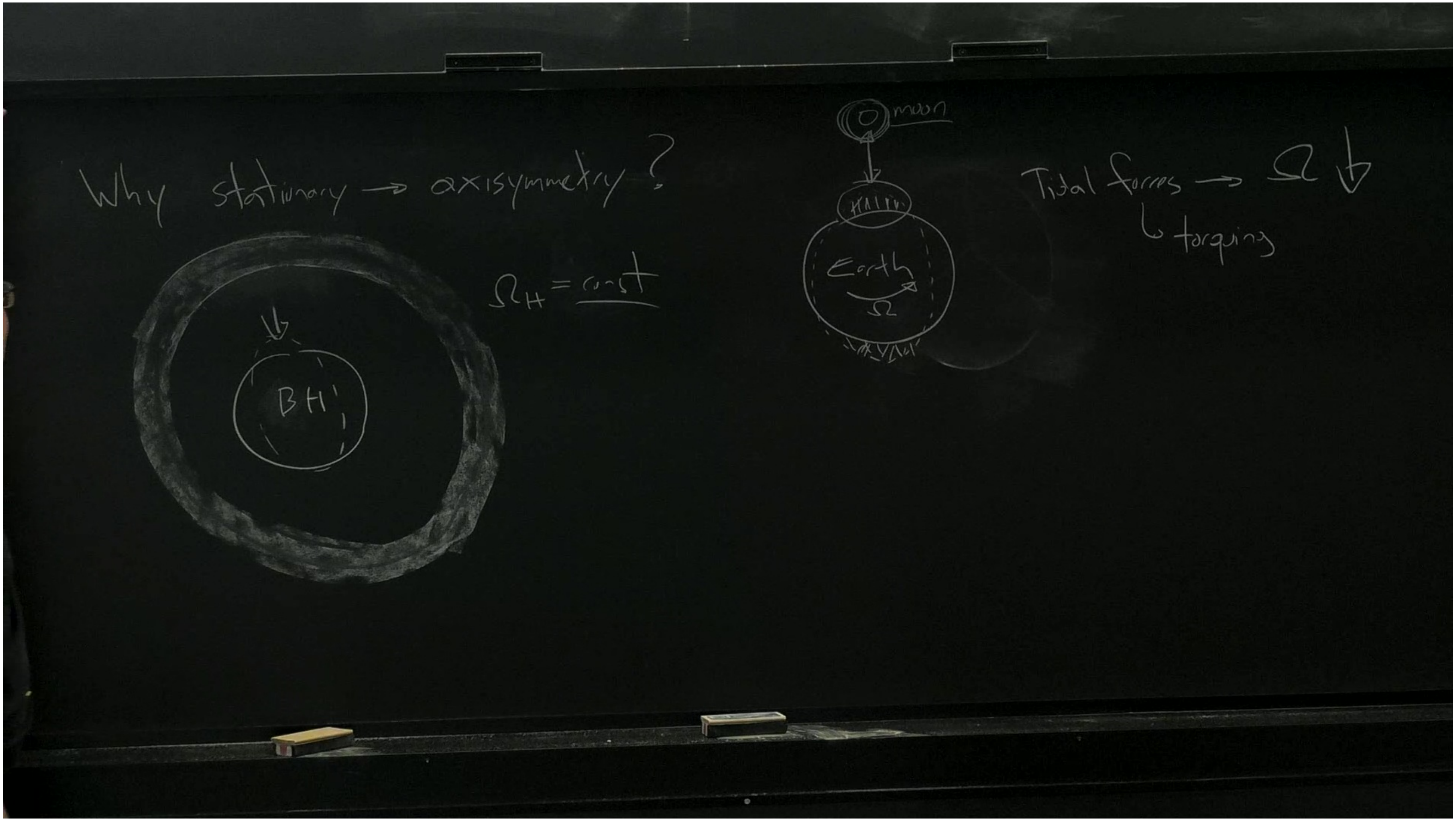
(t^α hypersurface orthogonal

$\rightarrow \omega_{\alpha\beta} = \nabla_{[\alpha} t_{\beta]} = 0$)

or axisymmetric

t^α not hypersurface orthogonal

φ^α axial killing vector.


Why stationary → axisymmetry ?
Ω_H = const
BH
moon
Earth
Ω
Tidal forces → Ω ↓
↳ torquing

$$K^\alpha = t^\alpha + \Omega_H \phi^\alpha = \text{null on event horizon}$$

Ω_H const

↳ tangent to null generators ; $\boxed{K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha}$

Stationary $\longrightarrow$ axisymmetry $\longrightarrow$ $K^\alpha = t^\alpha + \Omega_H \phi^\alpha$

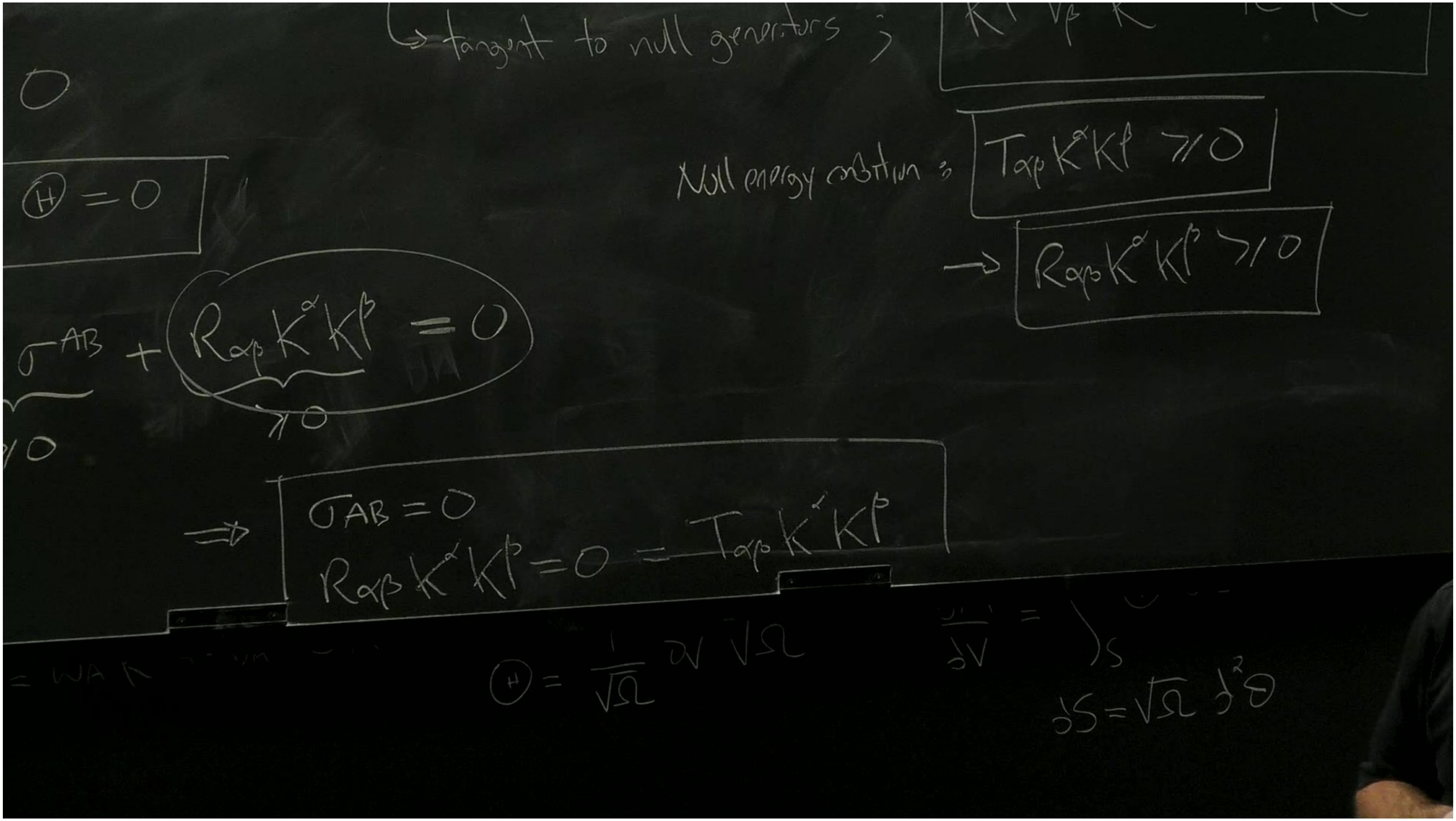
const

$\hookrightarrow$ tangent to null g

$\hookrightarrow$ ∂_v (anything) $= 0$

$\boxed{\Theta = 0}$ $\boxed{\partial_v \Theta = 0}$

Raychoudhuri : $\sigma_{AB}\sigma^{AB} + R_{\alpha\beta}K^\alpha K^\beta = 0$


tangent to null generators ;
Null energy condition :
$T_{\alpha\beta} K^\alpha K^\beta \geq 0$
$\rightarrow R_{\alpha\beta} K^\alpha K^\beta \geq 0$
$\Theta = 0$
$\sigma^{AB} + R_{\alpha\beta} K^\alpha K^\beta = 0$
≥ 0
≥ 0
$\Rightarrow \sigma_{AB} = 0$
$R_{\alpha\beta} K^\alpha K^\beta = 0 = T_{\alpha\beta} K^\alpha K^\beta$
$\Theta = \frac{1}{\sqrt{\Omega}} \partial_V \sqrt{\Omega}$
$dS = \sqrt{\Omega}\, d^2\theta$

$$\underbrace{R_{\alpha\beta}K^\alpha K^\beta}_{\geq 0}$$

$$\Rightarrow \boxed{\begin{array}{l}\sigma_{AB} = 0 \\ R_{\alpha\beta}K^\alpha K^\beta = 0 = T_{\alpha\beta}K^\alpha K^\beta\end{array}}$$

$e_A = \frac{\partial x}{\partial \theta^A}$, N_α

$N_\alpha N^\alpha = 0$, $N_\alpha K^\alpha = -1$, $N_\alpha e^\alpha_A = 0$

$\beta e^\alpha_A e^\beta_B$

$$B_{AB} = e^\alpha_A e^\beta_B \nabla_\alpha K_\beta = \frac{1}{2} \Theta \Omega_{AB} + \sigma_{AB}$$

$$= \partial_V \Omega_{AB}$$

$${}_A K^\alpha + B_A{}^B e^\alpha_A$$

$$\Theta = \frac{1}{\sqrt{\Omega}} \partial_V \sqrt{\Omega}$$

$$\frac{dA}{dV} = \int_S \Theta \, dS$$

$$dS = \sqrt{\Omega}\, d^2\theta$$