Title: Advanced General Relativity - 240320 (afternoon)

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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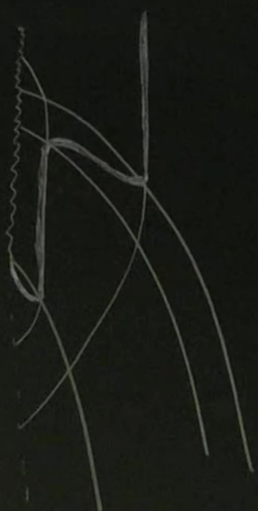


Vaidya spacetime (v, r, θ, φ)

$$ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$$
$$f = 1 - 2m(v)/r$$

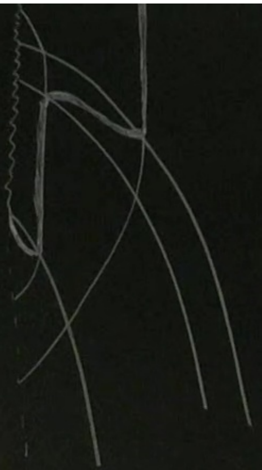
$$G_{vv} = 2\dot{m}/r^2$$
$$T_{vv} = \frac{\dot{m}}{4\pi r^2}$$

$$T_{\alpha\beta} = \frac{\dot{m}}{4\pi r^2}\, l_\alpha l_\beta$$
$$l_\alpha = -\nabla_\alpha v$$

null, tangent to ingoing light rays

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \cancel{p(g_{\alpha\beta} + u_\alpha u_\beta)}$$

pressureless null fluid



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→ null, tangent to ingoing light rays

energy conditions $\Rightarrow$ $\dot{m} > 0$

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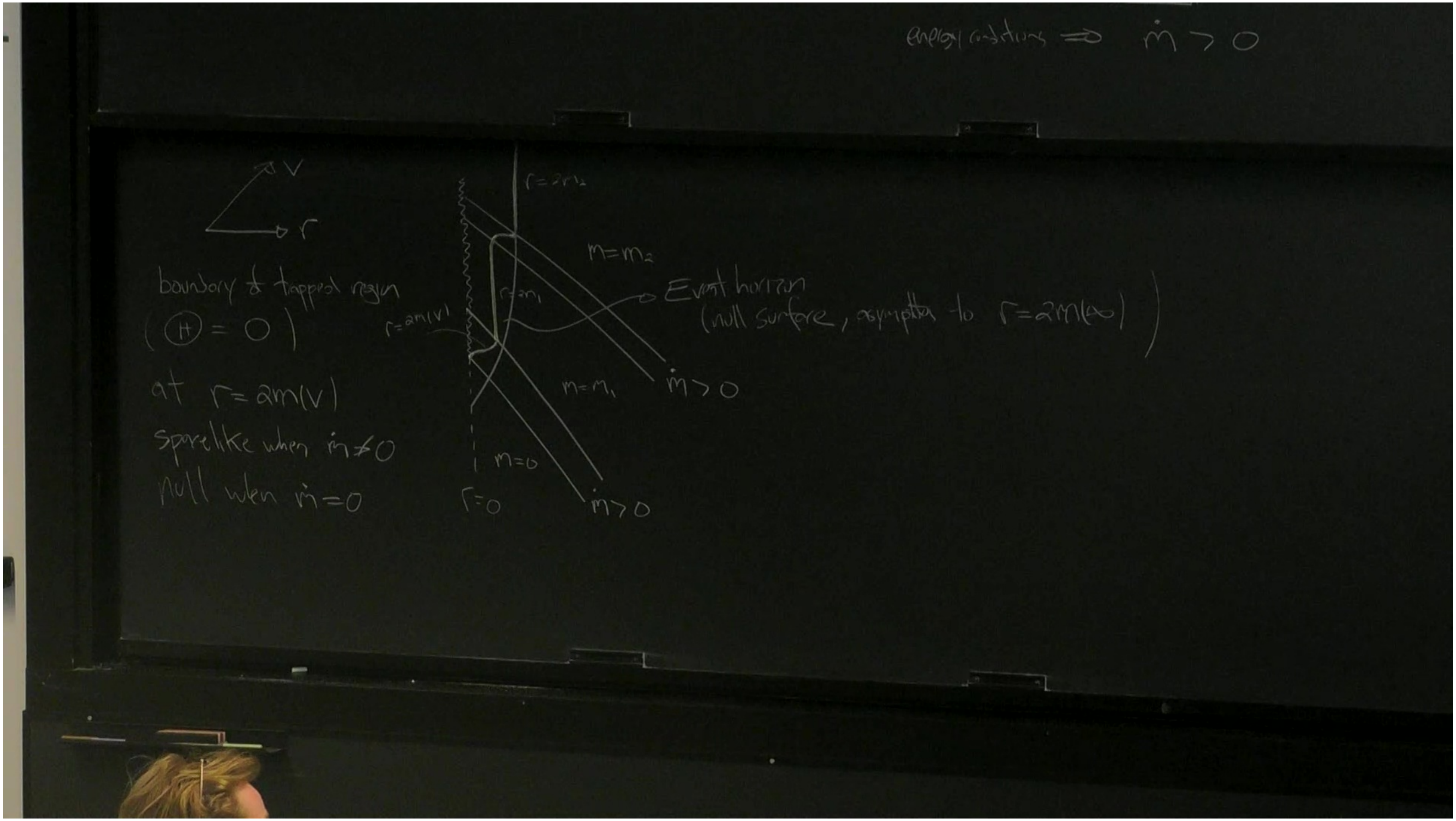
pressureless null fluid


energy conditions $\Rightarrow$ $\dot{m} > 0$
v
r
boundary of trapped region
$(\Theta = 0)$
at $r = 2m(v)$
spacelike when $\dot{m} \neq 0$
null when $\dot{m} = 0$
$r = 2m_2$
$m = m_2$
$r = 2m_1$
$r = 2m(v)$
Event horizon
(null surface, asymptotes to $r = 2m(\infty)$)
$m = m_1$
$\dot{m} > 0$
$m = 0$
$r = 0$
$\dot{m} > 0$

outgoing null geodesics : $K^\alpha = (1, \frac{1}{2} f, 0, 0)$

$$K_\alpha K^\alpha = 0$$

$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha \quad ; \quad \kappa = m(v)/r^2$$

$$\Theta = \nabla_\alpha K^\alpha - \kappa = \frac{r - 2m(v)}{r^2}$$

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$$\Theta = \nabla_\alpha K^\alpha - \kappa = \frac{r - 2m(v)}{r^2}$$

Surface $r = 2m(v) \longrightarrow \Phi = r - 2m(v) = 0$

normal vector: $\nabla_\alpha \Phi = (-2\dot{m}, 1, 0, 0)$

$$g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi = -4\dot{m} = \begin{cases} < 0 & \dot{m} > 0 \Rightarrow \text{normal is timelike} \Rightarrow \text{surface is sp} \\ = 0 & \dot{m} = 0 \Rightarrow \text{normal is null} \Rightarrow \text{surface is nul} \end{cases}$$

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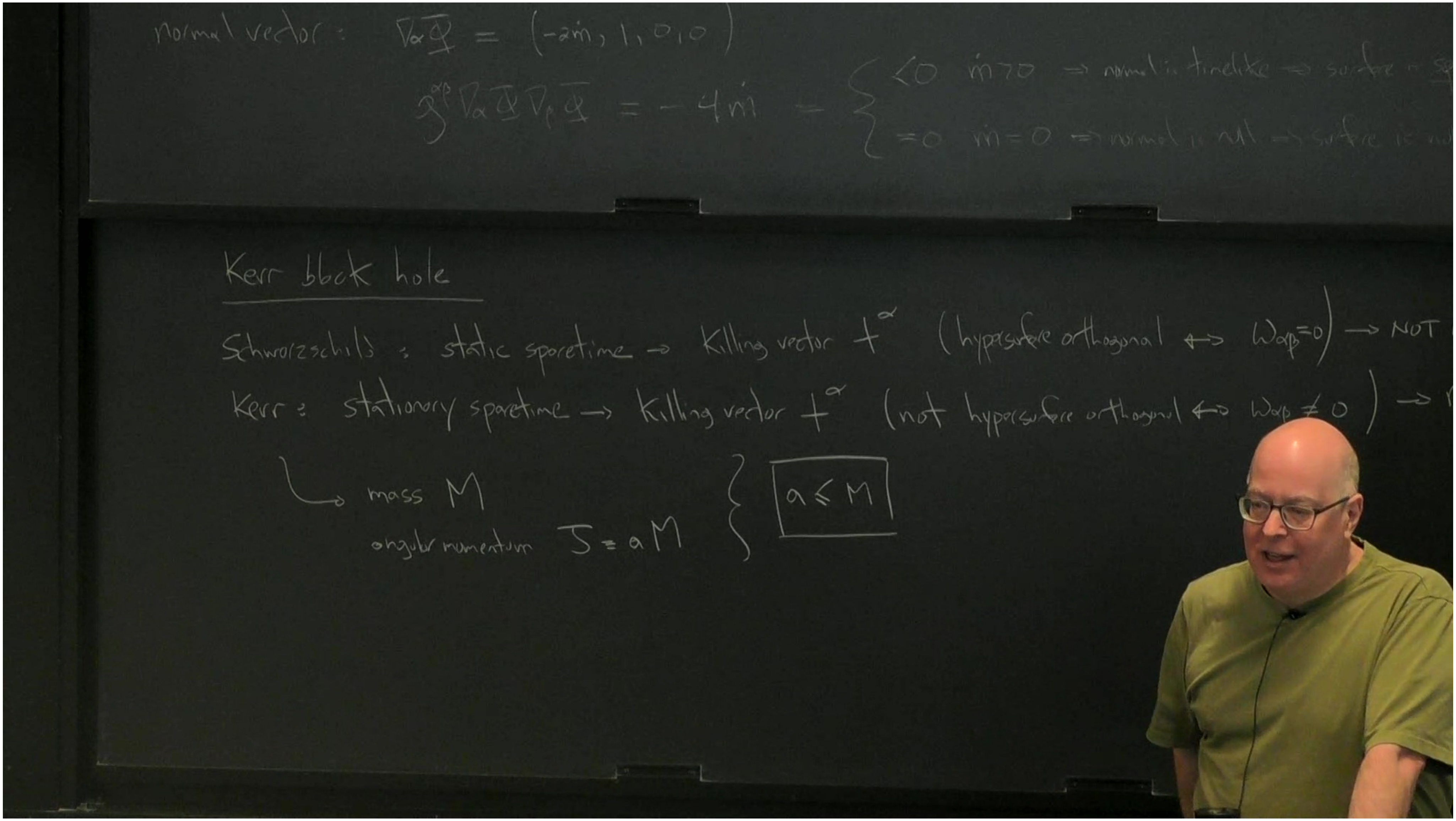
Kerr black hole

Schwarzschild: static spacetime $\rightarrow$ Killing vector t^α (hypersurface orthogonal $\leftrightarrow$ $\omega_{\alpha\beta} = 0$) $\rightarrow$ NOT

Kerr: stationary spacetime $\rightarrow$ Killing vector t^α (not hypersurface orthogonal $\leftrightarrow$ $\omega_{\alpha\beta} \neq 0$) $\rightarrow$


normal vector: $\nabla_\alpha \Phi = (-2\dot{m}, 1, 0, 0)$
$g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi = -4\dot{m}$
< 0 $\dot{m} > 0$ ⇒ normal is timelike ⇒ surface
$= 0$ $\dot{m} = 0$ ⇒ normal is null ⇒ surface is
Kerr black hole
Schwarzschild: static spacetime → Killing vector t^α (hypersurface orthogonal ↔ $\omega_{\alpha\beta} = 0$) → NOT
Kerr: stationary spacetime → Killing vector t^α (not hypersurface orthogonal ↔ $\omega_{\alpha\beta} \neq 0$) →
mass M
angular momentum $J = aM$
$\boxed{a \leq M}$

vector: $\nabla_\alpha \Phi = (-2\dot{m}, 1, 0, 0)$

$$g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi = -4\dot{m} = \begin{cases} <0 & \dot{m}>0 \Rightarrow \text{normal is timelike} \Rightarrow \text{surface is \underline{spacelike}} \\ =0 & \dot{m}=0 \Rightarrow \text{normal is null} \Rightarrow \text{surface is null} \end{cases}$$

Kerr black hole

Schwarzschild: static spacetime → Killing vector t^α (hypersurface orthogonal ⟺ $\omega_{\alpha\beta}=0$) → NOT ROTATING

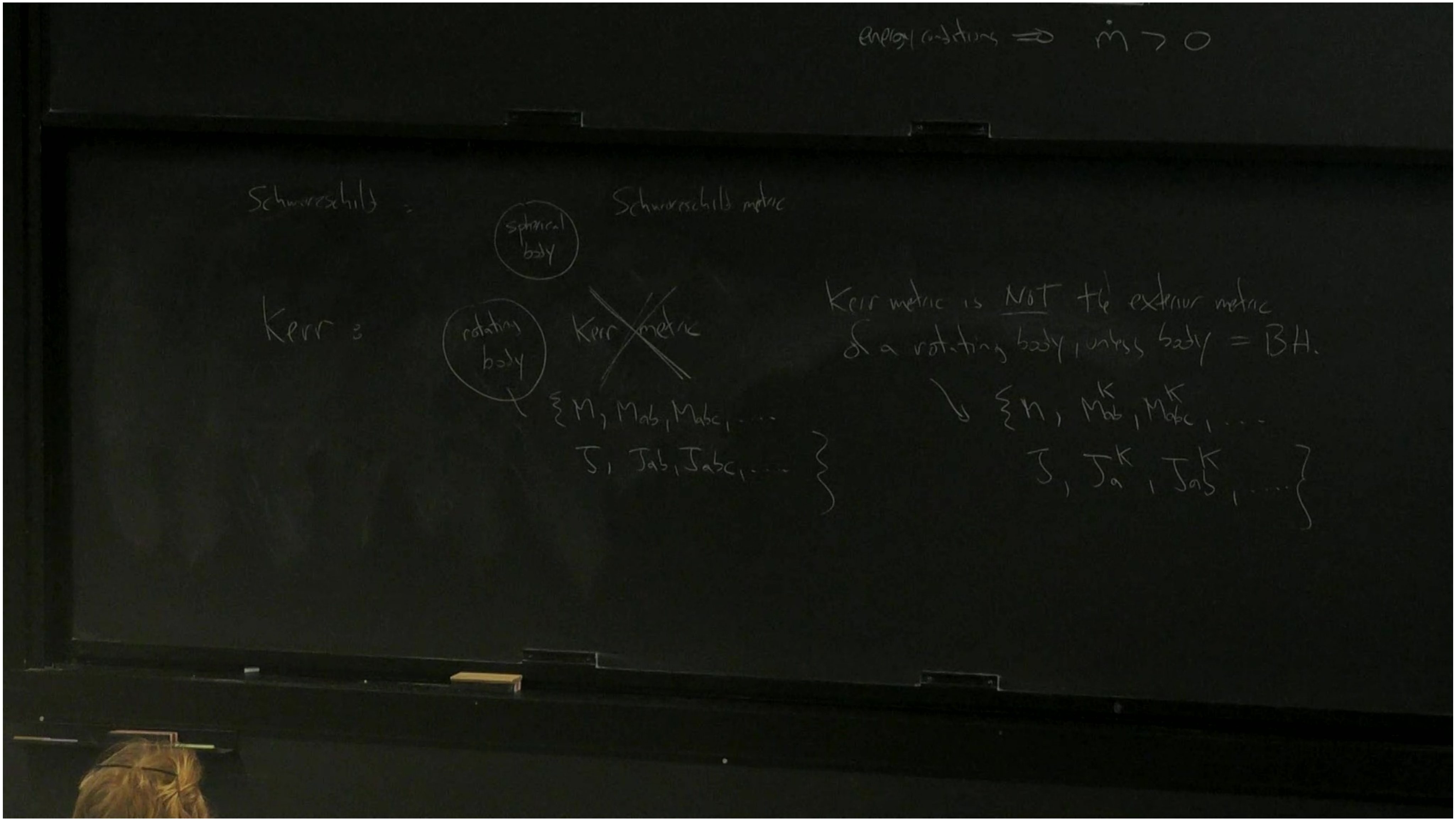
Kerr: stationary spacetime → Killing vector t^α (not hypersurface orthogonal ⟺ $\omega_{\alpha\beta} \neq 0$) → ROTATING

↳ mass M

angular momentum $J = aM$ $\Big\{ \boxed{a \leq M}$

Kerr metric is a solution to EFE in vacuum

uniqueness thm (Carter, ...): Kerr metric is unique sln to vacuum EFE for asymptotically flat spacetime, with a nonsingular event horizon.
(stationary)


energy conditions ⇒ ṁ > 0
Schwarzschild :
spherical body
Schwarzschild metric
Kerr :
rotating body
Kerr metric
{M, Mab, Mabc, ...
J, Jab, Jabc, ... }
Kerr metric is NOT the exterior metric
of a rotating body, unless body = BH.
{m, MKab, MKabc, ...
J, JKa, JKab, ... }

energy conditions $\Rightarrow$ $\dot{m} > 0$

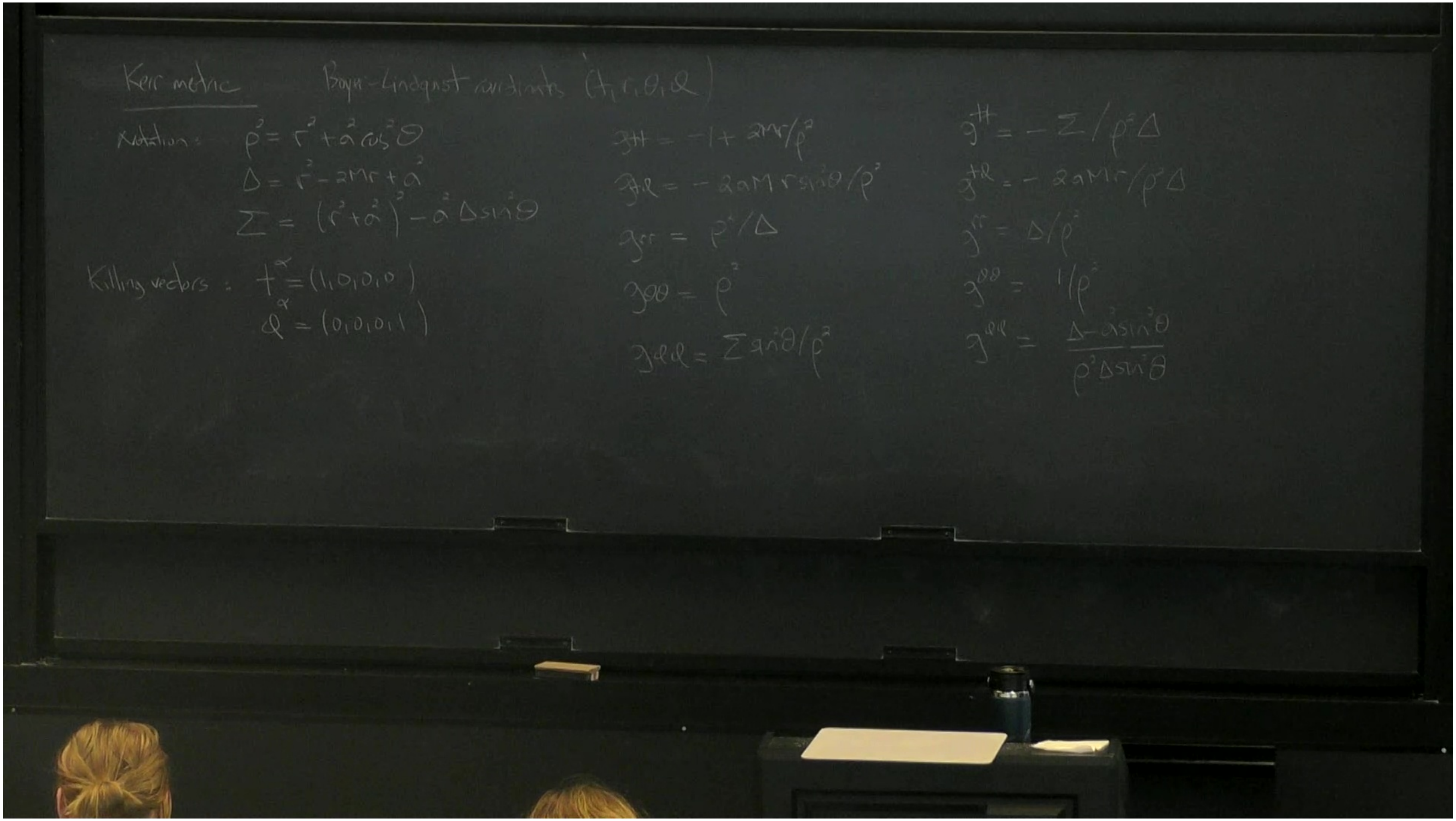
Schwarzschild = spherical body Schwarzschild metric

Kerr = rotating body ~~Kerr metric~~

$\{M, M_{ab}, M_{abc}, \ldots$
$J, J_{ab}, J_{abc}, \ldots\}$

Kerr metric is NOT the exterior metric of a rotating body, unless body = BH.

$\{M, M^K_{ab}, M^K_{abc}, \ldots$
$J, J^K_a, J^K_{ab}, \ldots\}$


Kerr metric
Boyer-Lindquist coordinates (t, r, θ, ϕ)
Notation: $\rho^2 = r^2 + a^2\cos^2\theta$
$\Delta = r^2 - 2Mr + a^2$
$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$
Killing vectors: $t^\alpha = (1,0,0,0)$
$\phi^\alpha = (0,0,0,1)$
$g_{tt} = -1 + 2Mr/\rho^2$
$g_{t\phi} = -2aMr\sin^2\theta/\rho^2$
$g_{rr} = \rho^2/\Delta$
$g_{\theta\theta} = \rho^2$
$g_{\phi\phi} = \Sigma\sin^2\theta/\rho^2$
$g^{tt} = -\Sigma/\rho^2\Delta$
$g^{t\phi} = -2aMr/\rho^2\Delta$
$g^{rr} = \Delta/\rho^2$
$g^{\theta\theta} = 1/\rho^2$
$g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}$

Observers #1 — ZAMOs (Zero angular momentum observers)

$$0 = \tilde{L} = U_\alpha \phi^\alpha = U_\phi = \text{const of motion.}$$

$$0 = U_\phi = g_{\phi\alpha} U^\alpha = g_{t\phi} U^t + g_{\phi\phi} U^\phi \Rightarrow \frac{U^\phi}{U^t} = -\frac{g_{t\phi}}{g_{\phi\phi}}$$

$$\frac{U^\phi}{U^t} = \frac{\dot\phi}{\dot t} = \frac{d\phi}{dt}$$

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \omega = \frac{2aM\,r\,\cancel{\sin^2\theta}}{\cancel{\rho^2}\,\Sigma\,\cancel{\sin^2\theta}}$$

$$\boxed{\frac{d\phi}{dt} = \omega = \frac{2aMr}{\Sigma}} \sim \frac{2J}{r^3}$$

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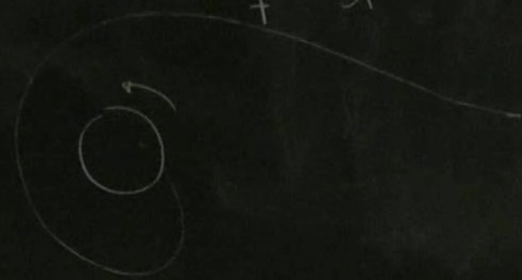
$$\frac{\dot\phi}{\dot t} = \frac{d\phi}{dt}$$

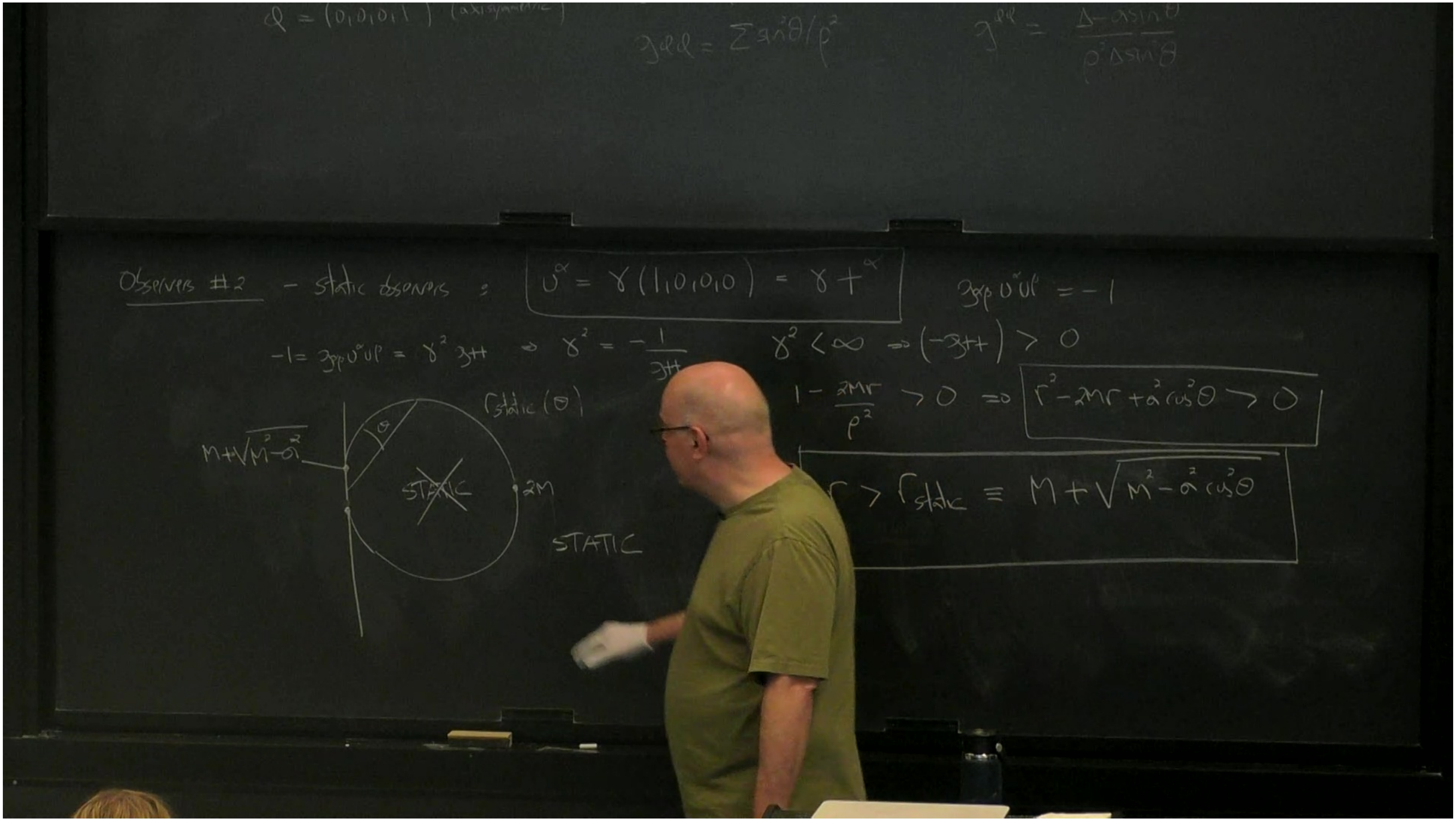
$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \omega = \frac{2aMr\cancel{\sin^2\theta}}{\Sigma\cancel{\sin^2\theta}}$$



$$\boxed{\frac{d\phi}{dt} = \omega = \frac{2aMr}{\Sigma}} \sim \frac{2J}{r^3}$$

"Lense-Thirring"

"dragging of inertial frames"


Observers #2 - static observers :
$u^\alpha = \gamma(1,0,0,0) = \gamma t^\alpha$
$g_{\alpha\beta} u^\alpha u^\beta = -1$
$-1 = g_{\alpha\beta} u^\alpha u^\beta = \gamma^2 g_{tt} \Rightarrow \gamma^2 = -\frac{1}{g_{tt}}$
$\gamma^2 < \infty \Rightarrow (-g_{tt}) > 0$
$1 - \frac{2Mr}{\rho^2} > 0 \Rightarrow r^2 - 2Mr + a^2\cos^2\theta > 0$
$r > r_{static} \equiv M + \sqrt{M^2 - a^2\cos^2\theta}$
$r_{static}(\theta)$
$M+\sqrt{M^2-a^2}$
θ
STATIC
2M
STATIC