Title: Advanced General Relativity - 240313 (afternoon)

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

Date: March 13, 2024 - 1:30 PM

URL: https://pirsa.org/24030005

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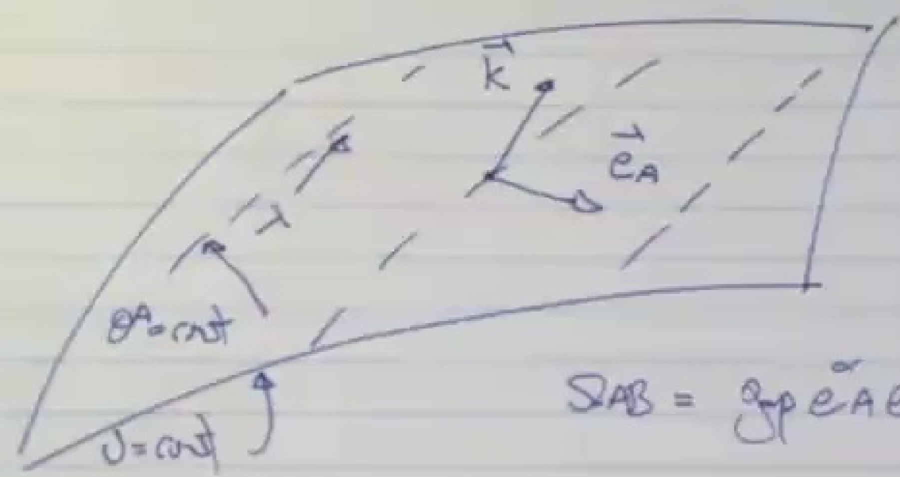

$$k^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A}$$

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda$$

$$\Omega_{AB} = g_{\alpha\beta}\, e^\alpha_A e^\beta_B = \text{transverse metric.}$$

$$ds^2 = -e^{\chi} du\,(V\,du + 2\,d\lambda) + \Omega_{AB}(d\theta^A + W^A du)(d\theta^B + W^B du)$$

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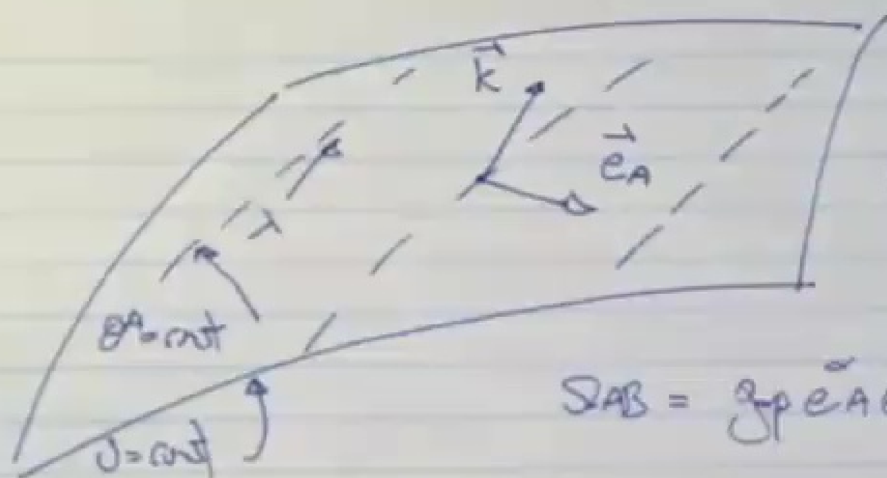

$$\left\{ \begin{aligned} k^\alpha &= \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} \\ e^\alpha_A &= \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda \end{aligned} \right.$$

$\Omega_{AB} = g_{\alpha\beta}\, e^\alpha_A e^\beta_B$ = transverse metric.

$$ds^2 = -e^{\chi} du\,(V du + 2\, d\lambda) + \Omega_{AB}(d\theta^A + w^A du)(d\theta^B + w^B du)$$

Lie identity : $\mathcal{L}_k e^\alpha_A = 0$

$$\boxed{k^\beta \nabla_\beta e^\alpha_A = e^\beta_A \nabla_\beta k^\alpha}$$

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Transport equations

$$\boxed{\begin{aligned} K^\beta \nabla_\beta K^\alpha &= \kappa K^\alpha \\ K^\beta \nabla_\beta e^\alpha_A &= \omega_A K^\alpha + B_A{}^B e^\gamma_B \end{aligned}}$$

no component along N

$$K_\alpha K^\beta \nabla_\beta e^\alpha_A = K_\alpha (e^\beta_A \nabla_\beta K^\alpha) = \tfrac{1}{2} e^\beta_A \nabla_\beta (\underbrace{K_\gamma K^\gamma}_{0}) = 0$$

$$\kappa = \partial_\lambda \chi$$

$$\omega_A = \tfrac{1}{2} (\partial_A \chi - e^\chi \Omega_{AB} \partial_\lambda W^B)$$

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Transport equations

$$\boxed{\begin{aligned} K^\beta \nabla_\beta K^\alpha &= \kappa K^\alpha \\ K^\beta \nabla_\beta e^\alpha_A &= \omega_A K^\alpha + B_A{}^B e^\alpha_B \end{aligned}}$$

→ no component along N^α

$$K_\alpha K^\beta \nabla_\beta e^\alpha_A = K_\alpha (e^\beta_A \nabla_\beta K^\alpha) = \tfrac{1}{2} e^\beta_A \nabla_\beta (\underbrace{K_\alpha K^\alpha}_{0}) = 0$$

$$\kappa = \partial_\lambda \chi$$

$$\omega_A = \tfrac{1}{2} (\partial_A \chi - e^\chi \Omega_{AB} \partial_\lambda w^B)$$

$$\boxed{B_{AB} = \tfrac{1}{2} \partial_\lambda \Omega_{AB}}$$

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Transport equations

$$\boxed{\begin{aligned} K^\beta \nabla_\beta K^\alpha &= \kappa K^\alpha \\ K^\beta \nabla_\beta e^\alpha_A &= \omega_A K^\alpha + B_A{}^B e^\alpha_B \end{aligned}}$$

no transport along N^α

$$K_\alpha K^\beta \nabla_\beta e^\alpha_A = K_\alpha (e^\beta_A \nabla_\beta K^\alpha) = \frac{1}{2} e^\beta_A \nabla_\beta (\underbrace{K_\alpha K^\alpha}_{0}) = 0$$

$$\kappa = \partial_\lambda \chi$$

$$\omega_A = \frac{1}{2} (\partial_A \chi - e^\chi \Omega_{AB} \partial_\lambda w^B)$$

$$\boxed{B_{AB} = \frac{1}{2} \partial_\lambda \Omega_{AB}}$$

$$\boxed{B_{AB} = e^\alpha_A e^\beta_B \nabla_\alpha K_\beta}$$

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Transport equations

$$\boxed{\begin{aligned} K^\beta \nabla_\beta K^\alpha &= k\, K^\alpha \\ K^\beta \nabla_\beta e^\alpha_A &= W_A K^\alpha + B_A{}^B e^\alpha_B \end{aligned}}$$

↳ no transport along N

$$K_\alpha K^\beta \nabla_\beta e^\alpha_A = K_\alpha \left(e^\beta_A \nabla_\beta K^\alpha \right) = \tfrac{1}{2} e^\beta_A \nabla_\beta \big(\underbrace{K_\alpha K^\alpha}_{0} \big) = 0$$

$$k = \partial_\lambda \chi$$

$$W_A = \tfrac{1}{2} \left(\partial_A \chi - e^{\chi} \Omega_{AB}\, \partial_\lambda W^B \right)$$

$$\boxed{B_{AB} = \tfrac{1}{2} \partial_\lambda \Omega_{AB}}$$

$$\boxed{B_{AB} = e^\alpha_A e^\beta_B \nabla_\alpha K_\beta} = \text{purely transverse part of } \nabla_\alpha K_\beta$$

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$$W_A = \frac{1}{2}\left(\partial_A \chi - e^{\chi}\,\Omega_{AB}\,\partial_\lambda W^B\right)$$

$$\boxed{B_{AB} = \frac{1}{2}\,\partial_\lambda \Omega_{AB}}$$

$$\boxed{B_{AB} = e^\alpha_A\, e^\beta_B\, \nabla_\alpha K_\beta} = \text{purely transverse part of } \nabla_\alpha K_\beta$$

$$= e^\beta_B\left(K\,\nabla_\alpha\, e_{A\beta}\right)$$

$$= e^\beta_B\left(\omega_A K_\beta + B_A{}^C e_{C\beta}\right)$$

$$= B_A{}^C\,(\underbrace{e_B{}^\beta e_{C\beta}}_{\Omega_{BC}})$$

$$= B_{AB}$$

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B_{AB} $\quad e^{\alpha}_{A}\, e^{\beta}_{B} \nabla_\alpha k_\beta = e^{\beta}_{B} \left(e^{\alpha}_{A} \nabla_\alpha k_\beta \right)$

$$= e^{\beta}_{B} \left(k^{\alpha} \nabla_\alpha e_{A\beta} \right)$$

$$= e^{\beta}_{B} \left(\omega_A k_\beta + B_A{}^{C} e_{C\beta} \right)$$

$$= B_A{}^{C} \left(\underbrace{e_B^{\beta} e_{C\beta}}_{\Omega_{BC}} \right)$$

$$= B_{AB}$$

Decomposition:

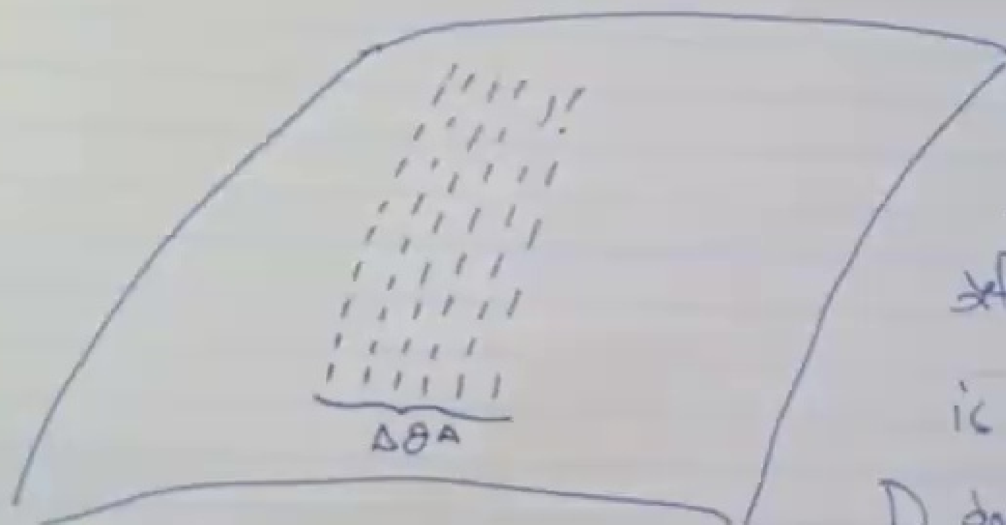
$$\boxed{B_{AB} = \frac{1}{2} \Omega_{AB} \underbrace{\Theta}_{\text{rate of expansion}} + \underbrace{\sigma_{AB}}_{\text{rate of shear}}}$$

$\Theta = \Omega^{AB} B_{AB}$, $\quad \sigma_{AB}\,\Omega^{AB} = 0$

$$\sigma_{AB} = B_{AB} - \frac{1}{2} \Theta\, \Omega_{AB}$$

$$\sigma_{AB} = B_{AB} - \frac{1}{2}\Theta\,\Omega_{AB}$$

Evolution of cross sections :



Domain D of the hypersurface, defined by a spread $\Delta\theta^A$ in generator labels.

D doesn't change with λ.

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Rate of change of cross-sectional area:

$$\underbrace{\text{cross-sectional area}}_{\text{to}} \quad A = \int_D dS$$

$$= \int_D \sqrt{\Omega}\, d^2\theta$$

$$\frac{dA}{d\lambda} = \ ?$$

$$\frac{dA}{d\lambda} = \frac{d}{d\lambda}\int_D \sqrt{\Omega}\, d^2\theta = \int_D \partial_\lambda \sqrt{\Omega}\, d^2\theta$$

(D is λ-independent).

Matrix M $\begin{pmatrix} \cdots & & \cdot \\ \cdot & & \\ & \cdot & \cdot \\ & \cdot & \cdot\ \cdot \end{pmatrix} \longrightarrow \det(M) = \begin{vmatrix} \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot \end{vmatrix}$

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Matrix M $\qquad \begin{pmatrix} \cdots & & \cdot \\ \cdot & & \\ & \cdot & \cdot \\ & \cdot & \cdot \ \cdot \end{pmatrix} \longrightarrow \det(M) = \begin{vmatrix} \cdots & \cdots \\ \cdot & \cdot\,\cdot \\ \cdot & \cdot\,\cdot \end{vmatrix}$

$$\partial_\lambda (\det M) = (\det M)\, \mathrm{Tr}\left(M^{-1} \partial_\lambda M \right)$$

$$\partial_\lambda \Omega = \Omega\, \Omega^{AB} \partial_\lambda \Omega_{AB}$$

$$\partial_\lambda \sqrt{\Omega} = \frac{1}{2} \frac{1}{\sqrt{\Omega}} \partial_\lambda \Omega = \frac{1}{2} \sqrt{\Omega}\, \Omega^{AB} \partial_\lambda \Omega_{AB}$$

$$\frac{\partial_\lambda \sqrt{\Omega}}{\sqrt{\Omega}} = \frac{1}{2} \Omega^{AB} \partial_\lambda \Omega_{AB}$$

$$= \Omega^{AB} B_{AB}$$

$$\frac{\partial A}{\partial \lambda} = \int_D \partial\lambda \sqrt{\sigma}\, d^2\theta$$

$$= \int_D \Theta \sqrt{\sigma}\, d^2\theta$$

$$= \int_S \Theta\, dS$$

$$\boxed{\frac{\partial A}{\partial \lambda} = \int_S \Theta\, dS}$$

Θ → fractional rate of change of cross-sectional area.

$$\Theta = \frac{1}{\delta S} \frac{d}{d\lambda} \delta S$$

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$$\Theta = \frac{1}{\partial S}\frac{\partial}{\partial \lambda}\,\delta S$$

Because λ is not affine, $\Theta \neq \nabla_\alpha K^\alpha$

$$\nabla_\alpha K^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\,K^\alpha\right)$$

$$\stackrel{*}{=} \frac{1}{e^{\chi}\sqrt{\Omega}}\partial_\lambda\left(e^{\chi}\sqrt{\Omega}\right)$$

$$\stackrel{*}{=} \underbrace{\partial_\lambda \chi}_{\kappa} + \underbrace{\frac{1}{\sqrt{\Omega}}\partial_\lambda\sqrt{\Omega}}_{\Theta}$$

$$\boxed{\Theta = \nabla_\alpha K^\alpha - \kappa}$$

Quick Camera: [IPEVO V4K]

Gauss- Codazzi eqtns

$^4R_{\alpha\beta\gamma\delta} \longrightarrow$ geometrical quantities on null hypersurface.

eg) $^4R_{\mu\nu\lambda\alpha} K^\mu N^\nu K^\lambda e^\alpha_A$

$$= \partial_\lambda \omega_A - D_A \kappa + B_A{}^B \omega_B$$

⋮

$$^4R_{\mu\nu} K^\mu K^\nu = -\partial_\lambda \Theta + \kappa \Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}$$

~~Raychaudhuri's eqn~~ !

Raychaudhuri's eqtn.

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eg) $^4R_{\mu\nu\lambda\alpha} K^\mu N^\nu K^\lambda e^\alpha_A$

$= \partial_\lambda \omega_A - D_A \kappa + B_A{}^B \omega_B$

$$\boxed{^4R_{\mu\nu} K^\mu K^\nu = -\partial_\lambda \Theta + \kappa \Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}}$$

~~Raychdri's eqn~~

Raychowdhuri's eqtn.

$$^4R_{\mu\alpha} K^\mu e^\alpha_A = \partial_\lambda \omega_A - D_A \kappa - \frac{1}{2} D_A \Theta + D_B \sigma_A{}^B$$

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eg) ${}^4R_{\mu\nu\lambda\alpha} K^\mu N^\nu K^\lambda e_A$

$$= \partial_\lambda \omega_A - \partial_A \kappa + B_A{}^B \omega_B$$

$$\boxed{{}^4R_{\mu\nu} K^\mu K^\nu = -\partial_\lambda \Theta + \kappa \Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}}$$

~~Raychduri's eqn~~

Raychudhuri's eqtn.

$$\boxed{{}^4R_{\mu\alpha} K^\mu e^\alpha_A = \partial_\lambda \omega_A - \partial_A \kappa - \frac{1}{2} D_A \Theta + D_B \sigma_A{}^B + \Theta \omega_A}$$

$D_B \equiv$ covariant derivative defined wrt Ω_{AB}

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BLACK HOLES

BH in cartoons

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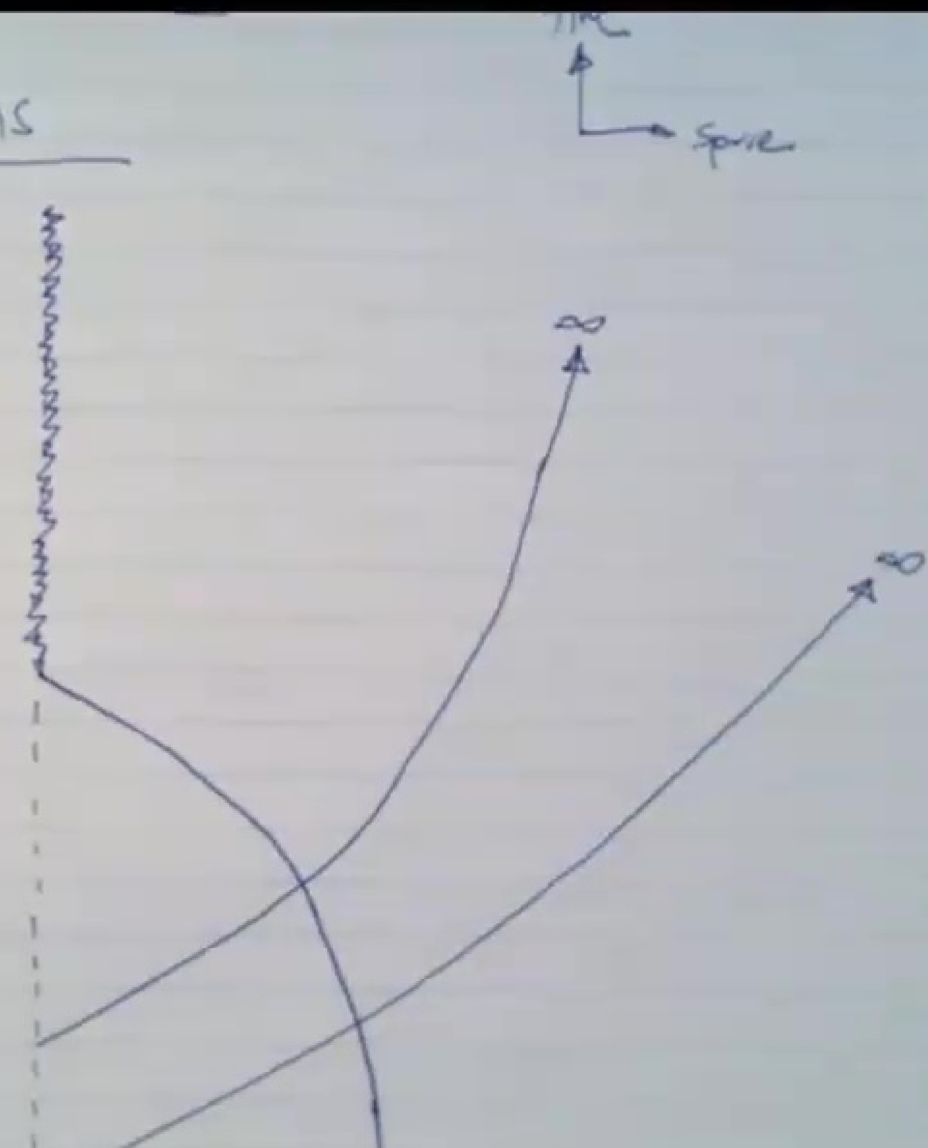
BH in cartoons



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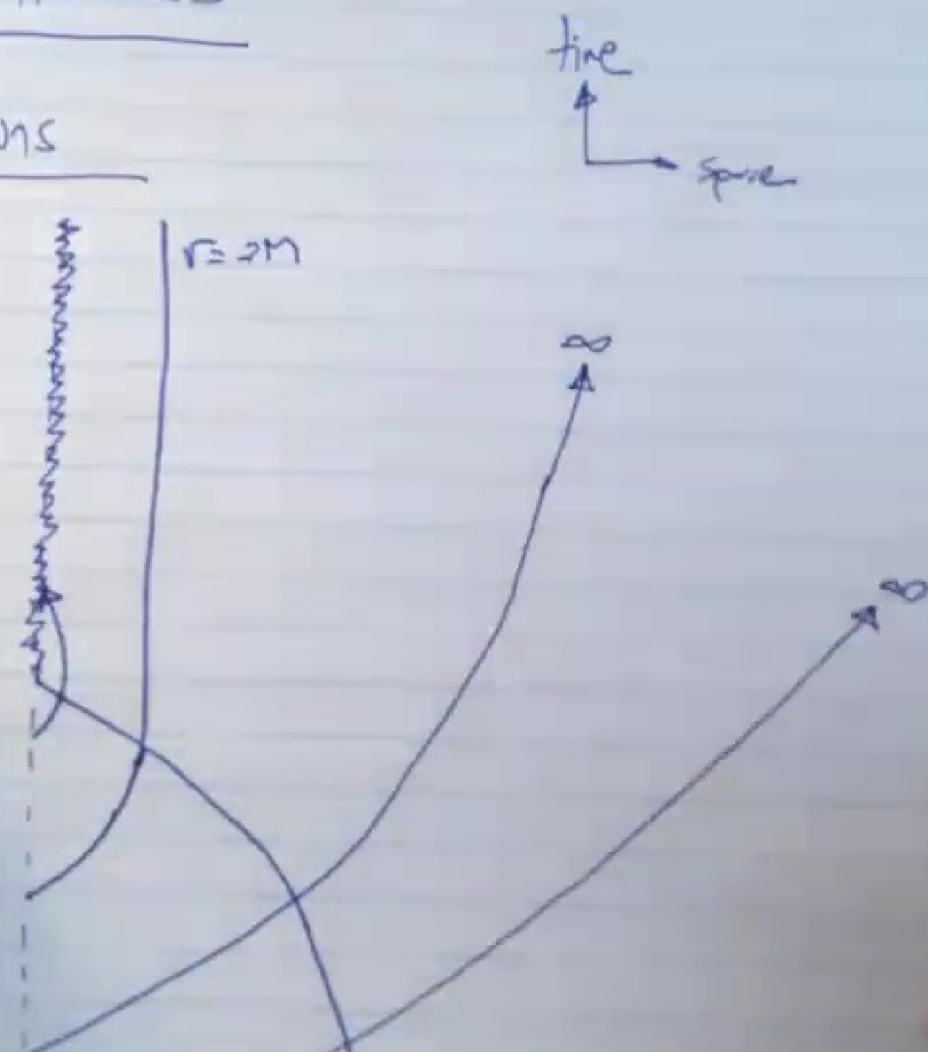
BH in cartoons



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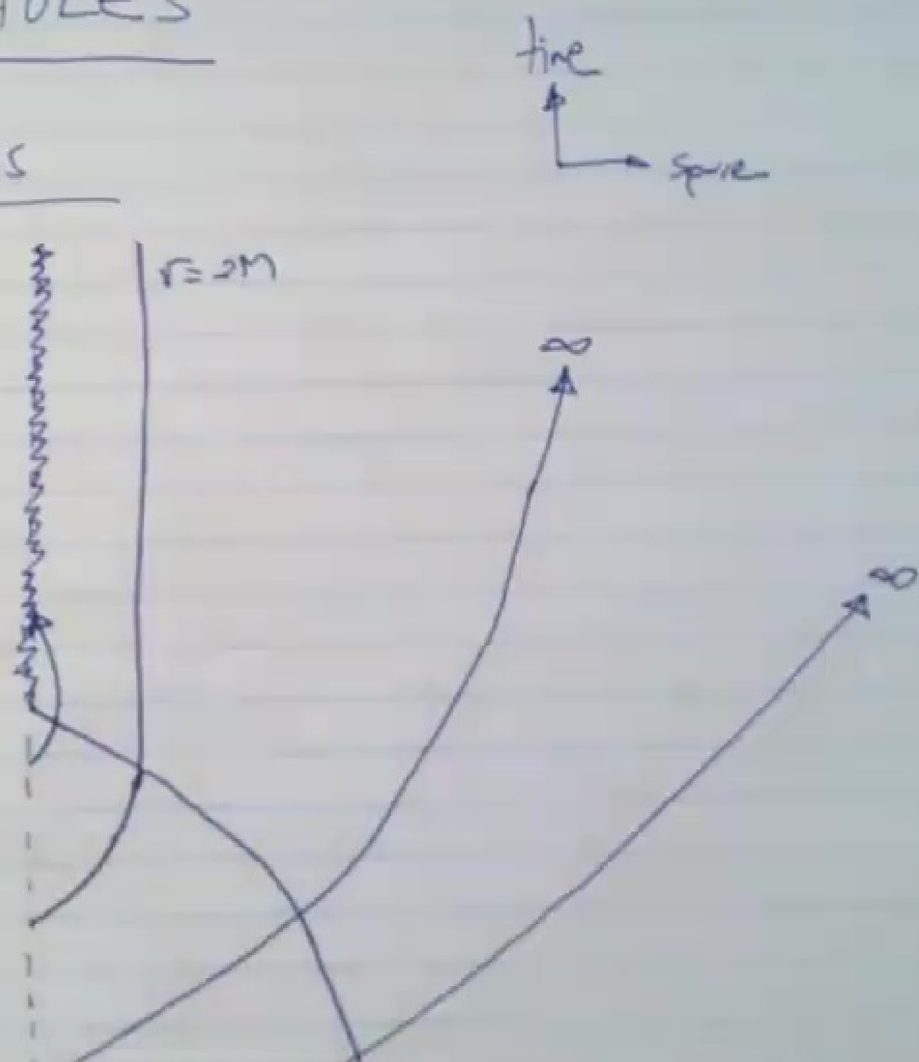

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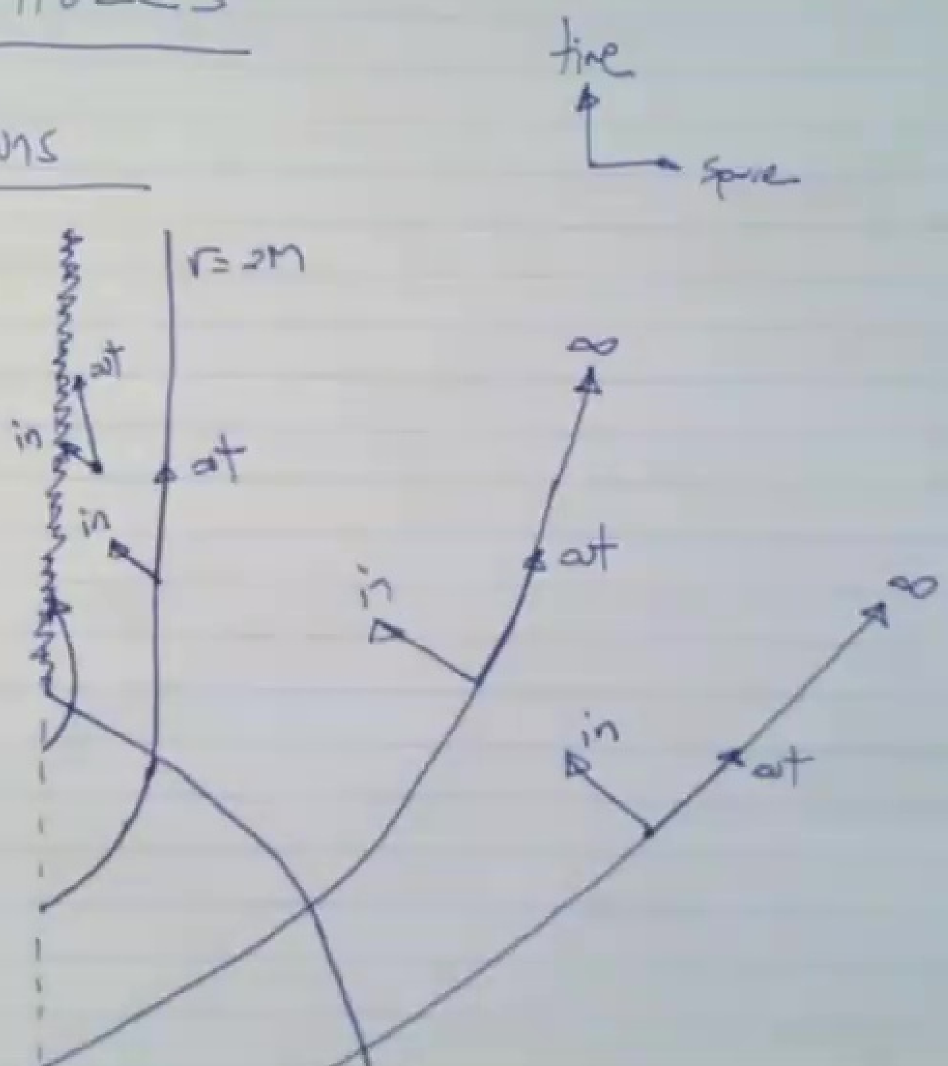

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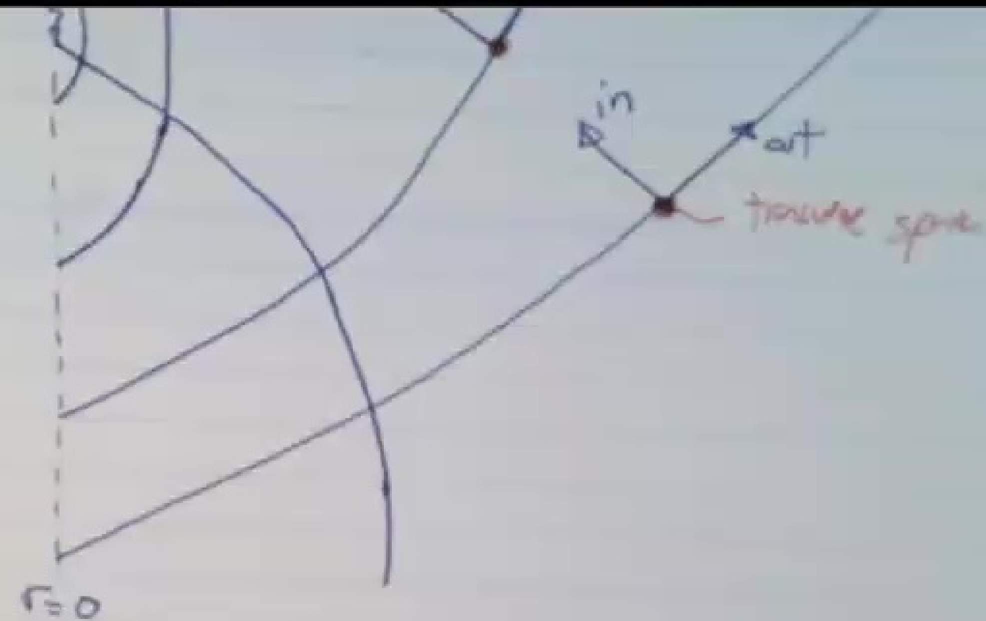
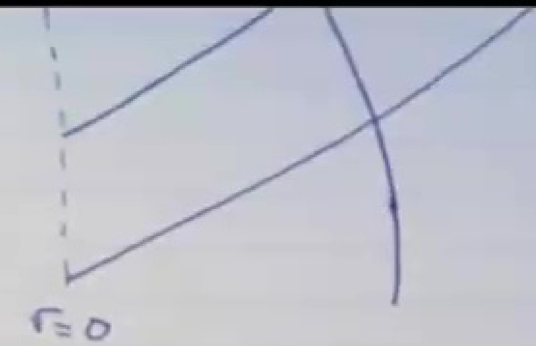


Trapped sphere — two null normal directions.

$\Theta_{out} = \begin{cases} +ve \text{ outside BH} \\ -ve \text{ inside BH} \end{cases}$

$\Theta_{in} < 0$



Transverse sphere — two null normal directions.

$$\Theta_{out} = \begin{cases} +ve \text{ outside BH} \\ -ve \text{ inside BH} \end{cases}$$

$$\Theta_{in} < 0$$

<u>Trapped surface</u>

A 2D, close surface with converging ($\Theta < 0$) null geodesics in the outgoing ~~and~~ normal direction.

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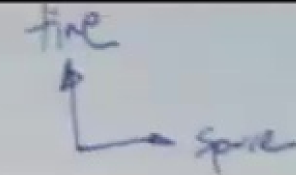
Marginally trapped surface

$\boxed{\text{A 2D, closed surface with stationary } (\Theta = 0) \text{ null geodesics in outgoing normal direction.}}$

→ boundary of trapped region

→ "apparent horizon".

BLACK HOLES



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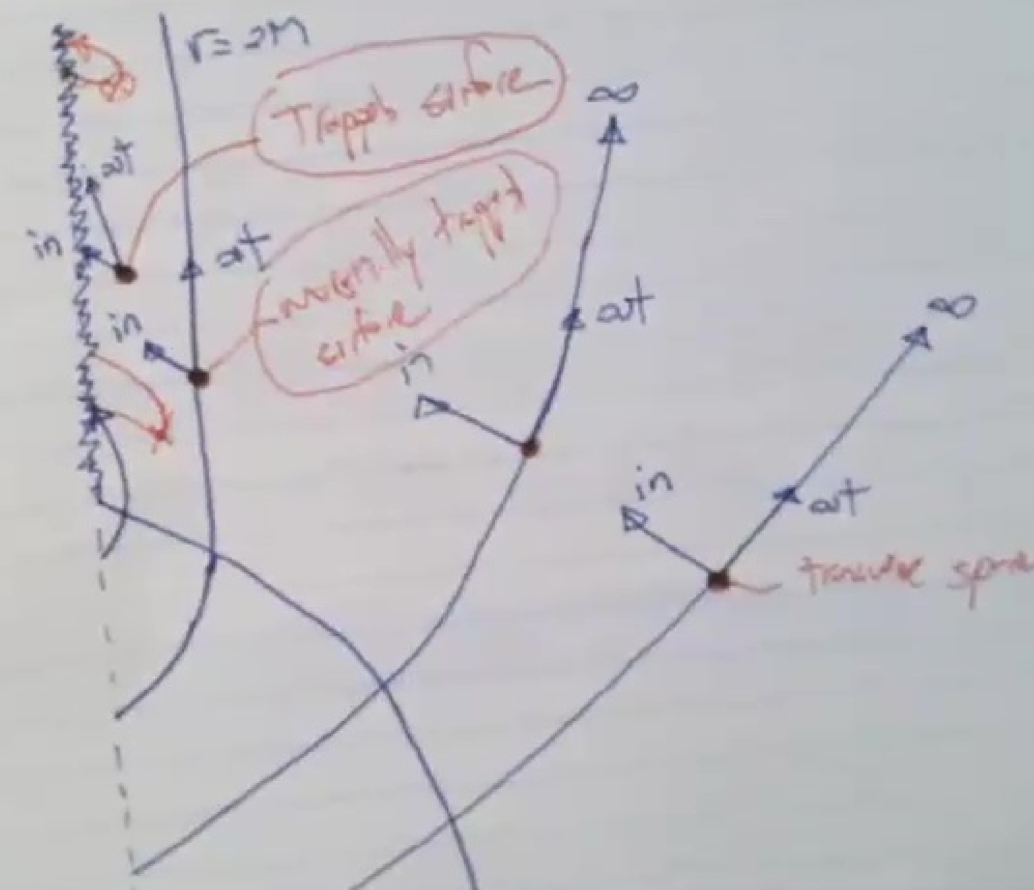

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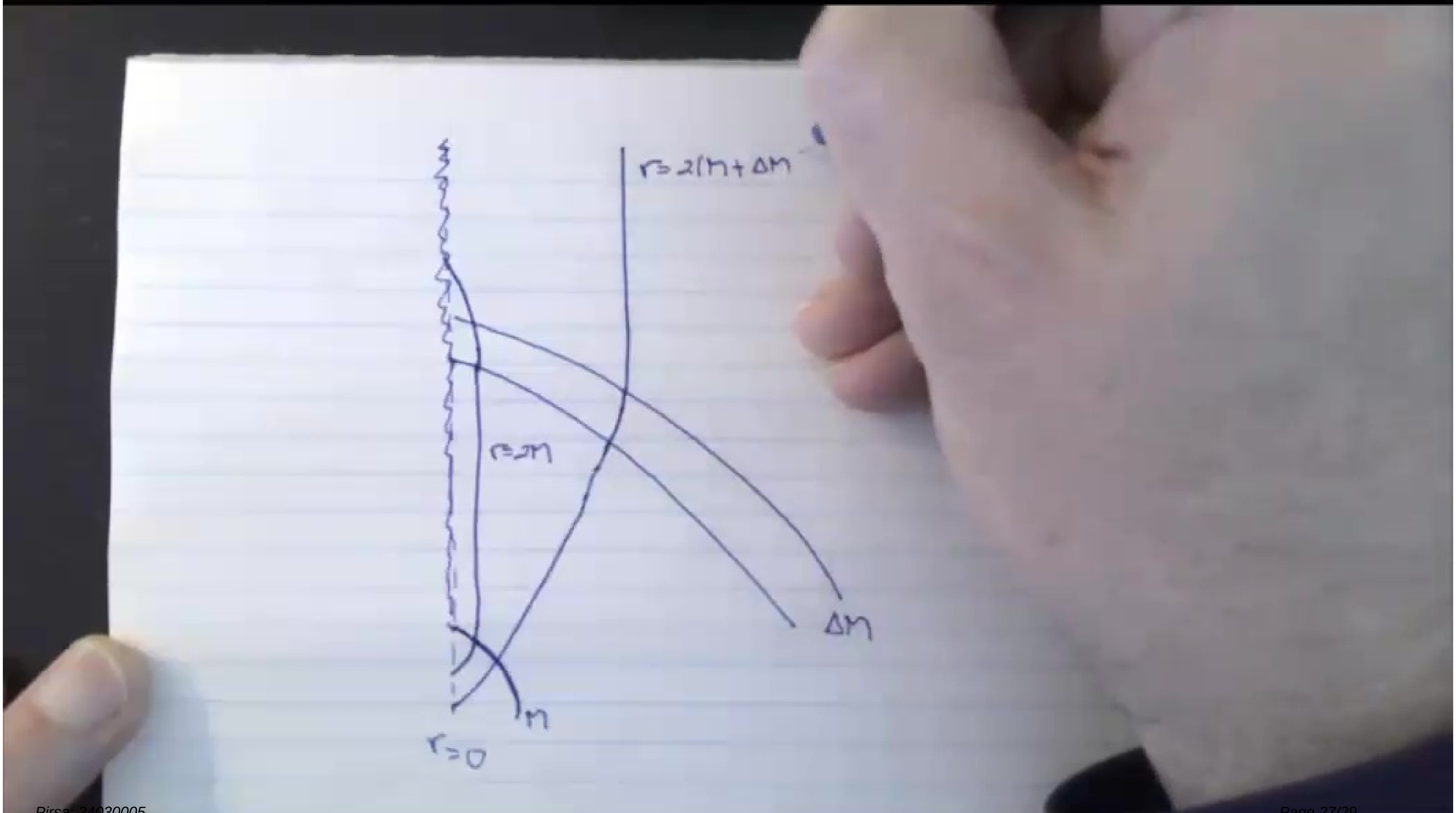

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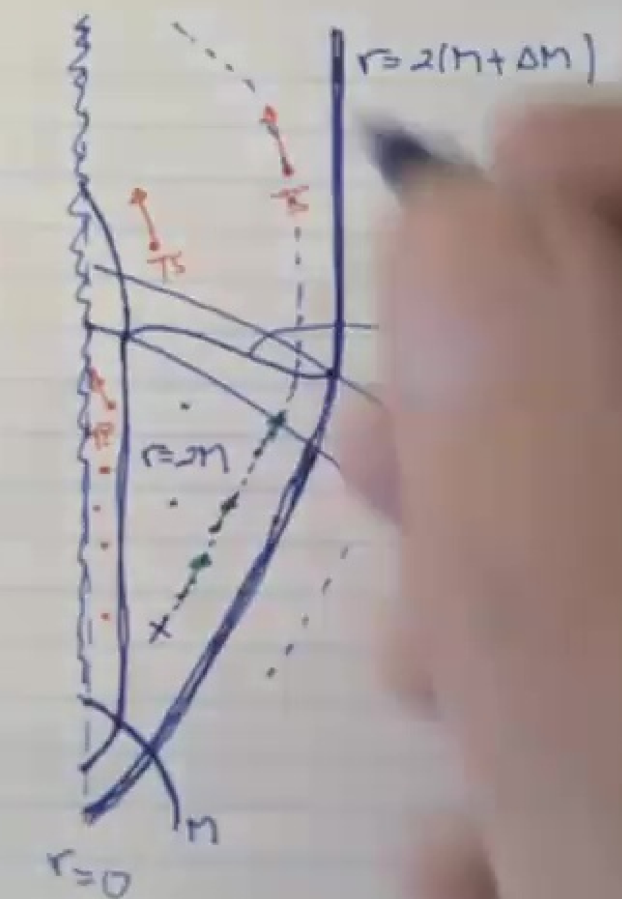

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