Title: Advanced General Relativity - 240327

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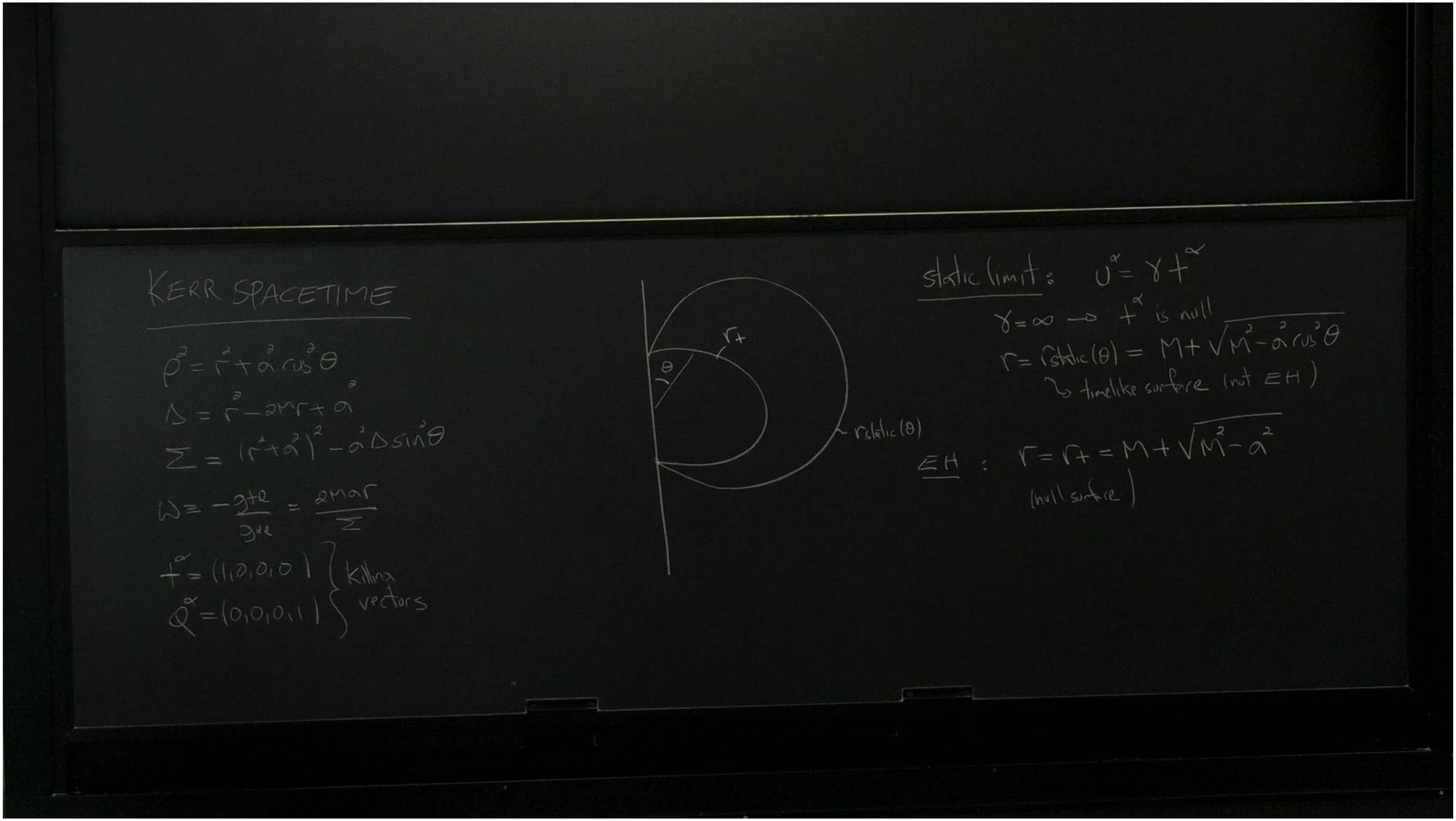
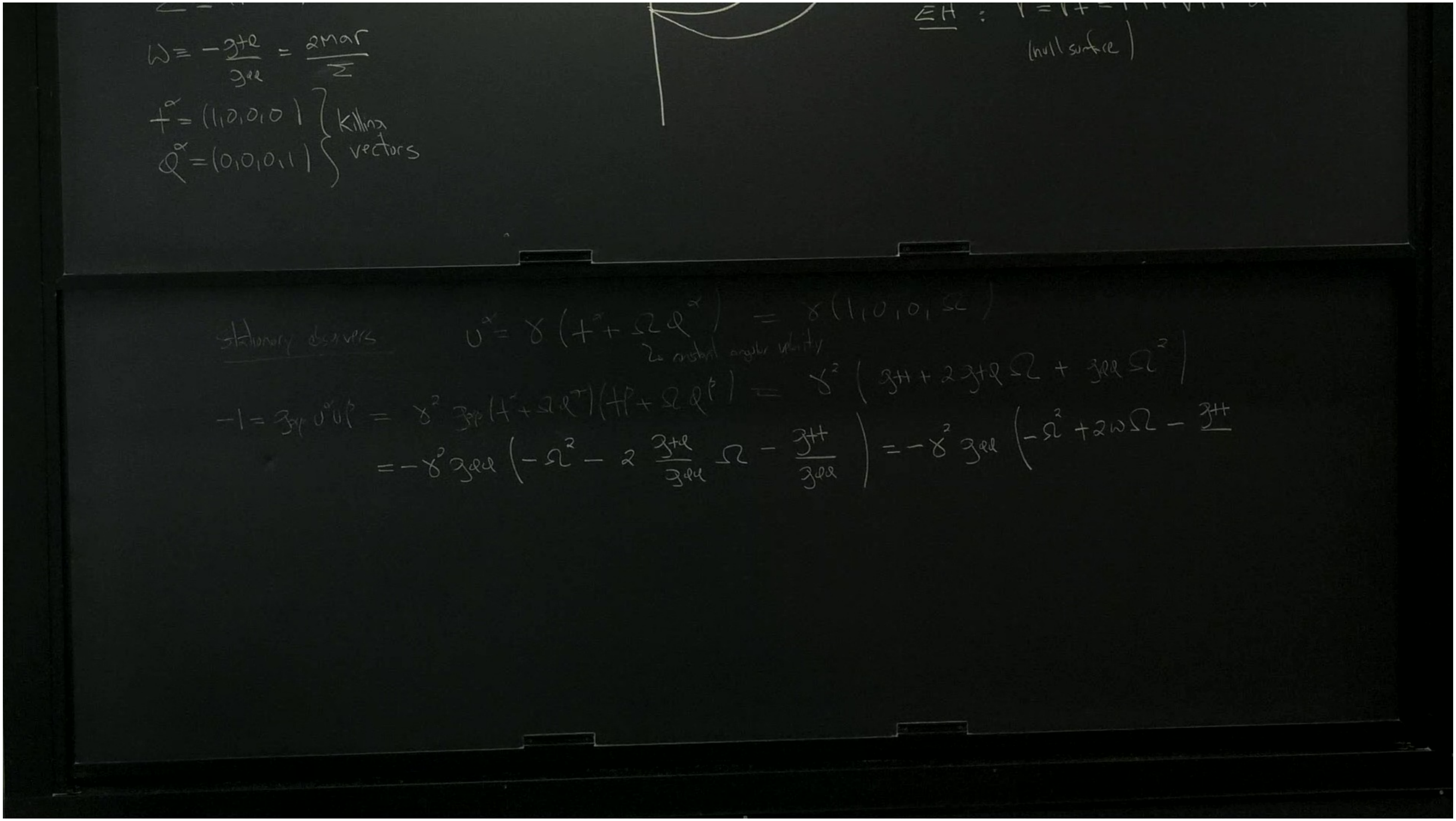
Collection: Advanced General Relativity (PHYS7840)

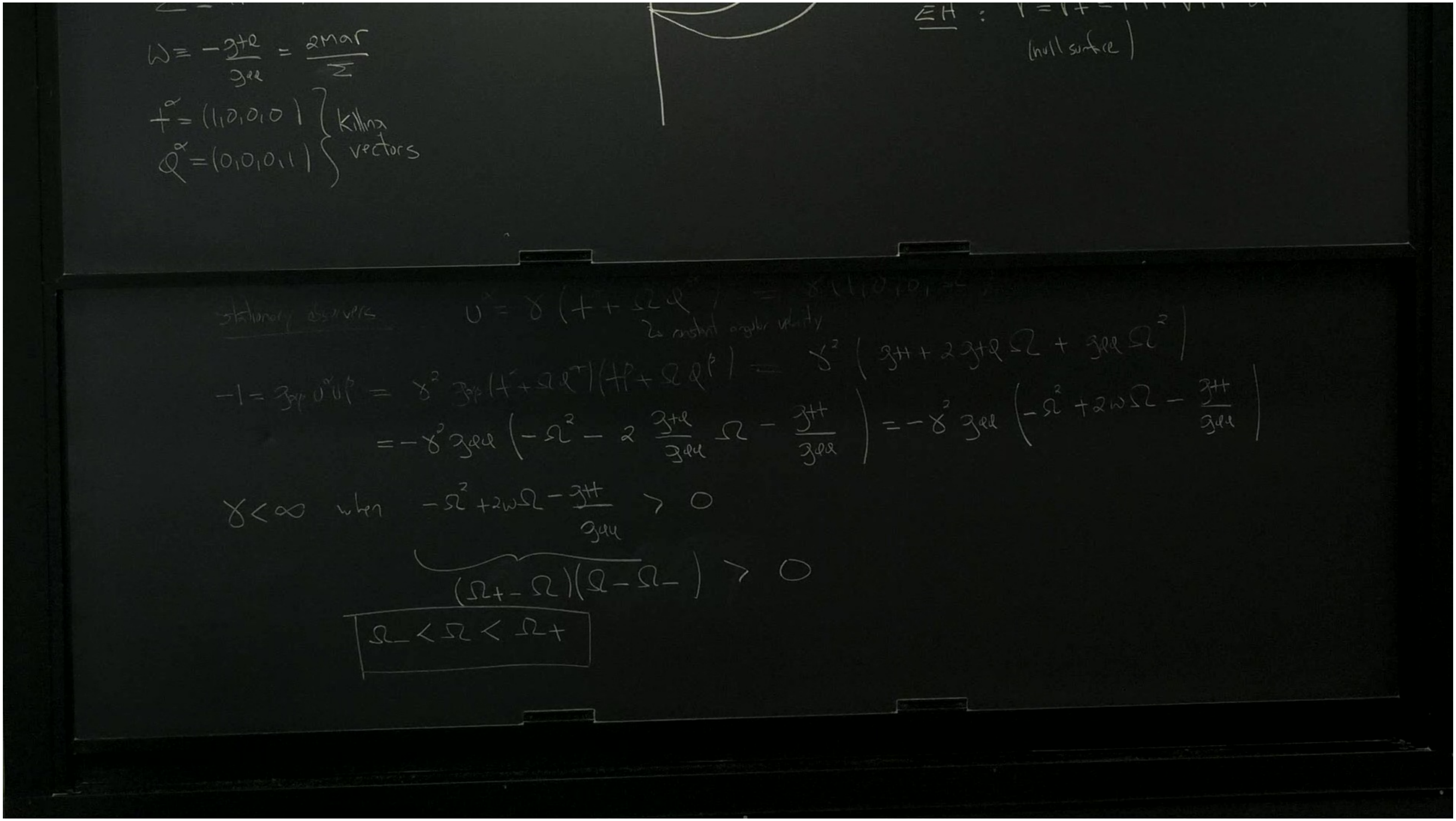
Date: March 27, 2024 - 10:30 AM

URL: https://pirsa.org/24030003

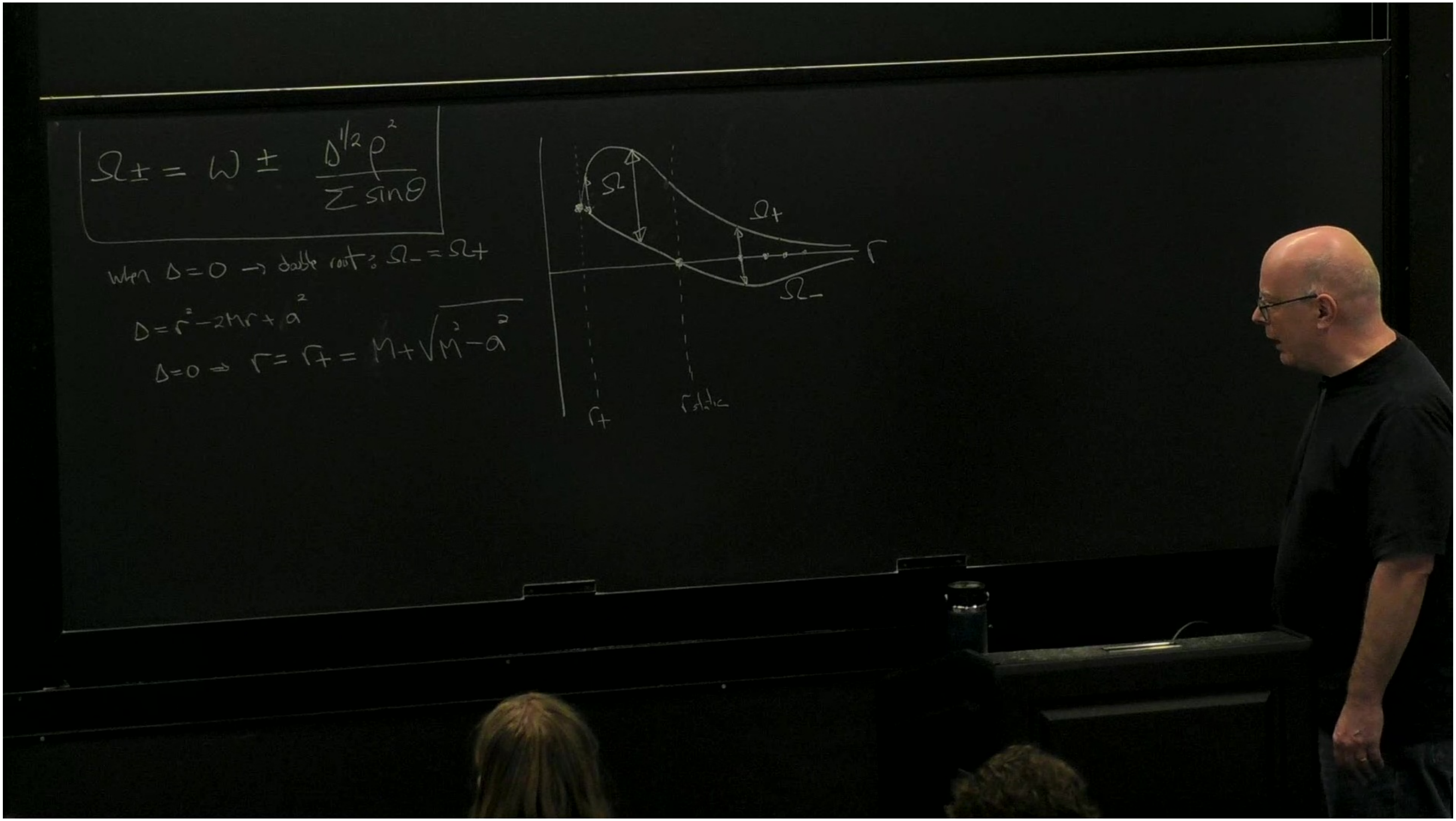

KERR SPACETIME
$\rho^2 = r^2 + a^2 \cos^2\theta$
$\Delta = r^2 - 2Mr + a^2$
$\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$
$\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Sigma}$
$t^\alpha = (1,0,0,0)$
$\phi^\alpha = (0,0,0,1)$
Killing vectors
θ
r_+
$r_{static}(\theta)$
static limit: $u^\alpha = \gamma t^\alpha$
$\gamma = \infty \to t^\alpha$ is null
$r = r_{static}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}$
timelike surface (not EH)
EH : $r = r_+ = M + \sqrt{M^2 - a^2}$
(null surface)

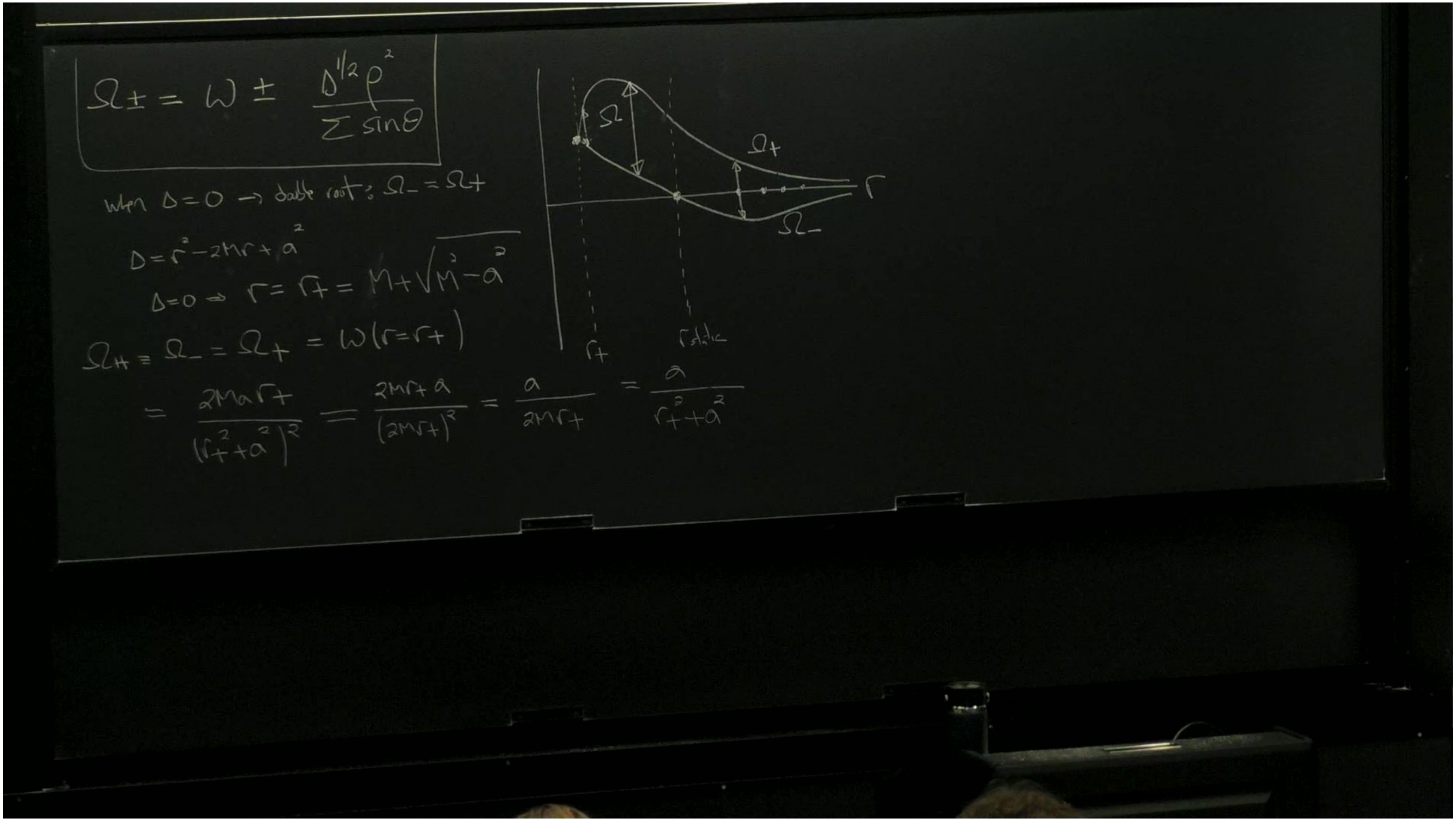


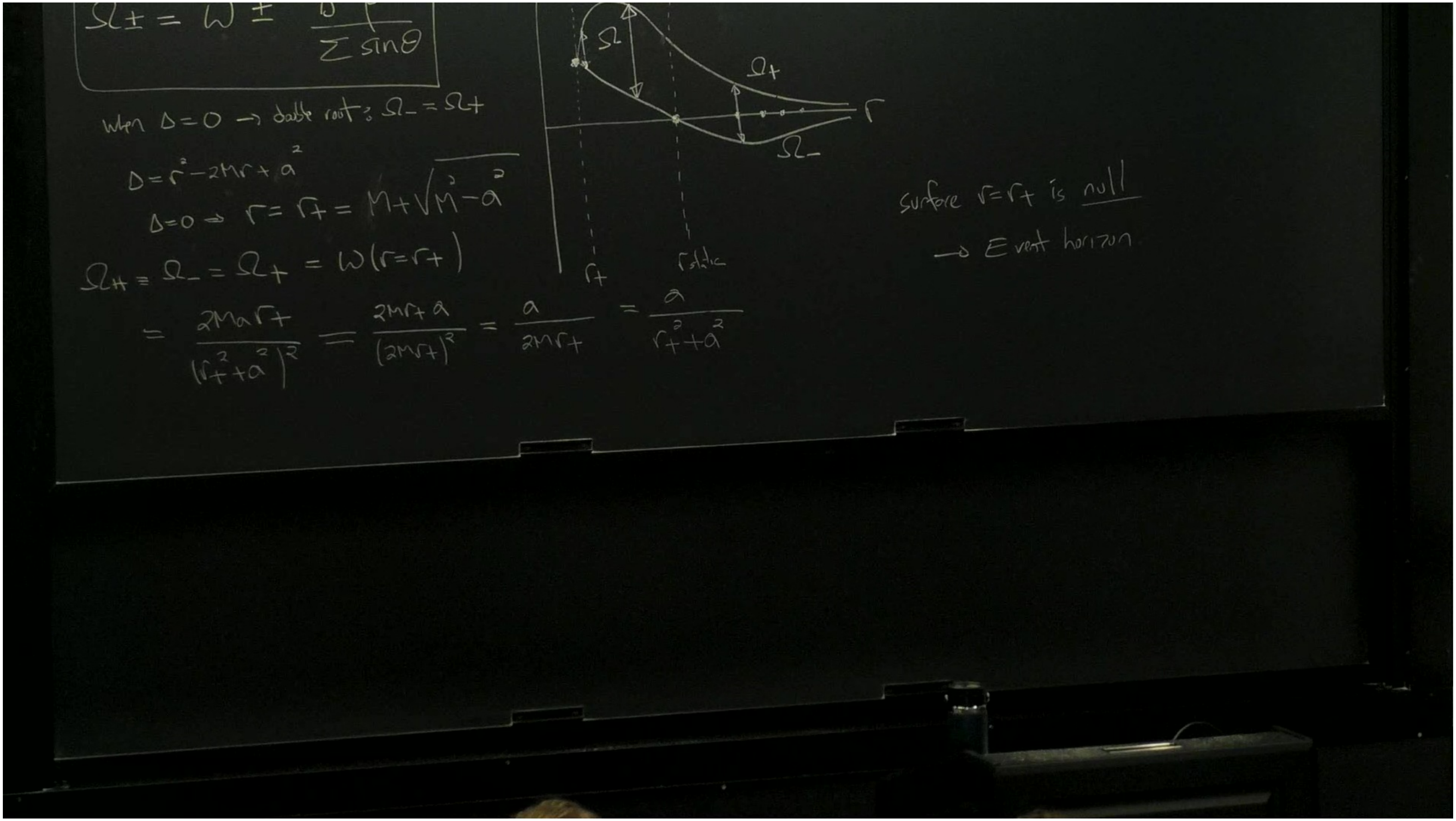
$\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Sigma}$
$t^\alpha = (1,0,0,0)$
$\phi^\alpha = (0,0,0,1)$
Killing vectors
EH :
(null surface)
stationary observers
$u^\alpha = \gamma (t^\alpha + \Omega \phi^\alpha) = \gamma (1,0,0,\Omega)$
Ω = constant angular velocity
$-1 = g_{\alpha\beta} u^\alpha u^\beta = \gamma^2 g_{\alpha\beta} (t^\alpha + \Omega \phi^\alpha)(t^\beta + \Omega \phi^\beta) = \gamma^2 (g_{tt} + 2 g_{t\phi} \Omega + g_{\phi\phi} \Omega^2)$
$= -\gamma^2 g_{\phi\phi} \left(-\Omega^2 - 2 \frac{g_{t\phi}}{g_{\phi\phi}} \Omega - \frac{g_{tt}}{g_{\phi\phi}}\right) = -\gamma^2 g_{\phi\phi} (-\Omega^2 + 2\omega\Omega - \frac{g_{tt}}{}$

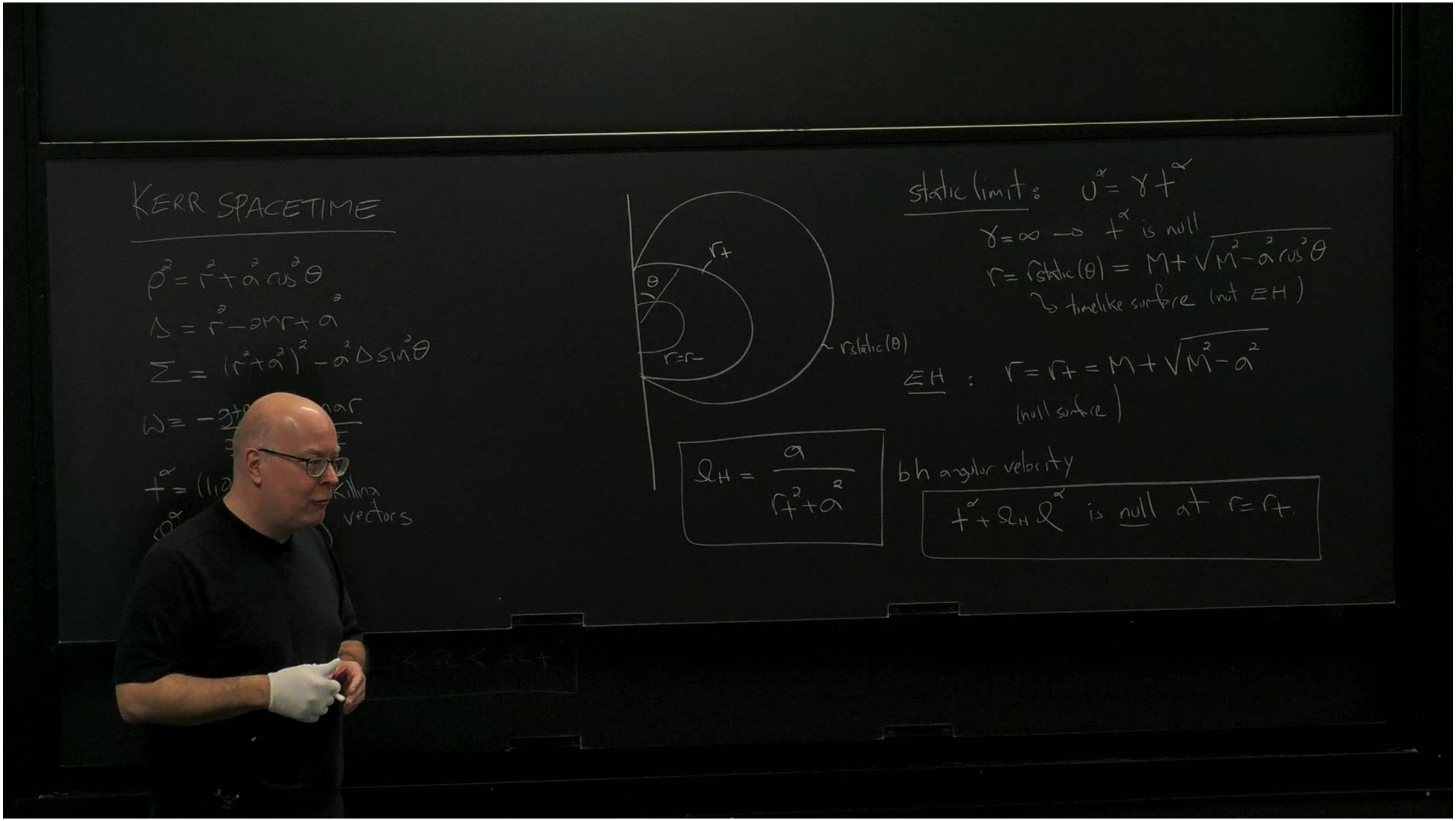


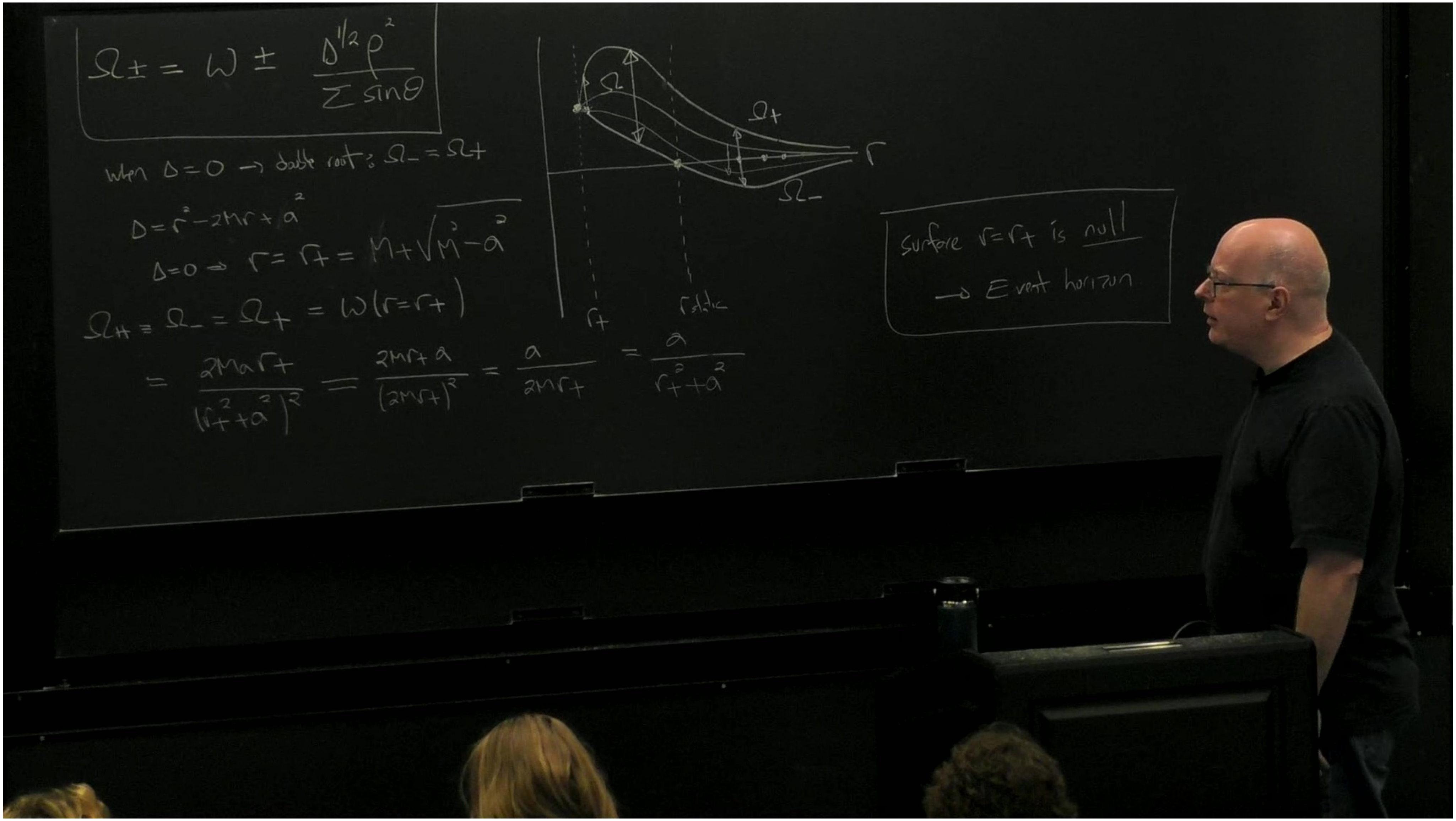
ω = −g_tφ / g_φφ = 2Mar / Σ
t^α = (1,0,0,0)
φ^α = (0,0,0,1)
Killing vectors
EH :
(null surface)
stationary observers
u^α = γ (t^α + Ω φ^α)
constant angular velocity
−1 = g_αβ u^α u^β = γ² g_αβ (t^α + Ω φ^α)(t^β + Ω φ^β) = γ² (g_tt + 2 g_tφ Ω + g_φφ Ω²)
= −γ² g_φφ (−Ω² − 2 (g_tφ / g_φφ) Ω − g_tt / g_φφ) = −γ² g_φφ (−Ω² + 2ωΩ − g_tt / g_φφ)
γ < ∞ when −Ω² + 2ωΩ − g_tt / g_φφ > 0
(Ω₊ − Ω)(Ω − Ω₋) > 0
Ω₋ < Ω < Ω₊


$\Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma \sin\theta}$
when $\Delta = 0 \to$ double root: $\Omega_- = \Omega_+$
$\Delta = r^2 - 2Mr + a^2$
$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$
Ω
Ω_+
Ω_-
r
r_+
r_{static}


$\Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma \sin\theta}$
when $\Delta = 0 \rightarrow$ double root: $\Omega_- = \Omega_+$
$\Delta = r^2 - 2Mr + a^2$
$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$
$\Omega_H \equiv \Omega_- = \Omega_+ = \omega(r = r_+)$
$= \frac{2Mar_+}{(r_+^2 + a^2)^2} = \frac{2Mr_+ a}{(2Mr_+)^2} = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$
Ω
Ω_+
Ω_-
r
r_+
r_{static}


Σ sinθ
when Δ=0 → double root: Ω− = Ω+
Δ = r² − 2Mr + a²
Δ=0 ⇒ r = r+ = M + √(M² − a²)
Ω_H ≡ Ω− = Ω+ = ω(r=r+)
= 2Mar+ / (r+² + a²)² = 2Mr+a / (2Mr+)² = a / 2Mr+ = a / (r+² + a²)
Ω
Ω+
Ω−
r
r+
r_static
surface r=r+ is null
→ Event horizon


KERR SPACETIME
ρ² = r² + a² cos²θ
Δ = r² − 2Mr + a²
Σ = (r² + a²)² − a²Δ sin²θ
static limit: u^α = γ t^α
γ = ∞ → t^α is null
r = r_static(θ) = M + √(M² − a² cos²θ)
timelike surface (not EH)
r_static(θ)
r+
r=r−
θ
EH : r = r+ = M + √(M² − a²)
(null surface)
Ω_H = a / (r+² + a²)
bh angular velocity
t^α + Ω_H ℓ^α is null at r = r+
vectors



$\Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma \sin\theta}$
when $\Delta = 0 \to$ double root: $\Omega_- = \Omega_+$
$\Delta = r^2 - 2Mr + a^2$
$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$
$\Omega_H \equiv \Omega_- = \Omega_+ = \omega(r = r_+)$
$= \frac{2Mar_+}{(r_+^2 + a^2)^2} = \frac{2Mr_+ a}{(2Mr_+)^2} = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$
Ω
Ω_+
Ω_-
r
r_+
r_{static}
surface $r = r_+$ is null
→ Event horizon

$$\Delta = r^2 - 2Mr + a^2$$

$$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$$

$$\Omega_H = \Omega_- = \Omega_+ = \omega(r = r_+)$$

$$= \frac{2Mar_+}{(r_+^2 + a^2)^2} = \frac{2Mr_+ a}{(2Mr_+)^2} = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$$

r_+ $\quad$ r_{static}

Surface $r = r_+$ is null

$\rightarrow$ Event horizon

Regular coordinates

behaviour of incoming light rays (principal null congruence)

$$V = t + \int \frac{r^2 + a^2}{\Delta} dr \quad \text{(constant on incoming light rays)}$$

$$\psi = \varphi + \int \frac{a}{\Delta} dr \quad (\text{" " " " "})$$

$$dt = dV - \frac{r^2 + a^2}{\Delta} dr$$

$$d\varphi = d\psi - \frac{a}{\Delta} dr$$

$g^{Kerr}_{\alpha\beta}(V, r, \theta, \psi)$

↳ regular the event horizon

(V, r, θ, ψ) :

$$t^\alpha = (1, 0, 0, 0)$$

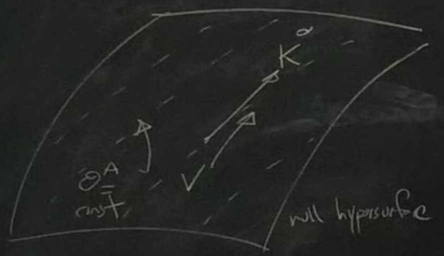
$$\varphi^\alpha = (0, 0, 0, 1)$$

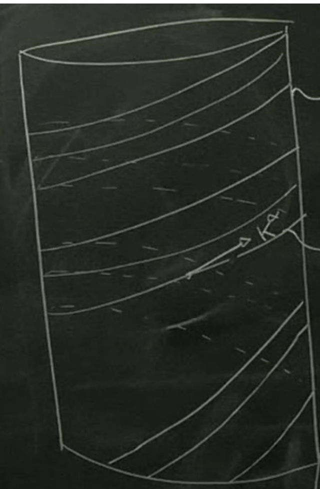
$$K^\alpha = t^\alpha + \Omega_H \varphi^\alpha$$

↳ null on EH

$$r = r_+ = M + \sqrt{M^2 - a^2}$$

Kinematics of Kerr horizon

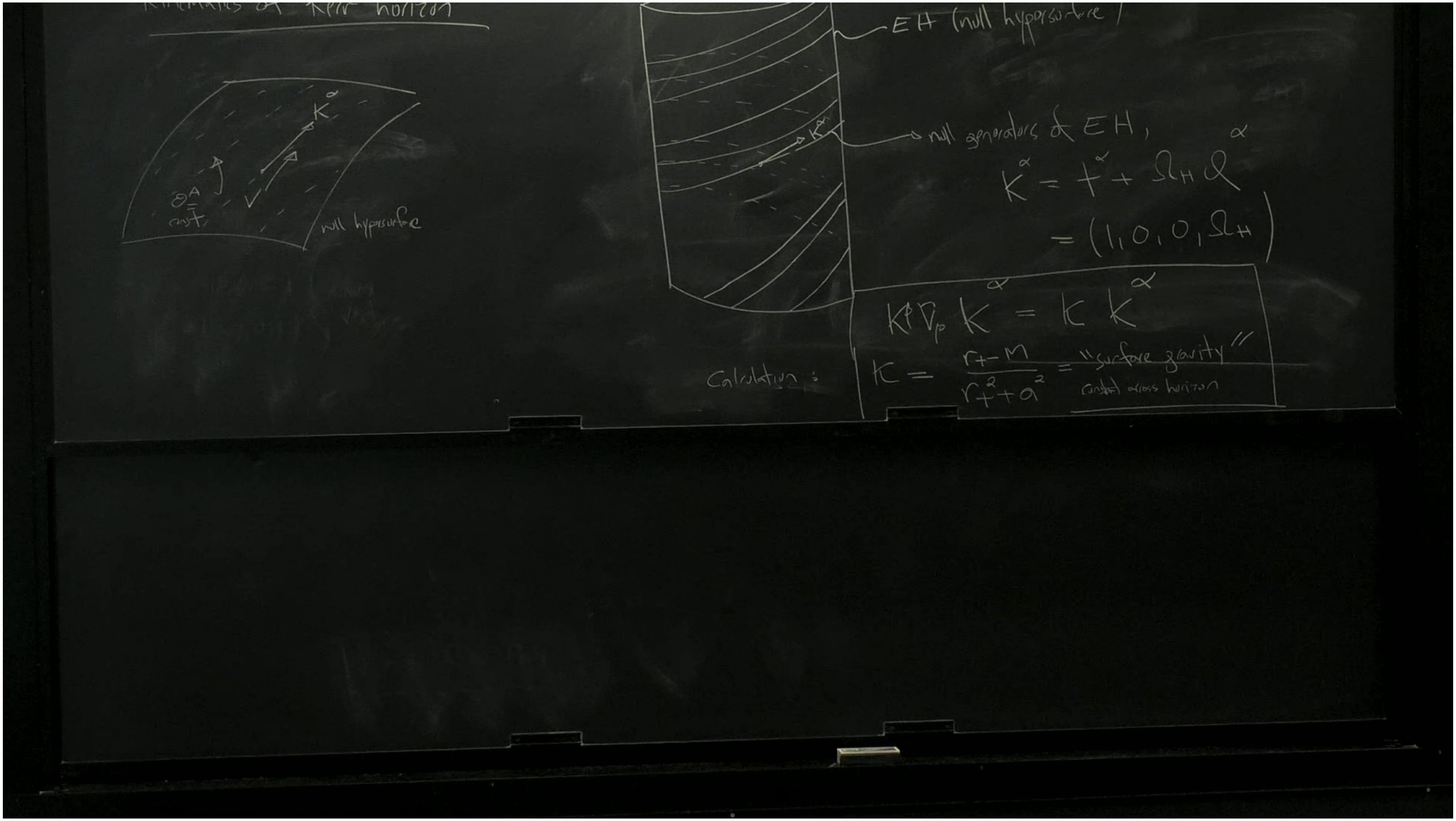


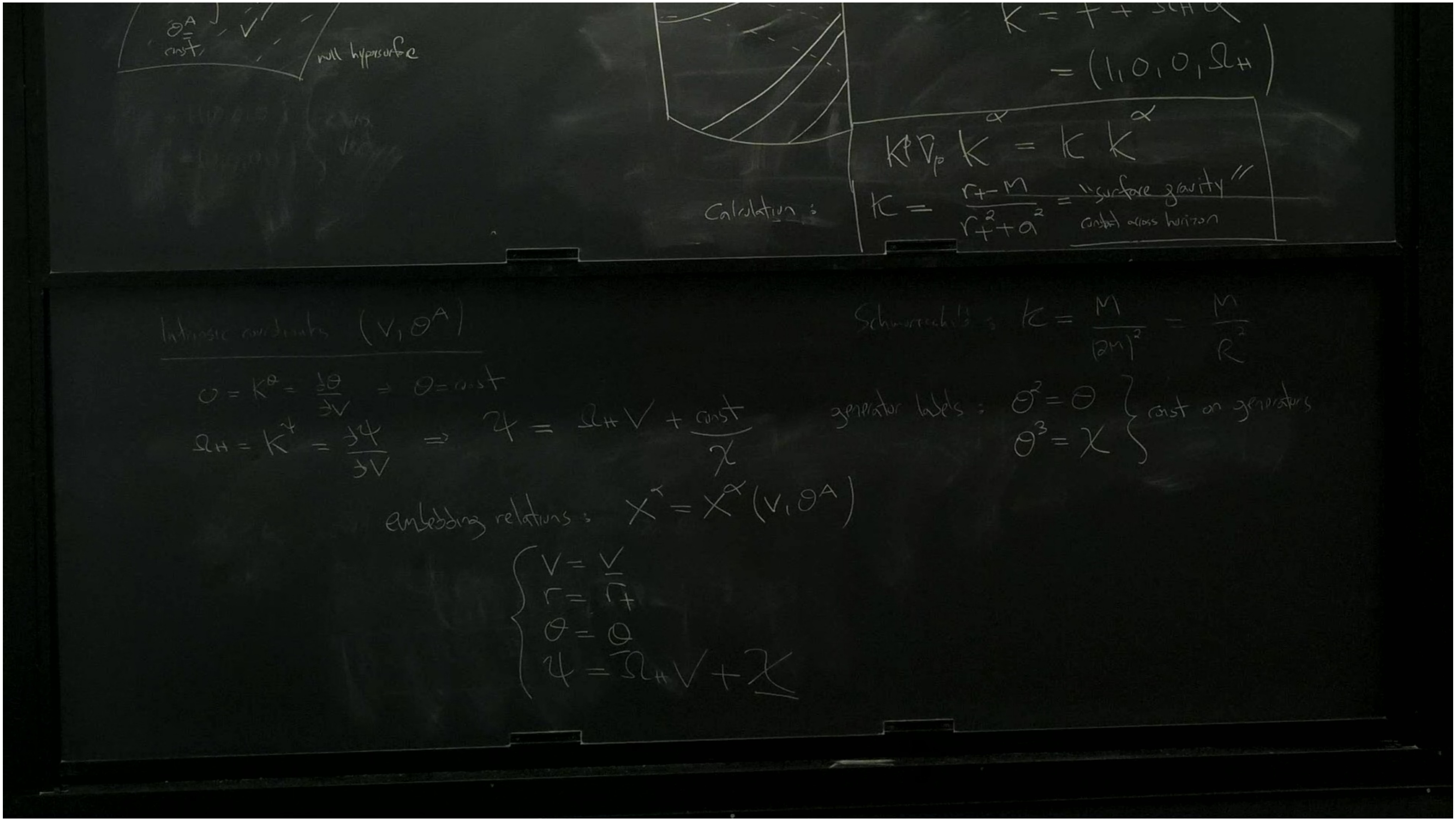


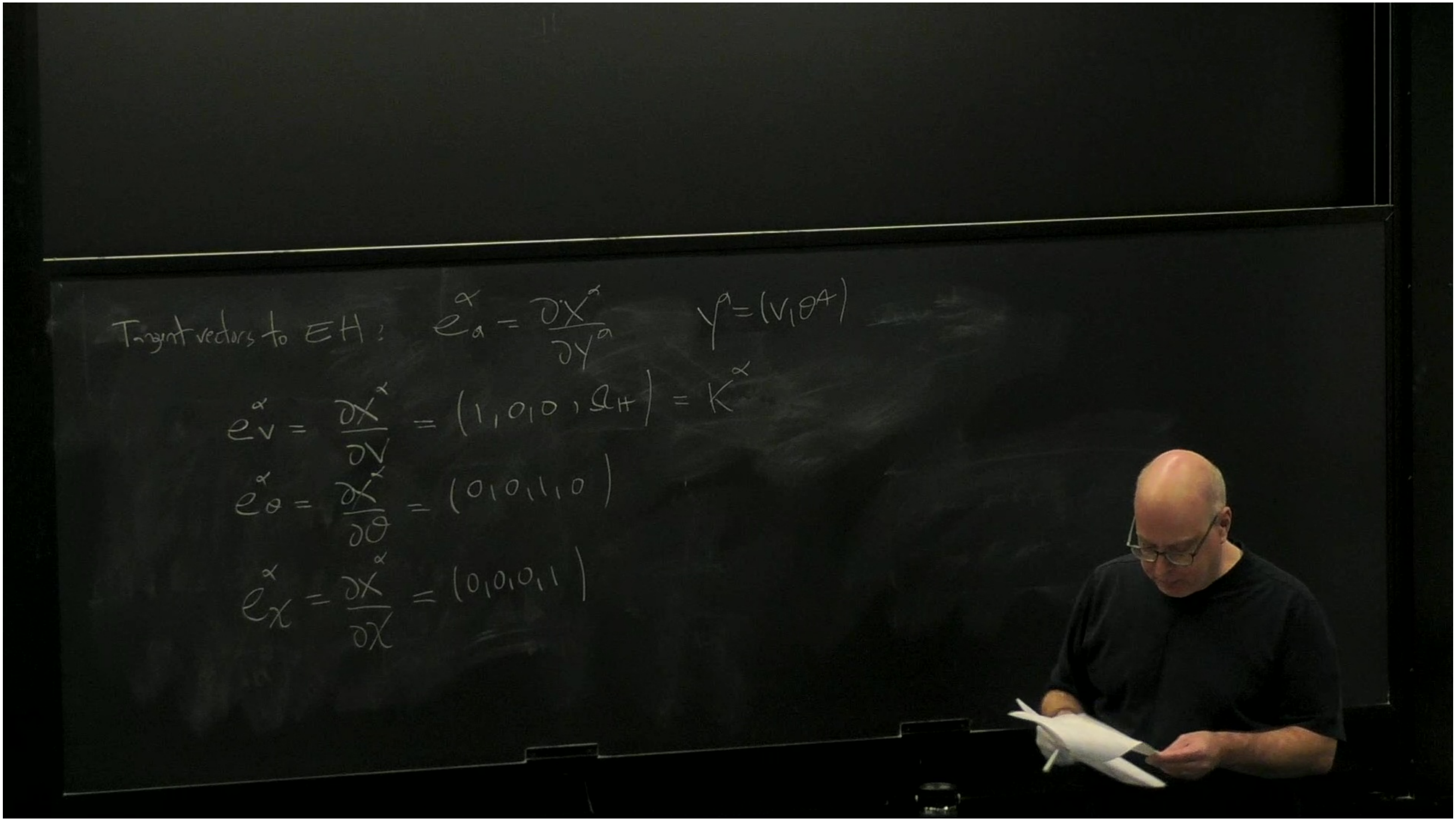
$$K^\alpha = t^\alpha + \Omega_H \phi^\alpha$$
$$= (1, 0, 0, \Omega_H)$$

$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$$

Calculation: $\kappa = \dfrac{r_+ - M}{r_+^2 + a^2}$ = "surface gravity"


Kinematics of Kerr horizon
null hypersurface
EH (null hypersurface)
null generators of EH,
$K^\alpha = t^\alpha + \Omega_H \phi^\alpha$
$= (1, 0, 0, \Omega_H)$
$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$
Calculation:
$\kappa = \frac{r_+ - M}{r_+^2 + a^2}$ = "surface gravity"
constant across horizon


θ^A = const
V
null hypersurface
K = t + Ω_H φ
= (1, 0, 0, Ω_H)
K^β ∇_β K^α = κ K^α
Calculation:
κ = (r_+ − M)/(r_+^2 + a^2) = "surface gravity"
constant across horizon
Intrinsic coordinates (V, θ^A)
0 = K^θ = ∂θ/∂V ⇒ θ = const
Ω_H = K^φ = ∂φ/∂V ⇒ φ = Ω_H V + const (χ)
Schwarzschild: κ = M/(2M)^2 = M/R^2
generator labels: θ^2 = θ, θ^3 = χ
const on generators
embedding relations: x^α = x^α(V, θ^A)
V = V
r = r_+
θ = θ
φ = Ω_H V + χ


Tangent vectors to EH: $e^\alpha_a = \frac{\partial X^\alpha}{\partial Y^a}$ $Y^a = (v, \theta^A)$
$e^\alpha_v = \frac{\partial X^\alpha}{\partial v} = (1, 0, 0, \Omega_H) = K^\alpha$
$e^\alpha_\theta = \frac{\partial X^\alpha}{\partial \theta} = (0, 0, 1, 0)$
$e^\alpha_\phi = \frac{\partial X^\alpha}{\partial \phi} = (0, 0, 0, 1)$