Title: Advanced General Relativity - 240327

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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KERR SPACETIME

$$\rho^2 = r^2 + a^2 \cos^2\theta$$

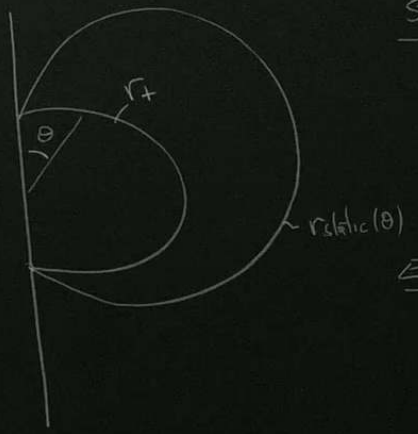
$$\Delta = r^2 - 2Mr + a^2$$

$$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$$

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Sigma}$$

$t^\alpha = (1,0,0,0)$ } Killing vectors

$\phi^\alpha = (0,0,0,1)$ }



static limit: $U^\alpha = \gamma t^\alpha$

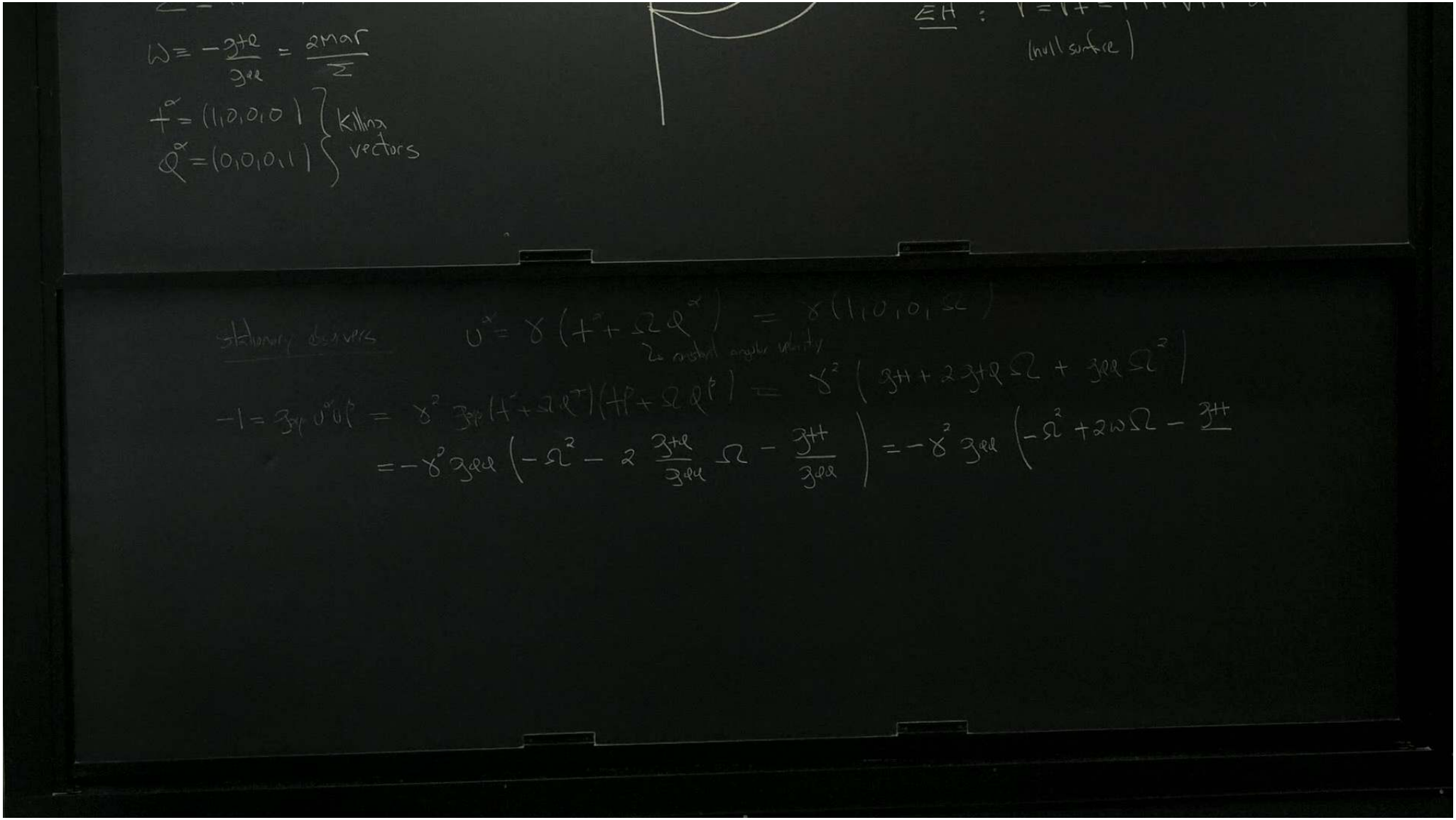
$\gamma = \infty \rightarrow t^\alpha$ is null

$r = r_{static}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}$

↳ timelike surface (not $\in H$)

$\in H$: $r = r_+ = M + \sqrt{M^2 - a^2}$

(null surface)


$\omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2Mar}{\Sigma}$
$t^\alpha = (1,0,0,0)$
$\varphi^\alpha = (0,0,0,1)$
Killing vectors
EH : $r = r_+ = M + \sqrt{M^2 - a^2}$
(null surface)
stationary observers
$u^\alpha = \gamma(t^\alpha + \Omega\varphi^\alpha) = \gamma(1,0,0,\Omega)$
Ω = orbital angular velocity
$-1 = g_{\mu\nu} u^\mu u^\nu = \gamma^2 g_{\mu\nu}(t^\mu + \Omega\varphi^\mu)(t^\nu + \Omega\varphi^\nu) = \gamma^2 (g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2)$
$= -\gamma^2 g_{\varphi\varphi}\left(-\Omega^2 - 2\frac{g_{t\varphi}}{g_{\varphi\varphi}}\Omega - \frac{g_{tt}}{g_{\varphi\varphi}}\right) = -\gamma^2 g_{\varphi\varphi}\left(-\Omega^2 + 2\omega\Omega - \frac{g_{tt}}{}\right.$

$$\omega \equiv -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2Mar}{\Sigma}$$

$$\left.\begin{aligned} t^\alpha &= (1,0,0,0) \\ \varphi^\alpha &= (0,0,0,1) \end{aligned}\right\} \text{Killing vectors}$$

EH : $r = r_+ = \ldots$

(null surface)

stationary observers $\quad u^\alpha = \gamma\,(t^\alpha + \Omega\,\varphi^\alpha) = \gamma\,(1,0,0,\Omega)$

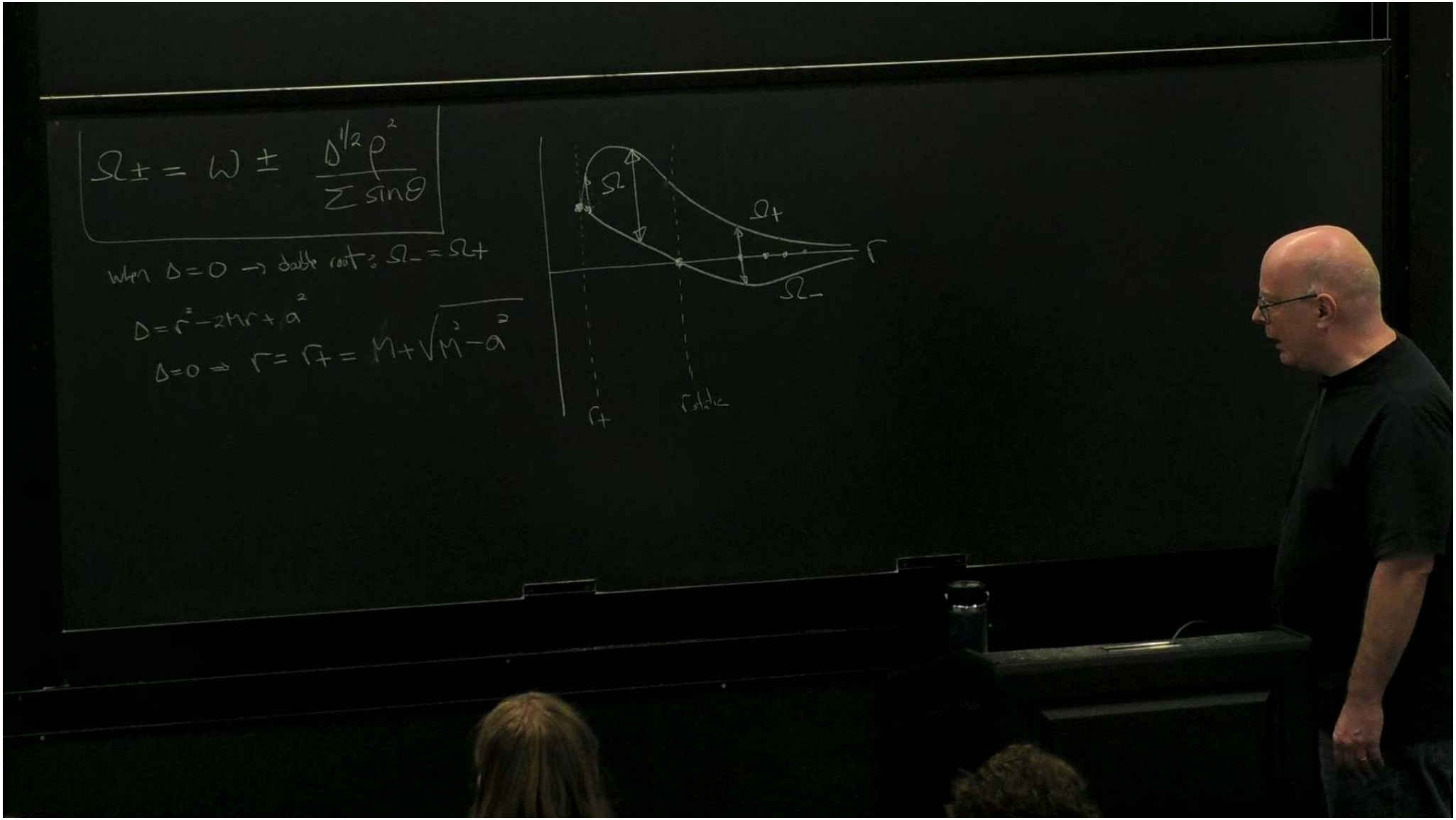
Ω constant angular velocity

$$-1 = g_{\alpha\beta}\,u^\alpha u^\beta = \gamma^2\, g_{\alpha\beta}\,(t^\alpha + \Omega\,\varphi^\alpha)(t^\beta + \Omega\,\varphi^\beta) = \gamma^2\left(g_{tt} + 2\,g_{t\varphi}\,\Omega + g_{\varphi\varphi}\,\Omega^2\right)$$

$$= -\gamma^2\, g_{\varphi\varphi}\left(-\Omega^2 - 2\,\frac{g_{t\varphi}}{g_{\varphi\varphi}}\,\Omega - \frac{g_{tt}}{g_{\varphi\varphi}}\right) = -\gamma^2\, g_{\varphi\varphi}\left(-\Omega^2 + 2\omega\Omega - \frac{g_{tt}}{g_{\varphi\varphi}}\right)$$

$\gamma < \infty$ when $\underbrace{-\Omega^2 + 2\omega\Omega - \frac{g_{tt}}{g_{\varphi\varphi}}}_{(\Omega_+ - \Omega)(\Omega - \Omega_-)} > 0$

$$\boxed{\Omega_- < \Omega < \Omega_+}$$


$\Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma \sin\theta}$
When $\Delta = 0$ → double root: $\Omega_- = \Omega_+$
$\Delta = r^2 - 2Mr + a^2$
$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$
Ω_+
Ω_-
r
r_+
r_{static}

$$\Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma \sin\theta}$$

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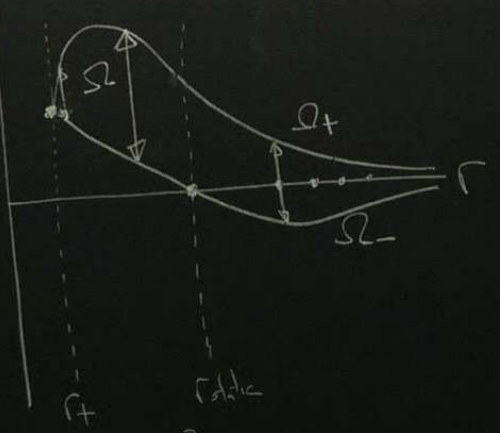
$$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$$

$$\Omega_H \equiv \Omega_- = \Omega_+ = \omega(r = r_+)$$

$$= \frac{2Mar_+}{(r_+^2 + a^2)^2} = \frac{2Mr_+ a}{(2Mr_+)^2} = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$$



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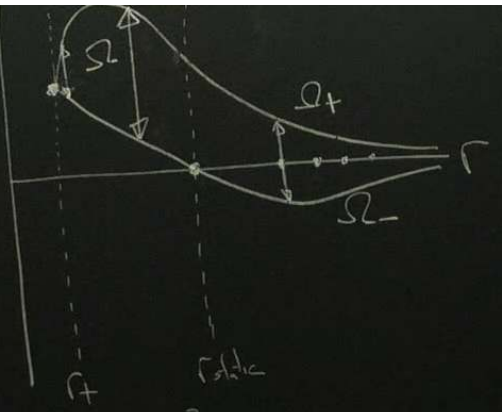
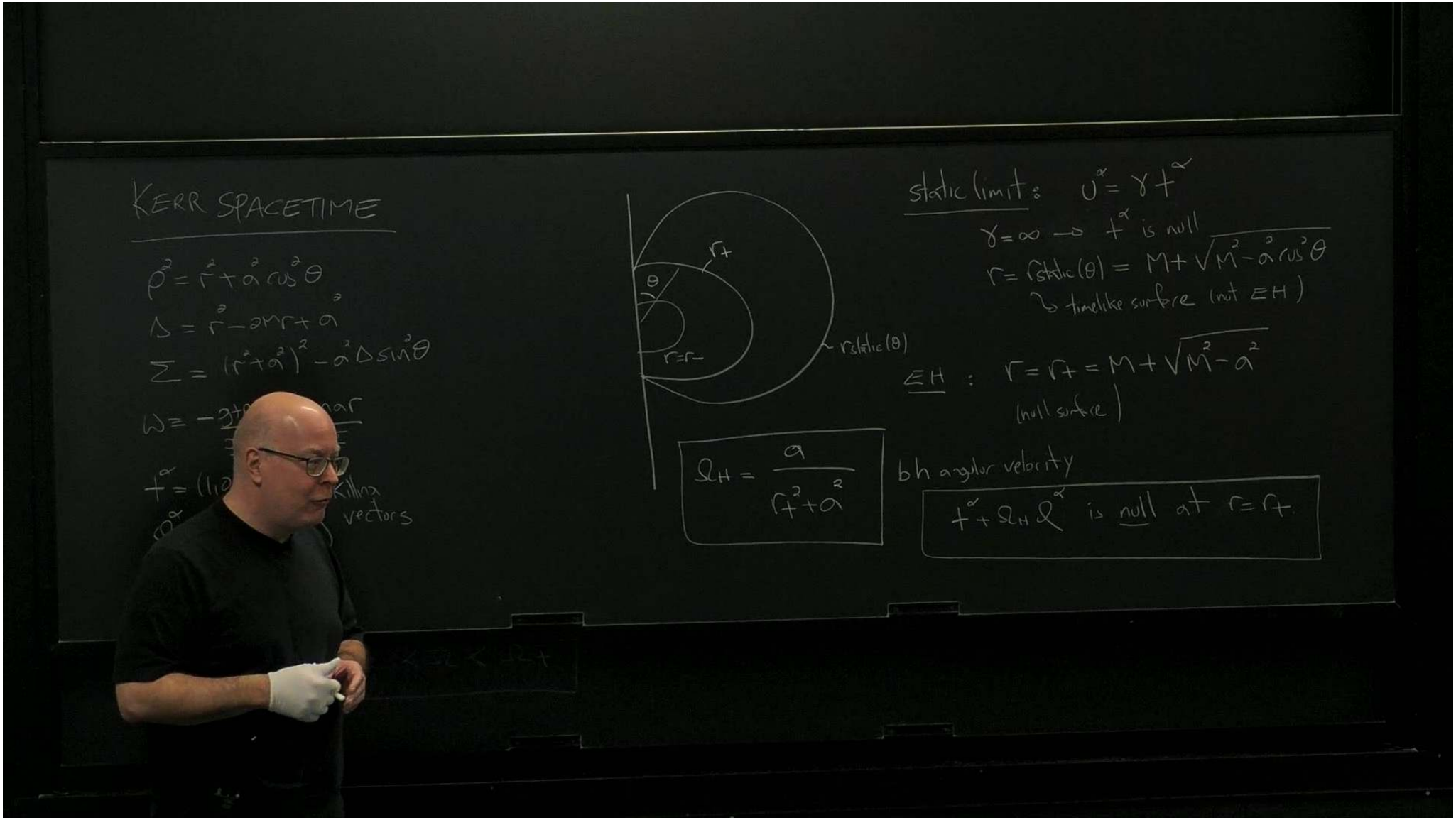


surface $r = r_+$ is null

→ Event horizon


KERR SPACETIME
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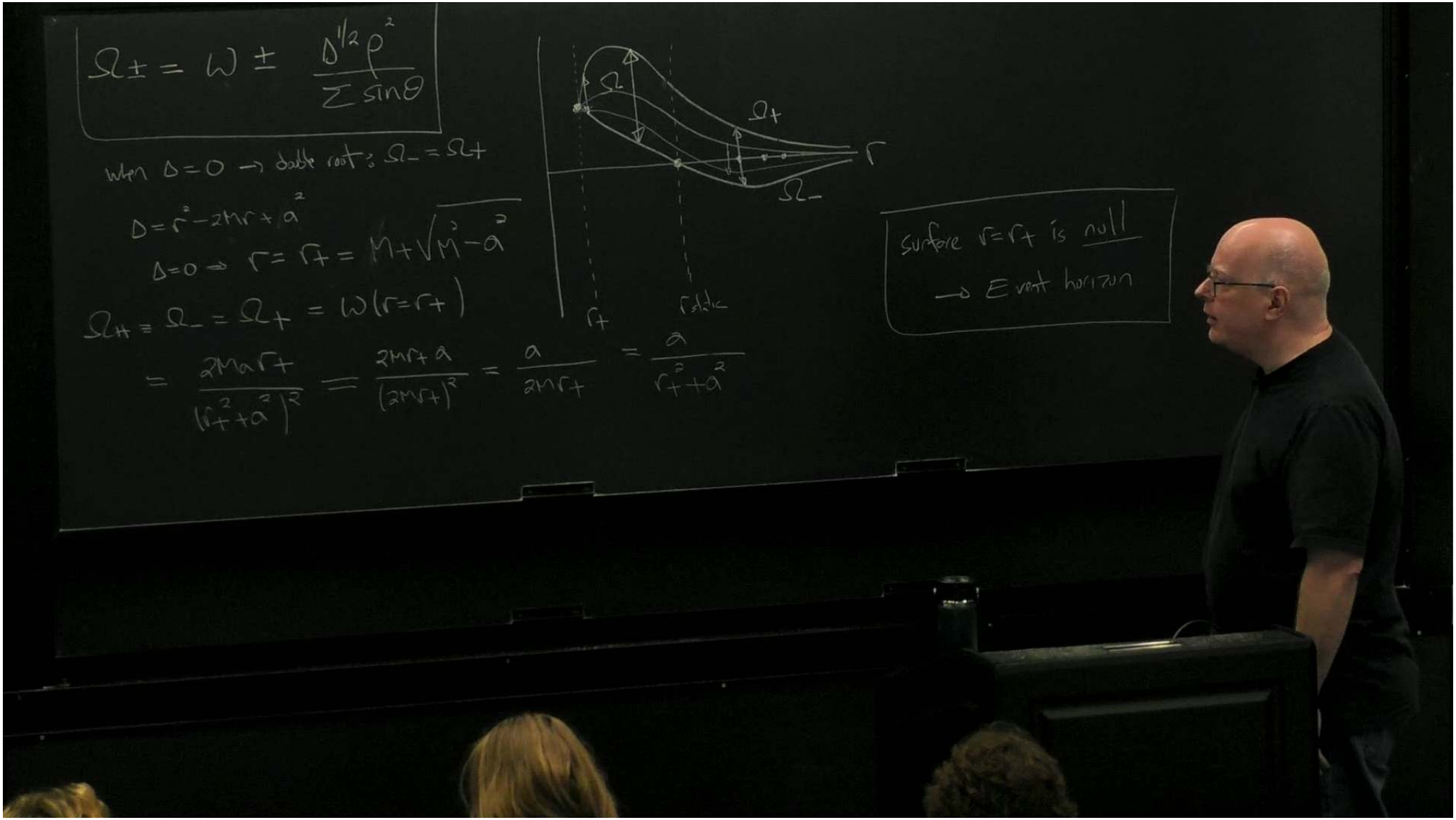
$\Delta = r^2 - 2Mr + a^2$
$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$
Killing vectors
r_+
θ
$r=r_-$
$r_{static}(\theta)$
static limit: $u^\alpha = \gamma t^\alpha$
$\gamma = \infty \to t^\alpha$ is null
$r = r_{static}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}$
timelike surface (not EH)
EH : $r = r_+ = M + \sqrt{M^2 - a^2}$
(null surface)
$\Omega_H = \frac{a}{r_+^2 + a^2}$
bh angular velocity
$t^\alpha + \Omega_H \ell^\alpha$ is null at $r = r_+$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When $\Delta = 0$ → double root: $\Omega_- = \Omega_+$
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$\Omega_H \equiv \Omega_- = \Omega_+ = \omega(r = r_+)$
$= \frac{2Mar_+}{(r_+^2 + a^2)^2} = \frac{2Mr_+}{(2Mr_+)^2} = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$
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r_+ r_{static}

Surface $r = r_+$ is null

→ Event horizon

Regular coordinates

behaviour of incoming light rays (principal null congruence)

$V = t + \int \frac{r^2 + a^2}{\Delta} dr$ (constant on incoming light rays)

$\psi = \varphi + \int \frac{a}{\Delta} dr$ (" " " " ")

$dt = dV - \frac{r^2 + a^2}{\Delta} dr$

$d\varphi = d\psi - \frac{a}{\Delta} dr$

(V, r, θ, ψ) : $t^\alpha = (1, 0, 0, 0)$, $\varphi^\alpha = (0, 0, 0, 1)$

$K^\alpha = t^\alpha + \Omega_H \varphi^\alpha$

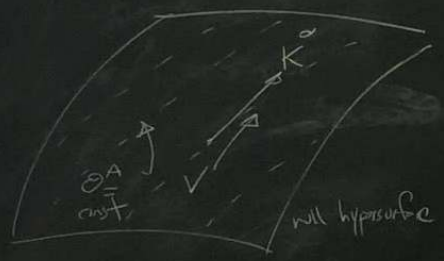
↳ null on EH

$r = r_+ = M + \sqrt{M^2 - a^2}$

$g_{typ}^{Kerr}(V, r, \theta, \psi)$

↳ regular to the event horizon

Kinematics of Kerr horizon



EH (null hypersurface)

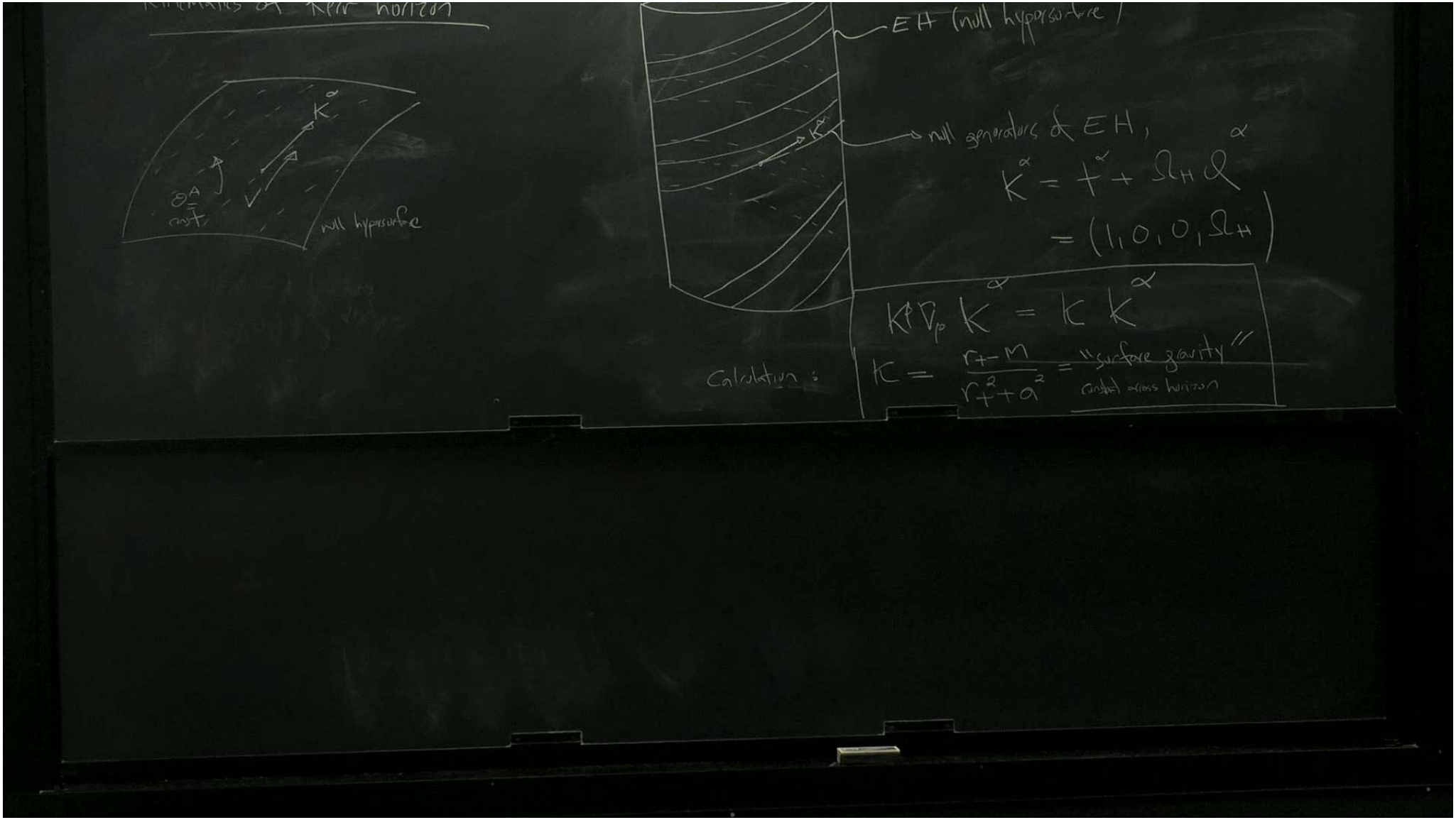
null generators of EH, ℓ^α

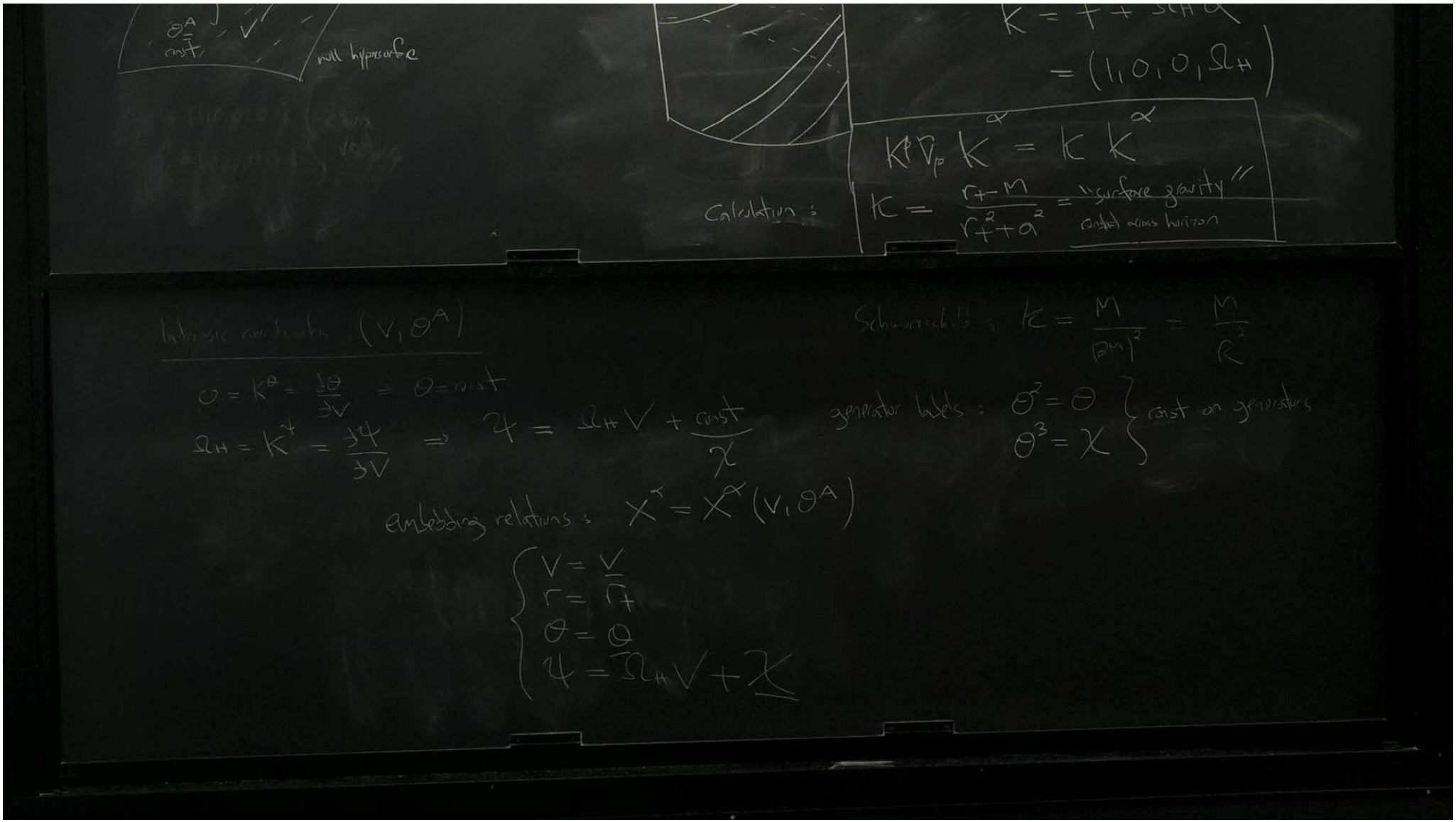
$$K^\alpha = t^\alpha + \Omega_H \ell^\alpha$$
$$= (1, 0, 0, \Omega_H)$$

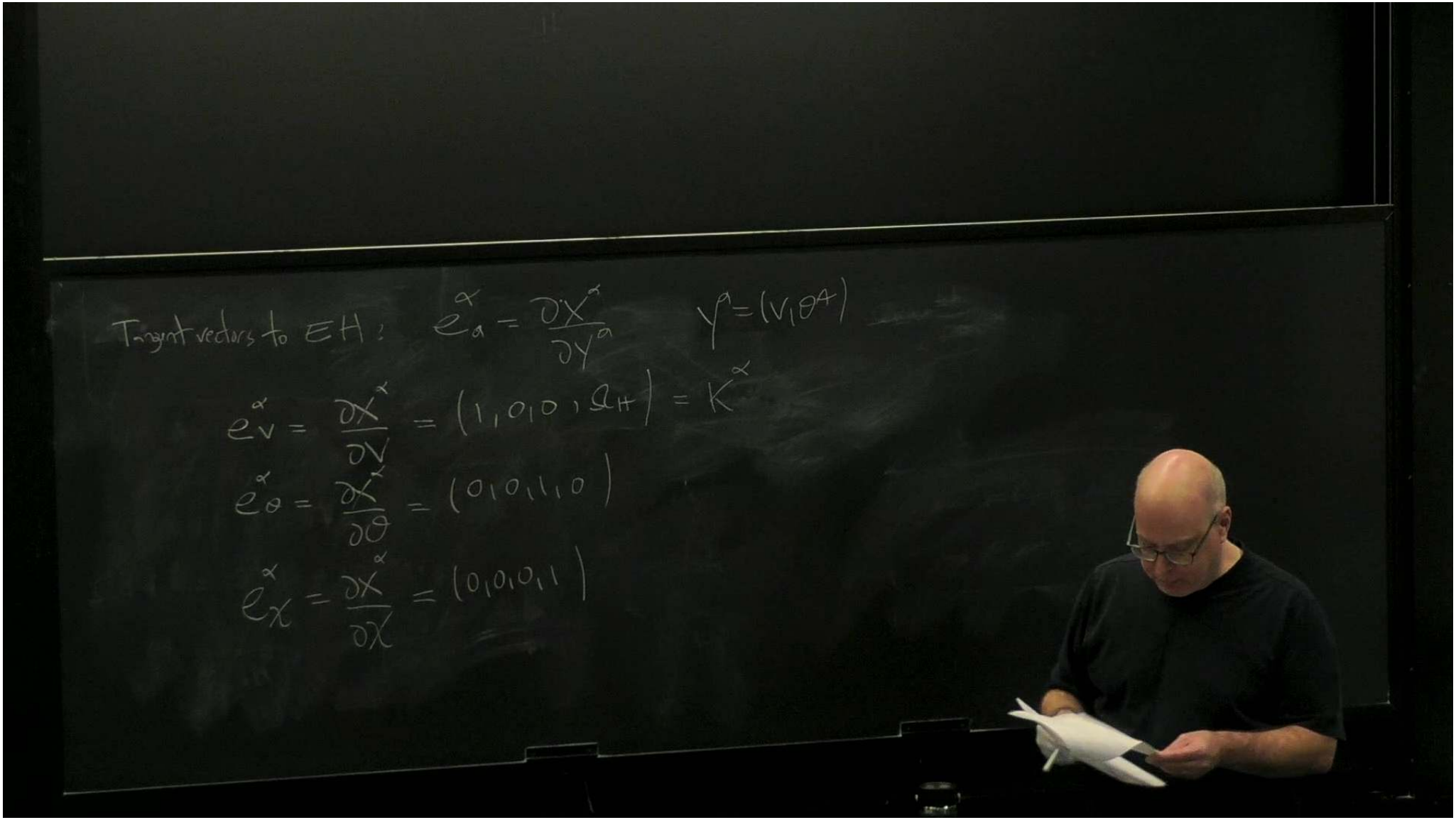
$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$$

Calculation:

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} = \text{"surface gravity"}$$


EH (null hypersurface)
null generators of EH,
$k^\alpha = t^\alpha + \Omega_H \phi^\alpha$
$= (1, 0, 0, \Omega_H)$
$k^\beta \nabla_\beta k^\alpha = \kappa k^\alpha$
$\kappa = \frac{r_+ - M}{r_+^2 + a^2}$ = "surface gravity"
Calculation:
null hypersurface


θ^A = const
V
null hypersurface
$k = t + \Omega_H \phi$
$= (1, 0, 0, \Omega_H)$
$k^\beta \nabla_\beta k^\alpha = \kappa k^\alpha$
Calculation:
$\kappa = \frac{r_+ - M}{r_+^2 + a^2}$ = "surface gravity" constant across horizon
Intrinsic coordinates (V, θ^A)
Schwarzschild: $\kappa = \frac{M}{(2M)^2} = \frac{M}{R^2}$
$v = k^\alpha \frac{\partial \theta}{\partial V} = 0 = \text{const}$
$\Omega_H = k^t = \frac{\partial t}{\partial V} \Rightarrow t = \Omega_H V + \frac{\text{const}}{\chi}$
generator labels: $\theta^2 = \theta$, $\theta^3 = \chi$ } const on generators
embedding relations: $X^\alpha = X^\alpha(V, \theta^A)$
$V = V$
$r = r_+$
$\theta = \theta$
$t = \Omega_H V + \chi$


Tangent vectors to EH: $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ $y^a = (v, \theta^A)$
$e^\alpha_v = \frac{\partial x^\alpha}{\partial v} = (1, 0, 0, \Omega_H) = K^\alpha$
$e^\alpha_\theta = \frac{\partial x^\alpha}{\partial \theta} = (0, 0, 1, 0)$
$e^\alpha_\chi = \frac{\partial x^\alpha}{\partial \chi} = (0, 0, 0, 1)$