Title: Advanced General Relativity - 240313

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

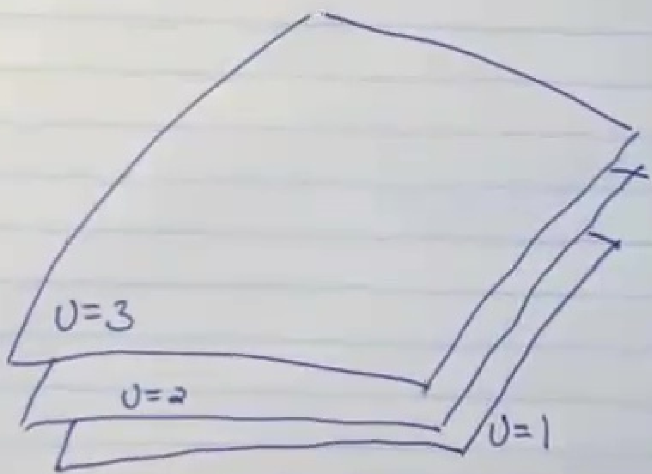
Date: March 13, 2024 - 10:30 AM

URL: https://pirsa.org/24030001

Quick Camera: [IPEVO V4K]

NULL HYPERSURFACES

Derivation



continuous stack (foliation) of null hypersurfaces.

$$u(x^\alpha) = \text{const}$$

$$k_\alpha \propto -\nabla_\alpha u$$

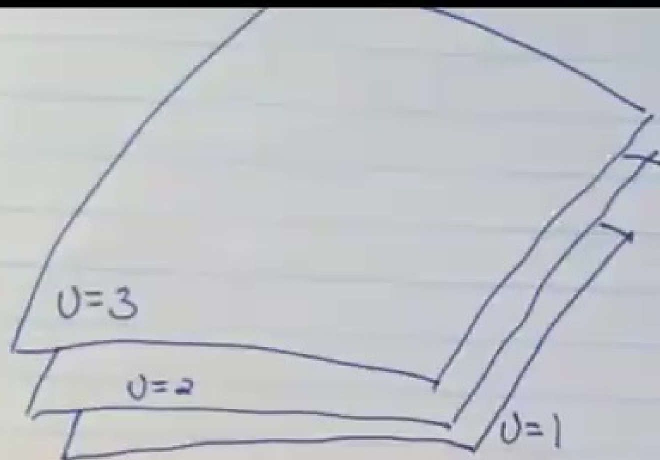
$$\boxed{k_\alpha = -e^{\chi}\,\nabla_\alpha u \quad ; \quad k_\alpha k^\alpha = 0}$$

$$\nabla_\beta k_\alpha = -e^{\chi}\left(\nabla_\beta \chi\, \nabla_\alpha u + \nabla_{\beta\alpha} u\right)$$

$$= \nabla_\beta \chi\, k_\alpha - e^{\chi}\, \nabla_{\beta\alpha} u$$

Quick Camera: [IPEVO V4K]

null hypersurfaces.



$$u(x^\alpha) = \text{const}$$

$$k_\alpha \propto -\nabla_\alpha u$$

$$\boxed{k_\alpha = -e^{\chi}\nabla_\alpha u \;;\; k_\alpha k^\alpha = 0}$$

$$\nabla_\beta k_\alpha = -e^{\chi}\left(\nabla_\beta\chi\,\nabla_\alpha u + \nabla_{\beta\alpha} u\right)$$

$$= \nabla_\beta\chi\, k_\alpha - e^{\chi}\nabla_{\beta\alpha} u$$

$$k^\beta\nabla_\beta k_\alpha = (k^\beta\nabla_\beta\chi)\,k_\alpha - e^{2\chi}\underbrace{\nabla^\beta u\,\nabla_{\beta\alpha} u}_{\nabla^\beta u\,\nabla_{\alpha\beta} u}$$

$$\boxed{k^\beta\nabla_\beta k_\alpha = \kappa\, k_\alpha}$$

Quick Camera: [IPEVO V4K]

$$\nabla_\rho K_\alpha = -e^{\chi}\left(\nabla_\rho \chi\, \nabla_\alpha U + \nabla_{\rho\alpha} U\right)$$

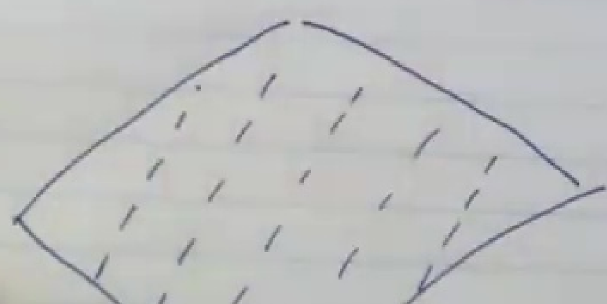
$$= \nabla_\rho \chi\, K_\alpha - e^{\chi}\, \nabla_{\rho\alpha} U$$

$$K^\rho \nabla_\rho K_\alpha = \left(K^\rho \nabla_\rho \chi\right) K_\alpha - e^{2\chi}\, \underbrace{\nabla^\rho U\, \nabla_{\rho\alpha} U}_{\nabla^\rho U\, \nabla_{\alpha\rho} U}$$

$$= \tfrac{1}{2} \nabla_\alpha \big(\underbrace{\nabla^\rho U\, \nabla_\rho U}_{0} \big)$$

$$\boxed{\begin{aligned} K^\rho \nabla_\rho K_\alpha &= \kappa\, K_\alpha \\ \kappa &= K^\rho \nabla_\rho \chi \end{aligned}}$$

geodesic eqn — non affine parametrization



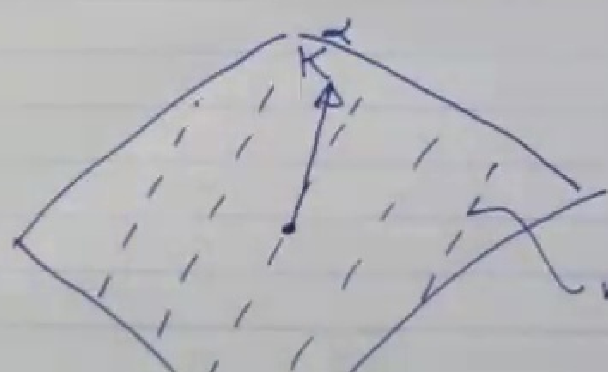
$$= \nabla_\beta \chi \, K_\alpha - e^{\chi} \nabla_{\beta\alpha} U$$

$$K^\beta \nabla_\beta K_\alpha = (K^\beta \nabla_\beta \chi) K_\alpha - e^{2\chi} \underbrace{\nabla^\beta U \nabla_{\beta\alpha} U}_{\nabla^\beta U \nabla_{\alpha\beta} U}$$

$$= \frac{1}{2} \nabla_\alpha (\underbrace{\nabla^\beta U \nabla_\beta U}_{0})$$

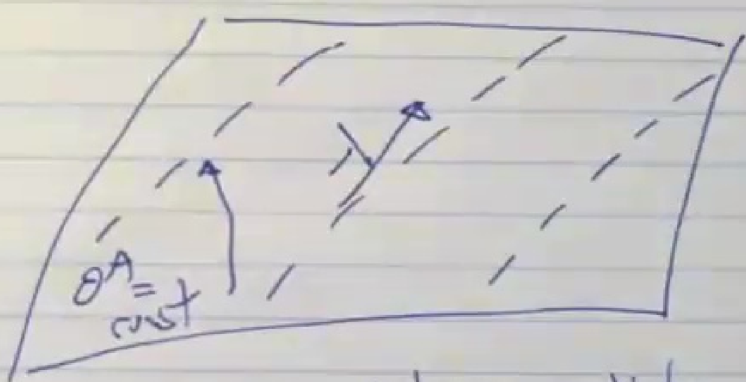
$$\boxed{\begin{gathered} K^\beta \nabla_\beta K_\alpha = \kappa K_\alpha \\ \kappa = K^\beta \nabla_\beta \chi \end{gathered}}$$

geodesic eqn – non affine parametrization



null generators of hypersurface

Quick Camera: [IPEVO V4K]



$\theta^A = (\theta^2, \theta^3)$

↳ const on each null generator

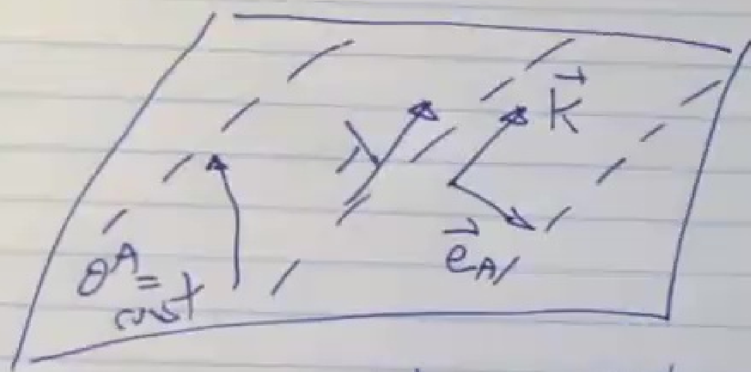
Intrinsic coordinates : $\boxed{y^a = (\lambda, \theta^A)}$

↳ nonaffine parameter.

Embedding relations : $\cancel{x^\alpha = x^\alpha(\lambda, \theta^A)}$

$\cancel{x^\alpha}$ on $\quad x^\alpha = x^\alpha(\lambda, \theta^A)$

Tangent vectors : $e^\alpha_\lambda = \left(\dfrac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} \equiv k^\alpha$



$\theta^A = (\theta^2, \theta^3)$

↳ const on each null generator

Intrinsic coordinates : $\boxed{y^a = (\lambda, \theta^A)}$

↳ nonaffine parameter.

Embedding relations : $\cancel{x^\alpha = x^\alpha(\lambda, \theta^A)}$

$\cancel{x^\alpha}$ on $\qquad x^\alpha = x^\alpha(\lambda, \theta^A)$

Tangent vectors :

$$\boxed{\begin{aligned} e^\alpha{}_\lambda &= \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} \equiv k^\alpha \\ e^\alpha{}_A &= \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda = \text{transverse vectors} \end{aligned}}$$

Quick Camera: [IPEVO V4K]

$$e^{\alpha}{}_{A} = \left(\frac{\partial x^{\alpha}}{\partial \theta^{A}}\right)_{\lambda} \equiv \text{transverse vectors}$$

$$\boxed{K_\alpha e^\alpha{}_A = 0} \qquad \boxed{K_\alpha K^\alpha = 0}$$

Second null vector :
$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \\ N_\alpha e^\alpha{}_A = 0 \end{cases}$$

Quick Camera: [IPEVO V4K]

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda \equiv \text{transverse vectors}$$

$$\boxed{K_\alpha e^\alpha_A = 0} \qquad \boxed{K_\alpha K^\alpha = 0}$$

Second null vector : $\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \\ N_\alpha e^\alpha_A = 0 \end{cases}$

Induced metric :

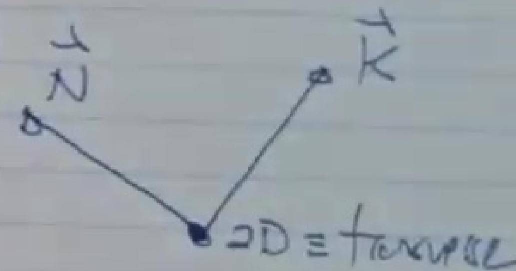
$$h_{\lambda\lambda} = g_{\alpha\beta} e^\alpha_\lambda e^\beta_\lambda = g_{\alpha\beta} K^\alpha K^\beta = 0$$

$$h_{\lambda A} = g_{\alpha\beta} e^\alpha_\lambda e^\beta_A = K_\alpha e^\alpha_A = 0$$

$$h_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B \equiv \Omega_{AB}$$

$$\partial\theta^A \Big)_\lambda$$

$$\boxed{K_\alpha e^\alpha{}_A = 0} \qquad \boxed{K_\alpha K^\alpha = 0}$$



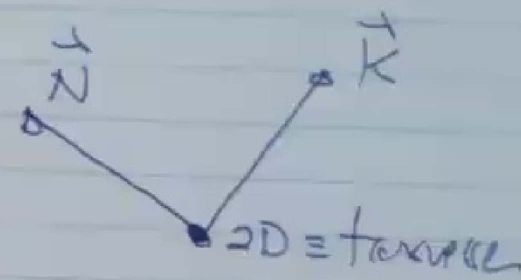
Second null vector :
$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \\ N_\alpha e^\alpha{}_A = 0 \end{cases}$$

Induced metric :
$$h_{\lambda\lambda} = g_{\alpha\beta}\, e^\alpha{}_\lambda e^\beta{}_\lambda = g_{\alpha\beta} K^\alpha K^\beta = 0$$
$$h_{\lambda A} = g_{\alpha\beta}\, e^\alpha{}_\lambda e^\beta{}_A = K_\alpha e^\alpha{}_A = 0$$
$$h_{AB} = g_{\alpha\beta}\, e^\alpha{}_A e^\beta{}_B = \Omega_{AB}$$

$$ds^2\big|_\Sigma = \Omega_{AB}\, d\theta^A d\theta^B$$

$$\partial\theta^A)_\lambda$$

$$\boxed{K_\alpha e^\alpha{}_A = 0} \qquad \boxed{K_\alpha K^\alpha = 0}$$



Second null vector :
$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \\ N_\alpha e^\alpha{}_A = 0 \end{cases}$$

Induced metric :

$$h_{\lambda\lambda} = g_{\alpha\beta}\, e^\alpha{}_\lambda e^\beta{}_\lambda = g_{\alpha\beta} K^\alpha K^\beta = 0$$

$$h_{\lambda A} = g_{\alpha\beta}\, e^\alpha{}_\lambda e^\beta{}_A = K_\alpha e^\alpha{}_A = 0$$

$$h_{AB} = g_{\alpha\beta}\, e^\alpha{}_A e^\beta{}_B = \Omega_{AB}$$

$$\boxed{ds^2\big|_\Sigma = \Omega_{AB}\, d\theta^A d\theta^B}$$

→ transverse, 2D metric

Quick Camera: [IPEVO V4K]

... cone in Schwarzschild

$$ds^2 = -f\,dt^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

light cones : $u = t - r^* = t - \int \frac{dr}{f} = \text{cst}$

$$k_\alpha = -\nabla_\alpha u = \left(-1, \frac{1}{f}, 0, 0\right)$$

$$k^\alpha = (f, 1, 0, 0) \qquad \frac{dt}{d\lambda} = f$$

$$y^a = (\lambda \equiv r, \theta, \varphi) \qquad \frac{dr}{d\lambda} = 1 \qquad (\lambda \equiv r)$$

$$x^\alpha(y^a) : \begin{cases} t = u + \int \frac{dr}{f} \\ r = r \\ \theta = \theta \\ \varphi = \varphi \end{cases} \qquad \frac{d\theta}{d\lambda} = 0 = \frac{d\varphi}{d\lambda}$$

Quick Camera: [IPEVO V4K]

$$\begin{cases} r = r \\ \theta = \theta \\ \varphi = \varphi \end{cases}$$

$$e_r^\alpha = \frac{\partial x^\alpha}{\partial r} = \left(\tfrac{1}{f}, 1, 0, 0\right) = k^\alpha$$

$$e_\theta^\alpha = \frac{\partial x^\alpha}{\partial \theta} = (0, 0, 1, 0)$$

$$e_\varphi^\alpha = \frac{\partial x^\alpha}{\partial \varphi} = (0, 0, 0, 1)$$

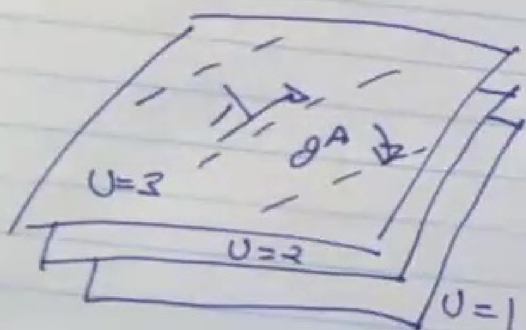
$$\Omega_{AB} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

Quick Camera: [IPEVO V4K]

Spacetime metric

→ neighborhood of the foliation

→ construct x^α



$$\boxed{x^\alpha \stackrel{*}{=} (U, \lambda, \theta^A)}$$

$$\begin{cases} U = \text{const on each null hypersurface} \\ \lambda = \text{parameter on each generator} \\ \theta^A = \text{const on each generator} \end{cases}$$

Quick Camera: [IPEVO V4K]

In the coordinates :

$$\cancel{K_\alpha \stackrel{*}{=} (1,0,0,0)}$$

$$\boxed{K_\alpha \stackrel{*}{=} (-e^{x},0,0,0)}$$

$$\boxed{K^\alpha \stackrel{*}{=} (0,1,0,0)}$$

$$\boxed{\begin{aligned} e^\alpha_\theta &= (0,0,1,0) \\ e^\alpha_\phi &= (0,0,0,1) \end{aligned}}$$

Quick Camera: [IPEVO V4K]

In the coordinates : $\cancel{K_\alpha \stackrel{*}{=} (1,0,0,0)}$

$$\boxed{K_\alpha \stackrel{*}{=} (-e^x, 0, 0, 0)}$$

$$\boxed{K^\alpha \stackrel{*}{=} (0, 1, 0, 0)}$$

$$\boxed{\begin{aligned} e^\alpha_\theta &= (0,0,1,0) \\ e^\alpha_\varphi &= (0,0,0,1) \end{aligned}}$$

Consequences : $K^\alpha = g^{\alpha\beta} K_\beta$

& $0 = K^u = g^{u\beta} K_\beta = -e^x g^{uu}$

Quick Camera: [IPEVO V4K]

In the coordinates :

$$\cancel{K_\alpha \triangleq (1,0,0,0)}$$

$$\boxed{K_\alpha \triangleq (-e^{x},0,0,0)}$$

$$\boxed{K^\alpha \triangleq (0,1,0,0)}$$

$$\boxed{\begin{aligned} e^\alpha_\theta &= (0,0,1,0) \\ e^\alpha_\varphi &= (0,0,0,1) \end{aligned}}$$

Consequences :

$$K^\alpha = g^{\alpha\beta} K_\beta$$

$$\Rightarrow \quad 0 = K^{v} = g^{v\beta} K_\beta = -e^{x} g^{vv} \quad \Rightarrow \quad g^{vv} = 0$$

$$1 = K^\lambda = g^{\lambda\beta} K_\beta = -e^{x} g^{v\lambda}$$

Quick Camera: [IPEVO V4K]

In the coordinates : $\cancel{K_\alpha \stackrel{*}{=} (1,0,0,0)}$

$$\boxed{K_\alpha \stackrel{*}{=} (-e^{X},0,0,0)}$$

$$\boxed{K^\alpha \stackrel{*}{=} (0,1,0,0)} \qquad \boxed{\begin{aligned} e^\alpha_\theta &= (0,0,1,0) \\ e^\alpha_\varphi &= (0,0,0,1) \end{aligned}}$$

Consequences : $K^\alpha = g^{\alpha\beta} K_\beta$

$$\Rightarrow \quad 0 = K^U = g^{U\beta} K_\beta = -e^{X} g^{UU} \quad \Rightarrow \quad g^{UU} = 0$$

$$1 = K^\lambda = g^{\lambda\beta} K_\beta = -e^{X} g^{U\lambda} \quad \Rightarrow \quad g^{U\lambda} = -e^{-X}$$

$$0 = K^A = -e^{X} g^{UA} \quad \Rightarrow \quad g^{UA} = 0$$

Quick Camera: [IPEVO V4K]

$$(0, 1, 0, 0)$$

$$e^\alpha_\theta = (0,0,1,0)$$
$$e^\alpha_\phi = (0,0,0,1)$$

Consequences:

$$k^\alpha = g^{\alpha\beta} k_\beta$$

$\Rightarrow$

$$0 = k^u = g^{u\beta} k_\beta = -e^{\chi} g^{uu} \Rightarrow g^{uu} = 0$$
$$1 = k^\lambda = g^{\lambda\beta} k_\beta = -e^{\chi} g^{u\lambda} \Rightarrow g^{u\lambda} = -e^{-\chi}$$
$$0 = k^A = -e^{\chi} g^{uA} \Rightarrow g^{uA} = 0$$

Inverse metric:

$$g^{uu} = 0 \qquad g^{\lambda\lambda} = e^{-\chi} V$$
$$g^{u\lambda} = -e^{-\chi} \qquad g^{\lambda A} = e^{-\chi} W^A$$
$$g^{uA} = 0 \qquad g^{AB} = \Omega^{AB}$$

Quick Camera: [IPEVO V4K]

$$e_\theta = (0,0,1,0)$$
$$e^\alpha_\phi = (0,0,0,1)$$

Consequences :

$$K^\alpha = g^{\alpha\beta} k_\beta$$

$$\Rightarrow \quad 0 = k^u = g^{u\beta} K_\beta = -e^x g^{uu} \Rightarrow g^{uu} = 0$$
$$1 = k^\lambda = g^{\lambda\beta} K_\beta = -e^x g^{u\lambda} \Rightarrow g^{u\lambda} = -e^{-x}$$
$$0 = k^A = -e^x g^{uA} \Rightarrow g^{uA} = 0$$

Inverse metric :

$$\boxed{\begin{array}{ll} g^{uu} = 0 & g^{\lambda\lambda} = e^{-x} V \\ g^{u\lambda} = -e^{-x} & g^{\lambda A} = e^{-x} W^A \\ g^{uA} = 0 & g^{AB} = \Omega^{AB} = \text{matrix inverse to } \Omega_{AB} \end{array}}$$

Quick Camera: [IPEVO V4K]

Invert the 4x4 matrix $g^{\alpha\beta} \rightarrow g_{\alpha\beta}$

$$\boxed{\begin{aligned} g_{uu} &= -e^{\chi} V + \Omega_{AB} W^A W^B & g_{\lambda\lambda} &= 0 \\ g_{u\lambda} &= -e^{\chi} & g_{\lambda A} &= 0 \\ g_{uA} &= \Omega_{AB} W^B \equiv W_A & g_{AB} &= \Omega_{AB} \end{aligned}}$$

$$\boxed{\begin{aligned} ds^2 = &- e^{\chi} V du^2 - 2e^{\chi} du\, d\lambda \\ &+ \Omega_{AB} (d\theta^A + W^A du) (d\theta^B + W^B du) \end{aligned}}$$

Quick Camera: [IPEVO V4K]

$$g \to g_{\alpha\beta}$$

$$\boxed{\begin{aligned} g_{uu} &= -e^{\chi} V + \Omega_{AB} W^A W^B & g_{\lambda\lambda} &= 0 \\ g_{u\lambda} &= -e^{\chi} & g_{\lambda A} &= 0 \\ g_{uA} &= \Omega_{AB} W^B \equiv W_A & g_{AB} &= \Omega_{AB} \end{aligned}}$$

$$\boxed{\begin{aligned} ds^2 = &-e^{\chi} V du^2 - 2e^{\chi} du\, d\lambda \\ &+ \Omega_{AB} (d\theta^A + W^A du)(d\theta^B + W^B du) \end{aligned}}$$

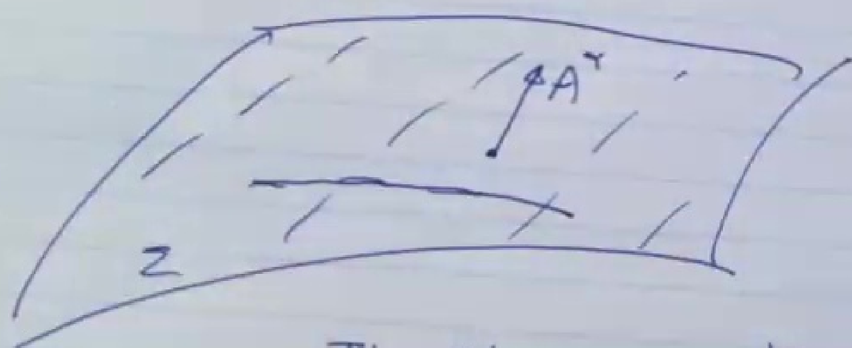
$$\det(g_{\alpha\beta}) \overset{*}{=} -e^{2\chi} \det(\Omega_{AB})$$

$$\boxed{\sqrt{-g} \overset{*}{=} e^{\chi} \sqrt{\Omega}}$$

$$\sqrt{-g} = e \quad \ldots$$

$$ds^2\Big|_\Sigma \overset{du=0}{=} \Omega_{AB}\, d\theta^A d\theta^B$$

Integration on null hypersurface



$$\int_\Sigma A^\alpha d\Sigma_\alpha \overset{?}{=}$$

$$\text{TL, SL} \rightarrow \quad d\Sigma_\alpha = \underbrace{\varepsilon n_\alpha d\Sigma}$$

Quick Camera: [IPEVO V4K]

More primitive :
$$d\Sigma_\alpha = \varepsilon_{\alpha\beta\gamma\delta}\, e^\beta_1 e^\gamma_2 e^\delta_3\, dy^1 dy^2 dy^3$$

$$y^1 = \lambda, \quad e^\beta_1 = k^\beta$$

$$y^2, y^3 \to \theta^2, \theta^3$$

$$\boxed{\begin{aligned} d\Sigma_\alpha &= \varepsilon_{\alpha\beta\gamma\delta}\, k^\beta e^\gamma_2 e^\delta_3\, d\lambda\, d^2\theta \\ &= dS_{\alpha\beta}\, k^\beta d\lambda \\ dS_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma\delta}\, e^\gamma_2 e^\delta_3\, d^2\theta \end{aligned}}$$

$\hookrightarrow$ surface element on cross-sectional surfaces

Quick Camera: [IPEVO V4K]

... surfaces

...ecialize to (v, λ, θ^A) coordinates.

$$d\Sigma_\alpha \stackrel{*}{=} e^{\chi} \sqrt{\Omega}\, [\alpha \beta \gamma \delta]\, k^\beta e^\gamma_2 e^\delta_3 \, d\lambda\, d^2\theta$$

$$\stackrel{*}{=} e^{\chi} \sqrt{\Omega}\, \underbrace{[\alpha \lambda 2 3]}_{\text{nonzero unless } \alpha = 0 = v} d\lambda\, d^2\theta$$

only nonvanishing component $=$ $d\Sigma_v = (e^{\chi} d\lambda) \sqrt{\Omega}\, d^2\theta$

Quick Camera: [IPEVO V4K]

$$d\Sigma_\alpha \overset{*}{=} e^{\chi}\sqrt{\Omega}\,[\alpha\,\beta\,\gamma\,\delta]\,k^\rho e^\gamma_2 e^\delta_3\, d\lambda\, d^2\theta$$

$$\overset{*}{=} e^{\chi}\sqrt{\Omega}\,\underbrace{[\alpha\,\lambda\,2\,3]}_{\text{zero unless } \alpha \,\equiv\, 0 \,\equiv\, v}\, d\lambda\, d^2\theta$$

only nonvanishing component $= \; d\Sigma_v = (e^{\chi} d\lambda)\sqrt{\Omega}\, d^2\theta$

$$k_v = -e^{\chi}$$

$$d\Sigma_\alpha \overset{*}{=} (-k_\alpha d\lambda)\underbrace{\sqrt{\Omega}\, d^2\theta}_{dS}$$

Equality holds in any coordinate system

$$\boxed{d\Sigma_\alpha = -k_\alpha\, d\lambda\, dS}$$

$$dS = \sqrt{\Omega}\, d^2\theta$$

Quick Camera: [IPEVO V4K]

$$dS_{\alpha\beta} \stackrel{*}{=} e^{\chi}\sqrt{\Omega}\,\underbrace{[\alpha\,\beta\,2\,3]}_{\text{zero unless } \alpha,\beta \neq 2\;3}\,d^2\theta$$

$\alpha, \beta \to u, \lambda$

$$dS_{u\lambda} \stackrel{*}{=} e^{\chi}\sqrt{\Omega}\;d^2\theta$$

$$k_u = -e^{\chi} \qquad N_\lambda = -1$$

$$dS_{u\lambda} \stackrel{*}{=} k_u N_\lambda\, dS$$

$$dS_{\lambda u} \stackrel{*}{=} -\,dS_{u\lambda} = -\,k_u N_\lambda\, dS$$

$$dS_{\alpha\beta} \stackrel{*}{=} \cancel{2(k_\alpha}$$

$$\stackrel{*}{=} 2\,k_{[\alpha}N_{\beta]}\,dS$$

Quick Camera: [IPEVO V4K]

$$dS_{\alpha\beta} \doteq \cancel{2\,[K_{\alpha}}$$

$$\overset{*}{=} 2\, K_{[\alpha} N_{\beta]}\, dS$$

$$\boxed{dS_{\alpha\beta} = 2\, K_{[\alpha} N_{\beta]}\, dS}$$

$$K_{\alpha} \overset{*}{=} (-e^{\chi}, 0, 0, 0)$$

$$K^{\alpha} \overset{*}{=} (0, 1, 0, 0) \qquad e^{\alpha}_{\theta} \overset{*}{=} (0, 0, 1, 0) \qquad e^{\alpha}_{\varphi} \overset{*}{=} (0, 0, 0, 1)$$

$$N_{\alpha} N^{\alpha} = 0\,, \quad N_{\alpha} K^{\alpha} = -1\,, \quad N_{\alpha} e^{\alpha}_{A} = 0$$

$$N_{\alpha} \overset{*}{=} \left(-\tfrac{1}{2} V, -1, 0, 0 \right)$$