Title: Advanced General Relativity - 240306

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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HYPERSURFACES — EXTRINSIC CURVATURE

↳ spacelike, timelike

Gaussian coordinates

$$ds^2 = \varepsilon\, d\ell^2 + g_{ab}(\ell, y)\, dy^a dy^b$$

ℓ = proper distance (or time)

$$\varepsilon = \begin{cases} -1 & \Sigma \text{ spacelike} \\ +1 & \Sigma \text{ timelike} \end{cases}$$

$h_{ab} = g_{ab}(\ell = 0)$

↳ induced metric

Intrinsic geometry of Σ

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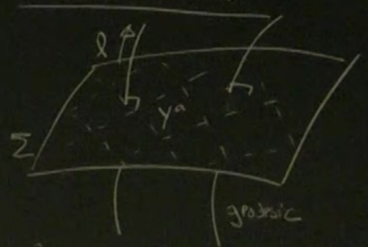

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Intrinsic geometry of Σ

— $h_{ab} \rightarrow D_a \rightarrow R^a{}_{bcd}$

Extrinsic geometry (how surface bends)

— K_{ab} (extrinsic curvature)

geodesic

$$ds^2 = \varepsilon\, dl^2 + g_{ab}(l,y)\, dy^a dy^b$$

$l \equiv$ proper distance (or time)

$$\varepsilon = \begin{cases} -1 & \Sigma \text{ spacelike} \\ +1 & \Sigma \text{ timelike} \end{cases} \qquad h_{ab} = g_{ab}(l=0)$$

induced metric

distances $\downarrow \Rightarrow h_{ab} \downarrow$

concave

distances $\uparrow \Rightarrow h_{ab} \uparrow$

convex

"bending" $\sim \partial_l h_{ab} \sim K_{ab}$

Def: $$\boxed{K_{ab} = \frac{1}{2} \partial_l g_{ab} \Big|_{l=0}}$$ (defined in terms of Gaussian coordinates)

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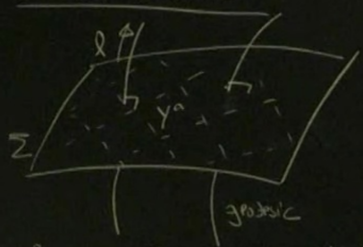

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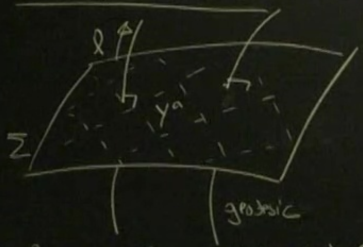
$$\Gamma^a_{bc} \equiv \Gamma^a_{bc}[h] \quad (3D) \quad \text{vs} \quad {}^4\Gamma^\alpha_{\beta\gamma}[g]$$



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Extrinsic geometry (how surface bends)

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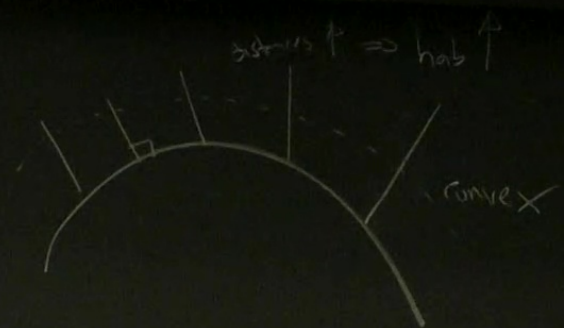
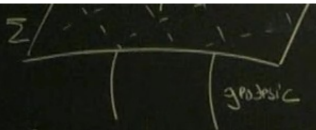
$$\Gamma^a_{bc} \equiv \Gamma^a_{bc}[h] \quad (3D) \quad \text{vs} \quad {}^4\Gamma^\alpha_{\beta\gamma}[g]$$

$$R^a{}_{bcd} \equiv R^a{}_{bcd}[h] \quad \text{vs} \quad {}^4R^\alpha{}_{\beta\gamma\delta}[g]$$

$$R_{ab} \equiv R_{ab}[h] \qquad {}^4R_{\alpha\beta}[g]$$

$$D_a[h] \quad \text{vs} \quad \nabla_\alpha[g]$$





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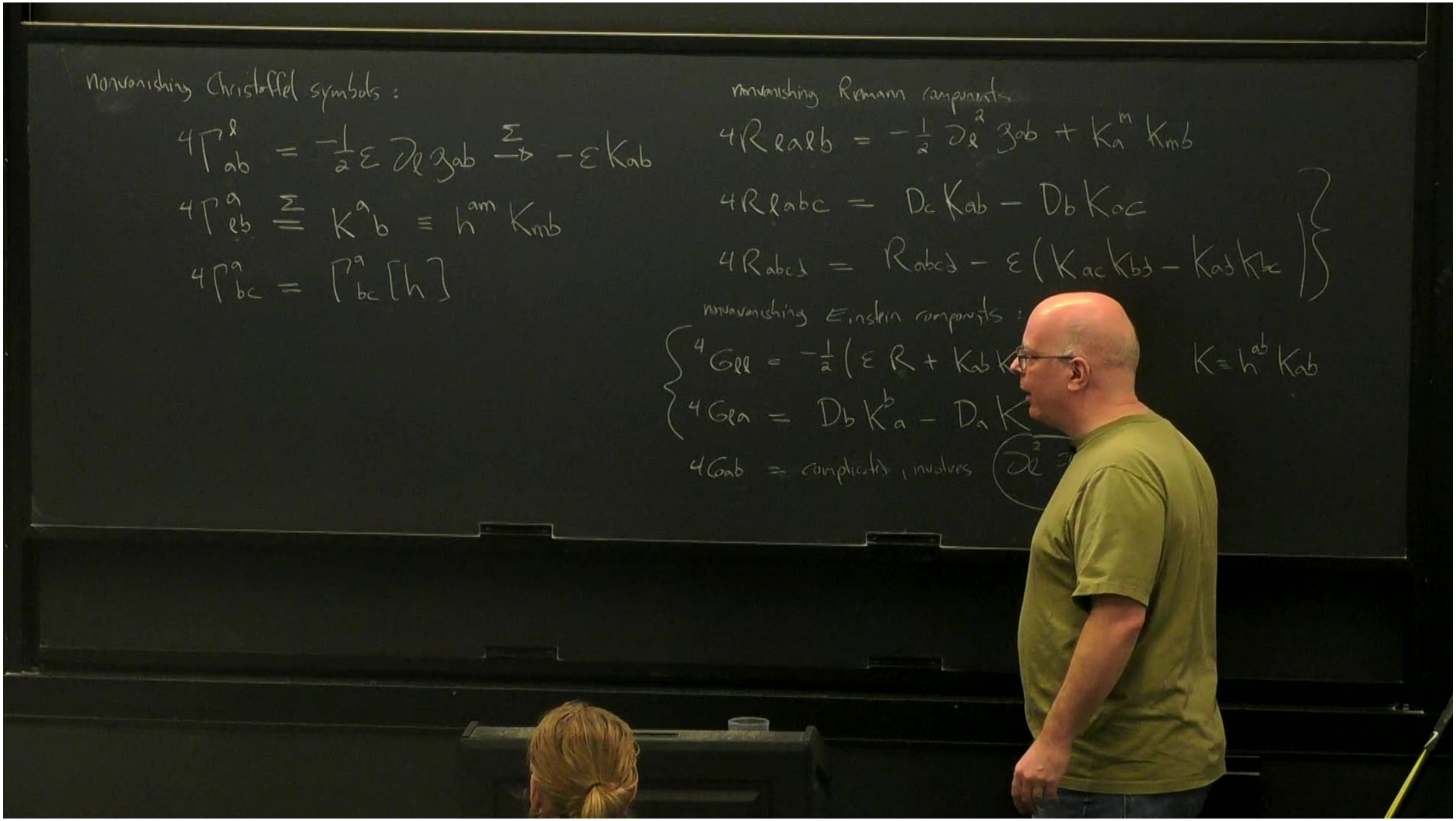
concave

distances $\uparrow \Rightarrow h_{ab} \uparrow$

Def: $\boxed{K_{ab} = \frac{1}{2} \partial_\ell g_{ab} \Big|_{\ell=0}}$ (defined in terms of Gaussian coordinates)


nonvanishing Christoffel symbols :
${}^4\Gamma^\ell_{ab} = -\frac{1}{2}\varepsilon\,\partial_\ell g_{ab} \overset{\Sigma}{\rightarrow} -\varepsilon K_{ab}$
${}^4\Gamma^a_{\ell b} \overset{\Sigma}{=} K^a{}_b \equiv h^{am} K_{mb}$
${}^4\Gamma^a_{bc} = \Gamma^a_{bc}[h]$
nonvanishing Riemann components
${}^4R_{\ell a \ell b} = -\frac{1}{2}\partial_\ell^2 g_{ab} + K_a{}^m K_{mb}$
${}^4R_{\ell abc} = D_c K_{ab} - D_b K_{ac}$
${}^4R_{abcd} = R_{abcd} - \varepsilon(K_{ac}K_{bd} - K_{ad}K_{bc})$
nonvanishing Einstein components :
${}^4G_{\ell\ell} = -\frac{1}{2}(\varepsilon R + K_{ab}K$
$K = h^{ab}K_{ab}$
${}^4G_{\ell a} = D_b K^b{}_a - D_a K$
${}^4G_{ab}$ = complicated, involves ∂_ℓ^2

nonvanishing Christoffel symbols:

$$^4\Gamma^\ell_{ab} = -\frac{1}{2}\varepsilon\, \partial_\ell g_{ab} \xrightarrow{\Sigma} -\varepsilon K_{ab}$$

$$^4\Gamma^a_{\ell b} \stackrel{\Sigma}{=} K^a{}_b \equiv h^{am} K_{mb}$$

$$^4\Gamma^a_{bc} = \Gamma^a_{bc}[h]$$

nonvanishing Riemann components

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$$^4R_{\ell abc} = D_c K_{ab} - D_b K_{ac}$$

$$^4R_{abcd} = R_{abcd} - \varepsilon\left(K_{ac}K_{bd} - K_{ad}K_{bc}\right)$$

nonvanishing Einstein components:

$$^4G_{\ell\ell} = -\frac{1}{2}\left(\varepsilon R + K_{ab}K^{ab} - K^2\right) \qquad K = h^{ab}K_{ab}$$

$$^4G_{\ell a} = D_b K^b{}_a - D_a K$$

$^4G_{ab}$ = complicated, involves $\partial_\ell^2 g_{ab}$

$$G_{\ell\ell} = -\tfrac{1}{2}\left(\varepsilon R + K_{ab}K^{ab} - K^2\right) \qquad K = h^{ab}K_{ab}$$

$$^4G_{\ell a} = D_b K^b{}_a - D_a K$$

$^4G_{ab}$ = complicated, involves $\partial_\ell^2 g_{ab}$

Covariant definition of K_{ab}

Claim: $\boxed{K_{ab} = e^\alpha{}_a e^\beta{}_b \nabla_\alpha n_\beta}$

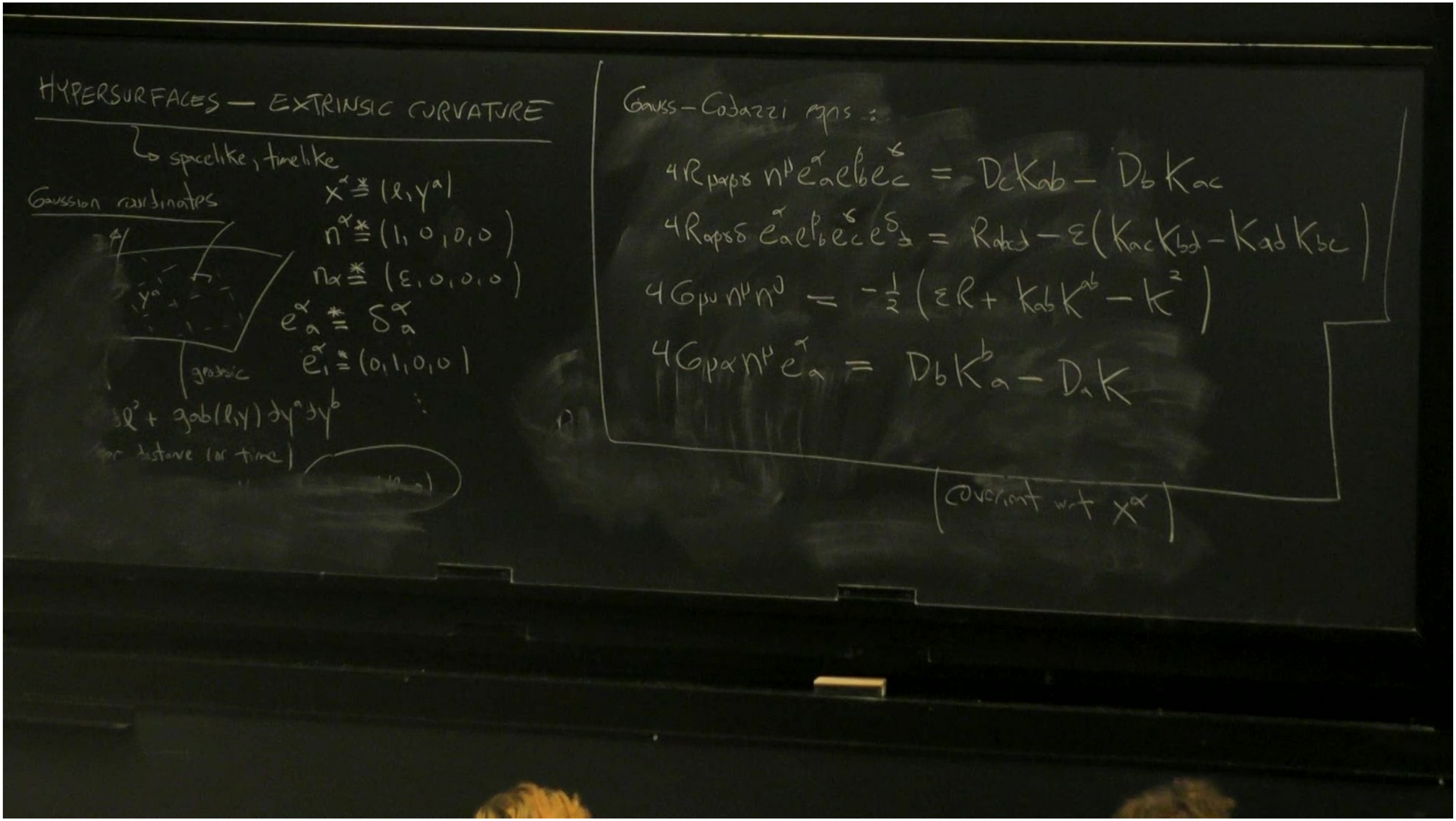
4D: scalars (same in all coordinate systems x^α)

3D: tensor (symmetric)

Gaussian coordinates: $K_{ab} \overset{*}{=} \nabla_a n_b = \left(\cancel{\partial_a n_b} - {}^4\Gamma^\alpha_{ab}\, n_\alpha\right) = -\varepsilon\, {}^4\Gamma^\ell_{ab}$

$$\overset{*}{=} (-\varepsilon)(-\varepsilon K_{ab}) = K_{ab}$$ (equality in Gaussian coordinates)

scalars $\rightarrow$ equality in all coordinates.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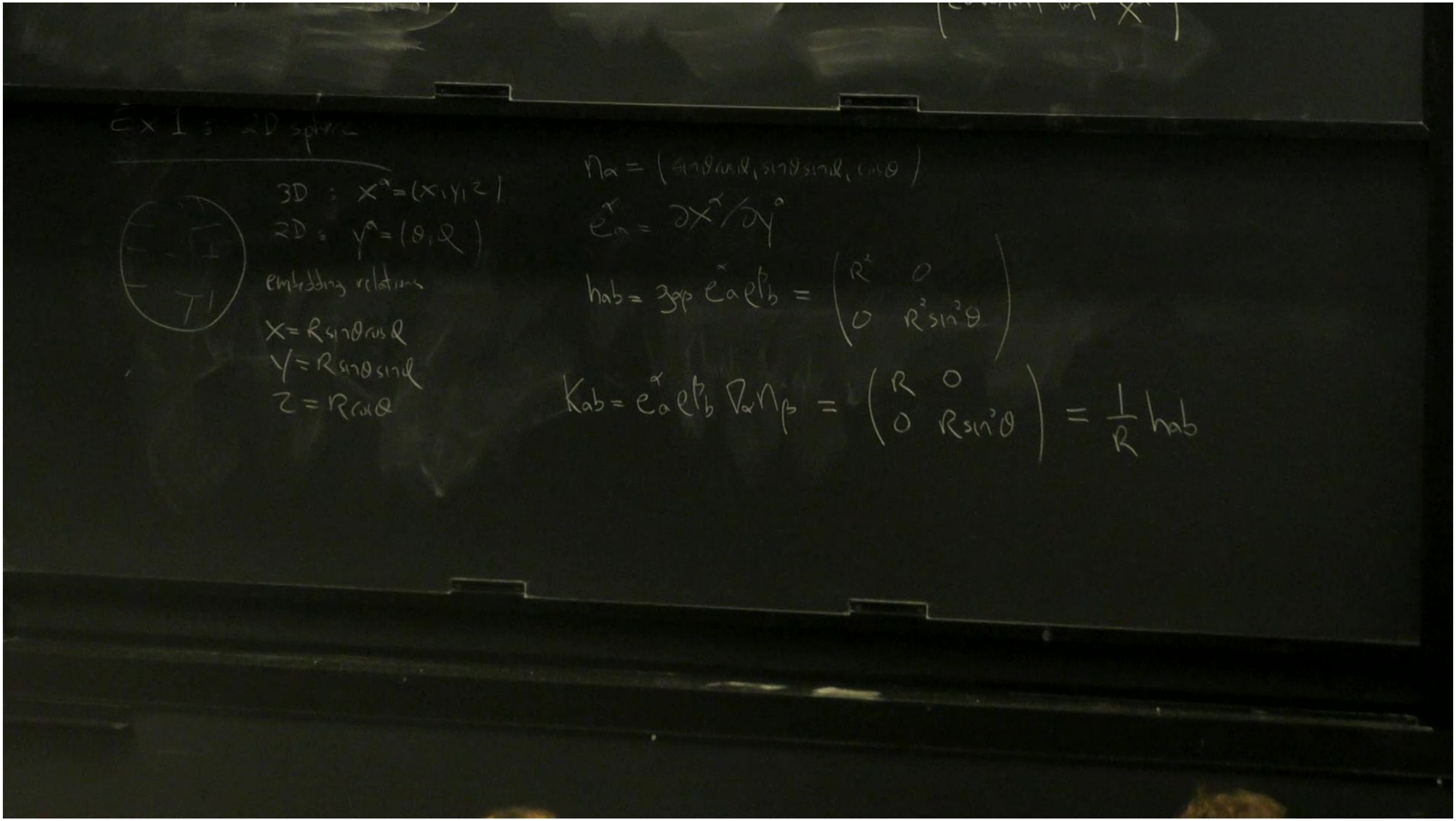
Gaussian coordinates
$x^\alpha \stackrel{*}{=} (\ell, y^a)$
$n^\alpha \stackrel{*}{=} (1,0,0,0)$
$n_\alpha \stackrel{*}{=} (\varepsilon,0,0,0)$
$e^\alpha_a \stackrel{*}{=} \delta^\alpha_a$
$e^\alpha_1 \stackrel{*}{=} (0,1,0,0)$
y^a
geodesic
$d\ell^2 + g_{ab}(\ell,y)\,dy^a dy^b$
distance (or time)
Gauss–Codazzi eqns:
${}^4R_{\mu\alpha\beta\gamma}\, n^\mu e^\alpha_a e^\beta_b e^\gamma_c = D_c K_{ab} - D_b K_{ac}$
${}^4R_{\alpha\beta\gamma\delta}\, e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} - \varepsilon(K_{ac}K_{bd} - K_{ad}K_{bc})$
${}^4G_{\mu\nu}\, n^\mu n^\nu = -\frac{1}{2}(\varepsilon R + K_{ab}K^{ab} - K^2)$
${}^4G_{\mu\alpha}\, n^\mu e^\alpha_a = D_b K^b_a - D_a K$
(covariant wrt x^α)



Ex 1 : 2D sphere
3D : $X^\alpha = (x, y, z)$
2D : $y^a = (\theta, \varphi)$
embedding relations
$x = R\sin\theta\cos\varphi$
$y = R\sin\theta\sin\varphi$
$z = R\cos\theta$
$n_\alpha = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$
$e^\alpha_a = \partial X^\alpha / \partial y^a$
$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$
$K_{ab} = e^\alpha_a e^\beta_b \nabla_\alpha n_\beta = \begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix} = \frac{1}{R} h_{ab}$

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$x = R\sin\theta\cos\varphi$

$y = R\sin\theta\sin\varphi$

$z = R\cos\theta$

$n_\alpha = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

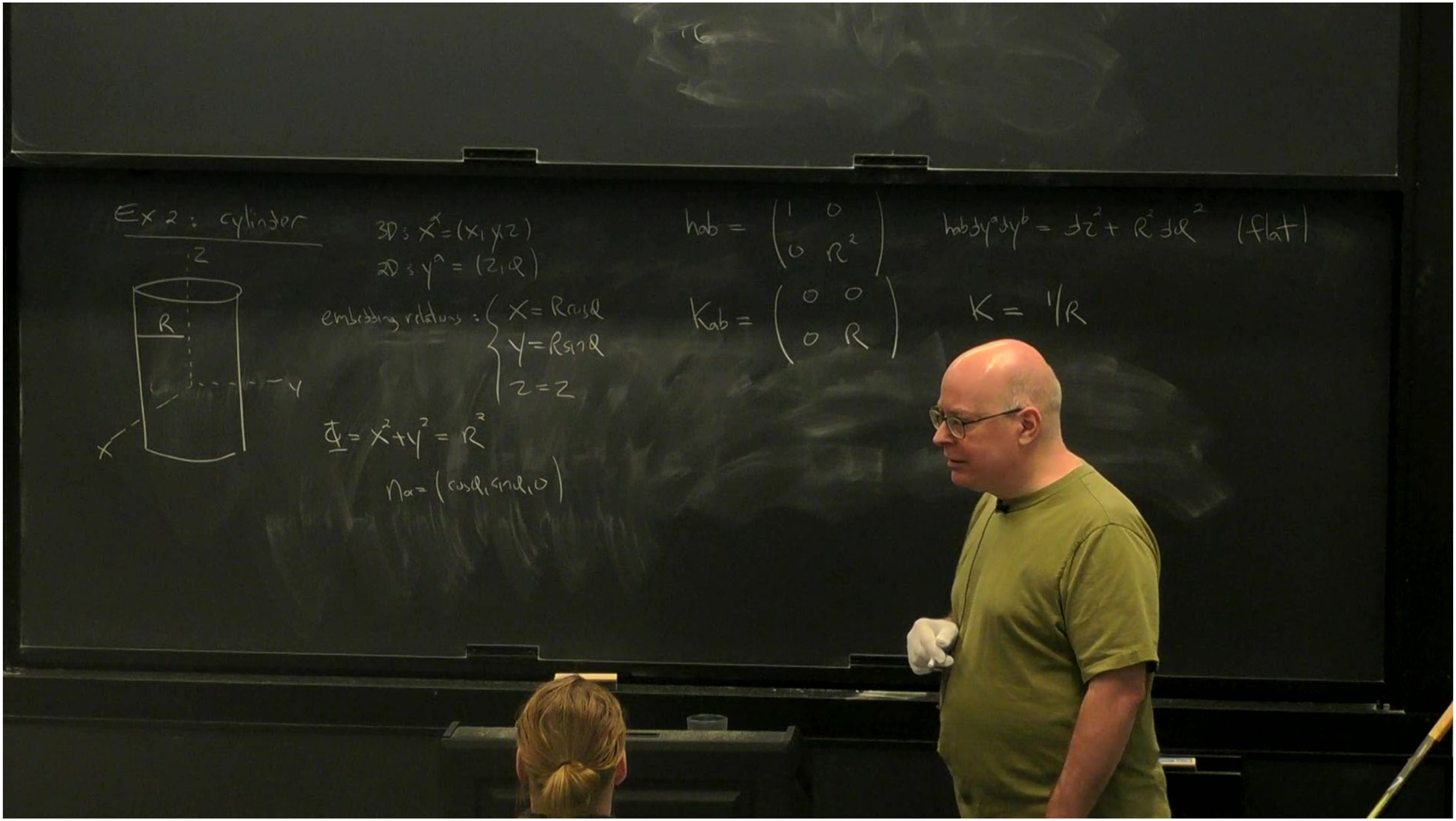
$e^\alpha_a = \partial X^\alpha / \partial y^a$

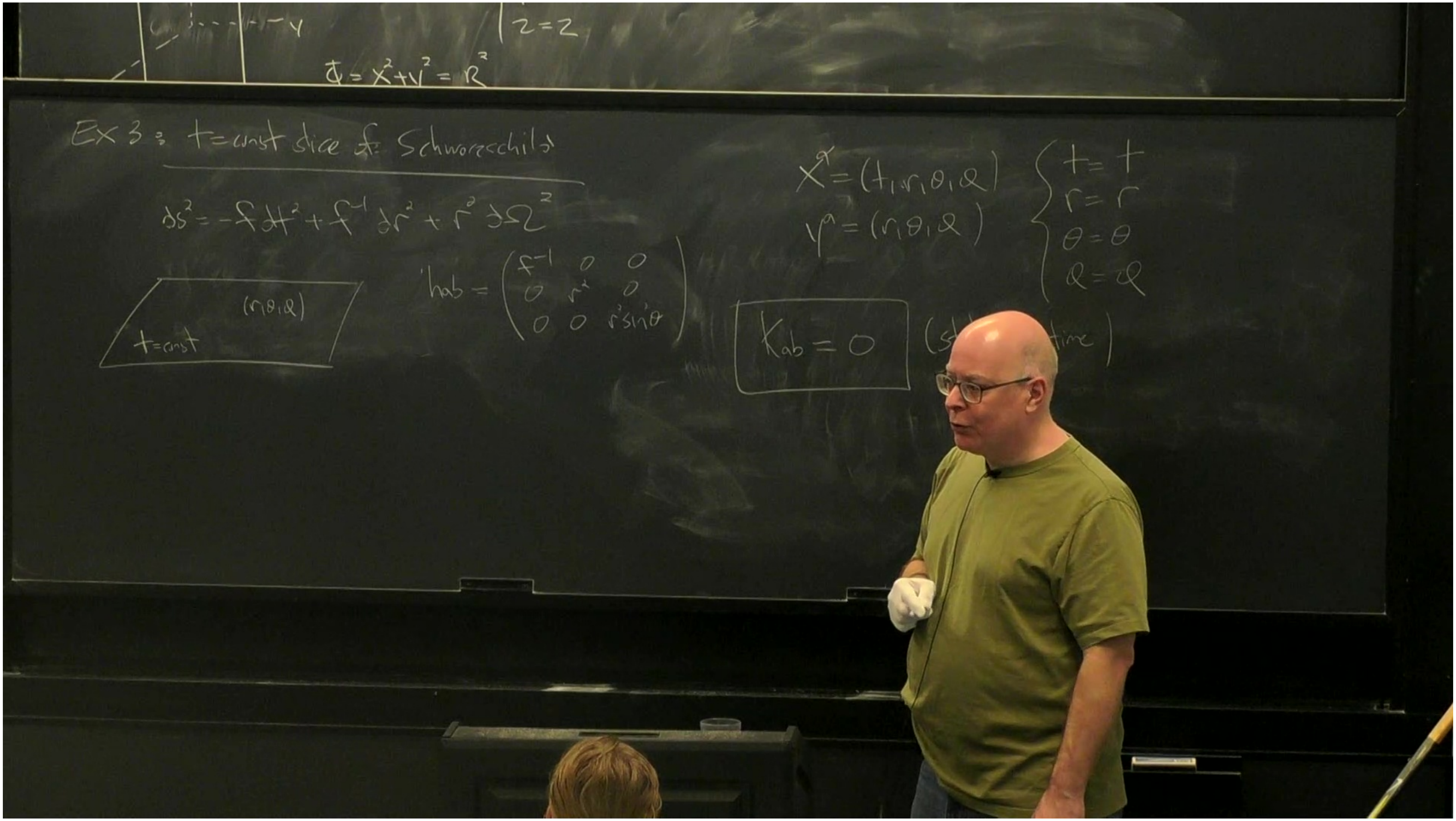
$$h_{ab} = g_{\alpha\beta}\, e^\alpha_a e^\beta_b = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$K_{ab} = e^\alpha_a e^\beta_b \nabla_\alpha n_\beta = \begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix} = \frac{1}{R} h_{ab}$$

$K = 2/R$

check this!


Ex 2 : cylinder
3D : X^α = (x, y, z)
2D : y^a = (z, φ)
embedding relations : X = R cos φ
Y = R sin φ
Z = Z
Φ = X² + Y² = R²
n_α = (cos φ, sin φ, 0)
h_ab = (1 0 / 0 R²)
h_ab dy^a dy^b = dz² + R² dφ² (flat)
K_ab = (0 0 / 0 R)
K = 1/R


z=z
Φ = x²+y² = R²
Ex 3: t=const slice of Schwarzschild
ds² = −f dt² + f⁻¹ dr² + r² dΩ²
(r,θ,φ)
t=const
h_ab = (f⁻¹ 0 0 / 0 r² 0 / 0 0 r² sin θ)
x^α = (t, r, θ, φ)
y^a = (r, θ, φ)
t = t
r = r
θ = θ
φ = φ
K_ab = 0