Title: Strong Gravity Lecture

Speakers: William East

Collection: Strong Gravity 2023/24

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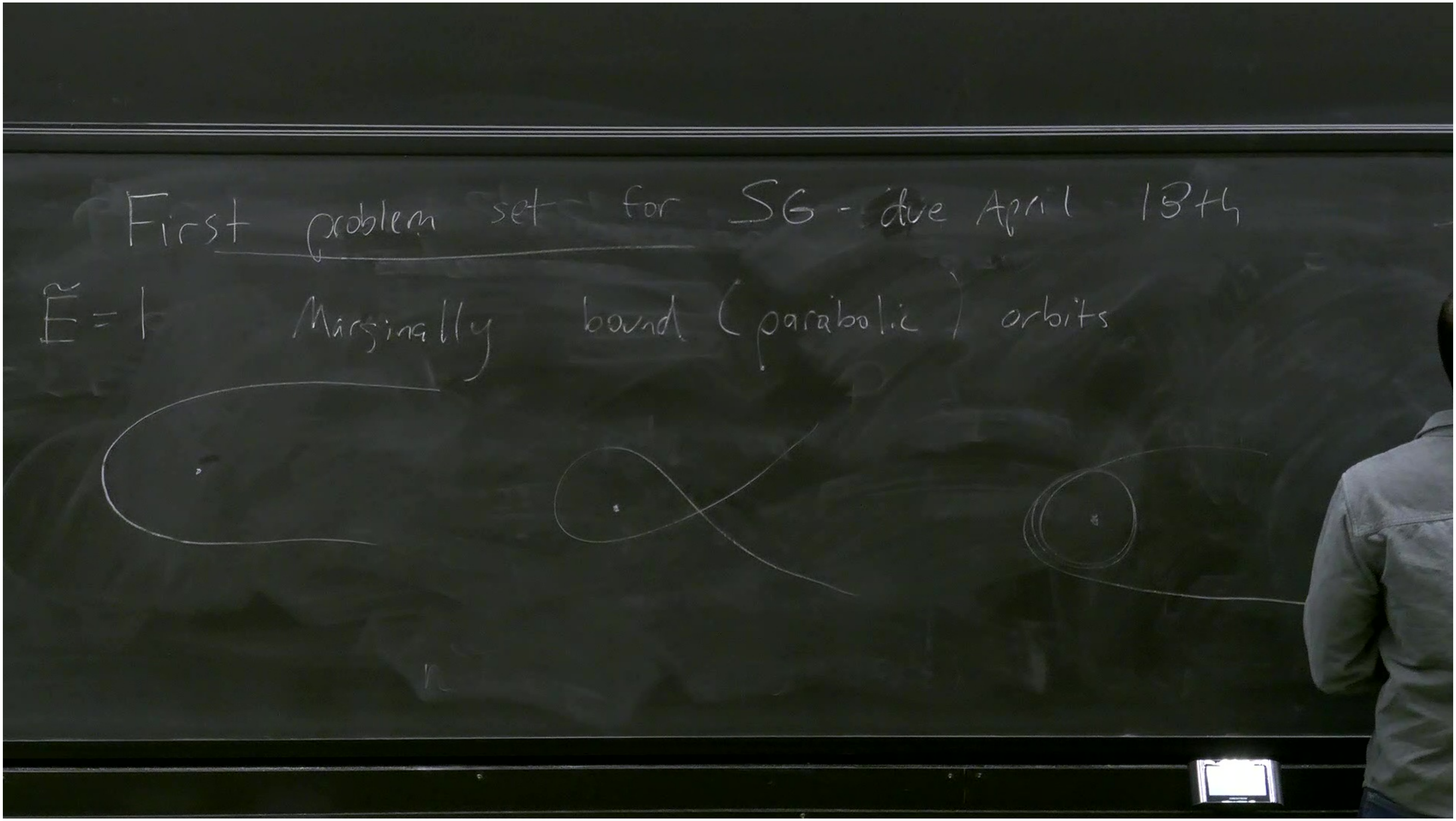
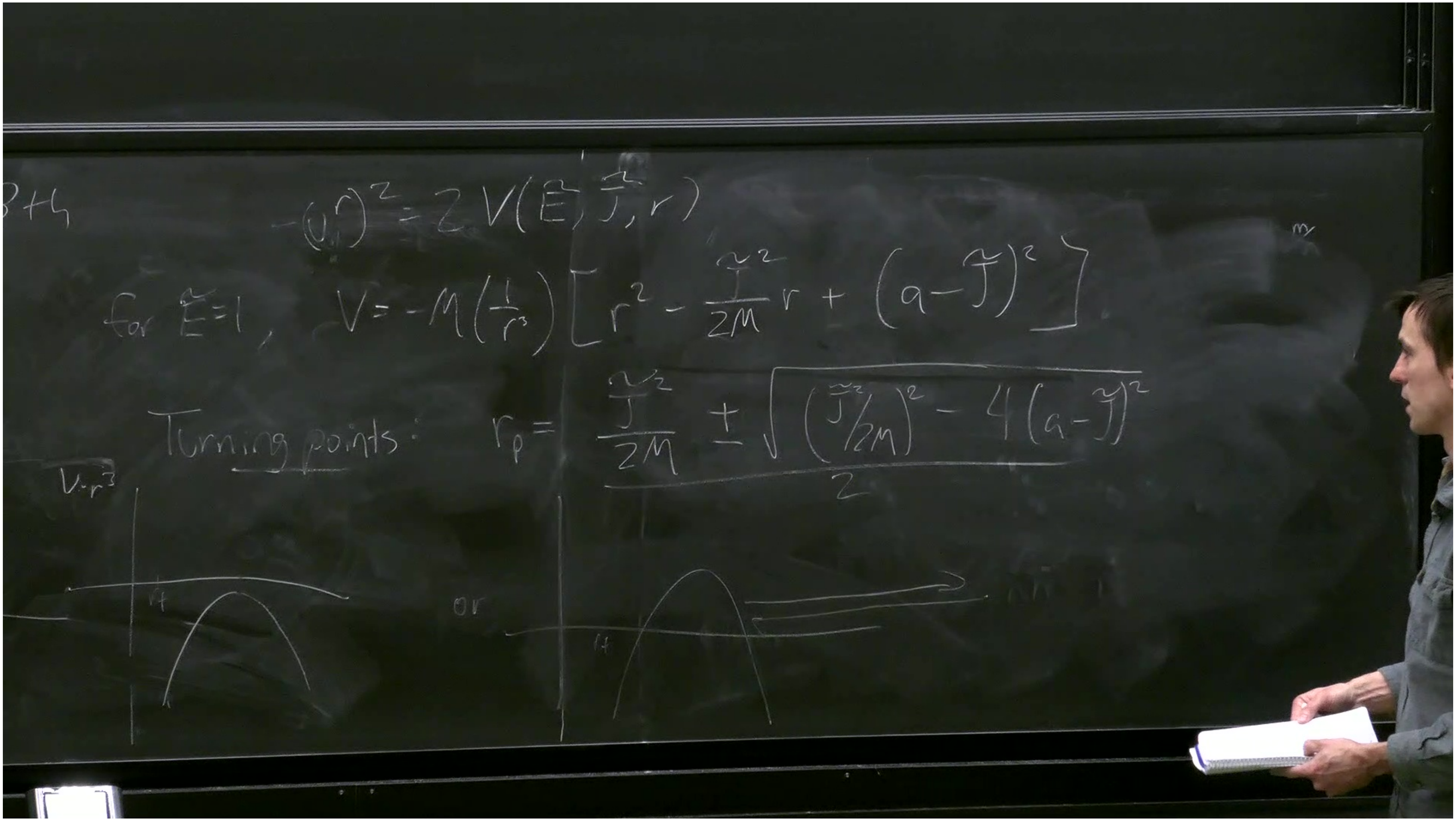
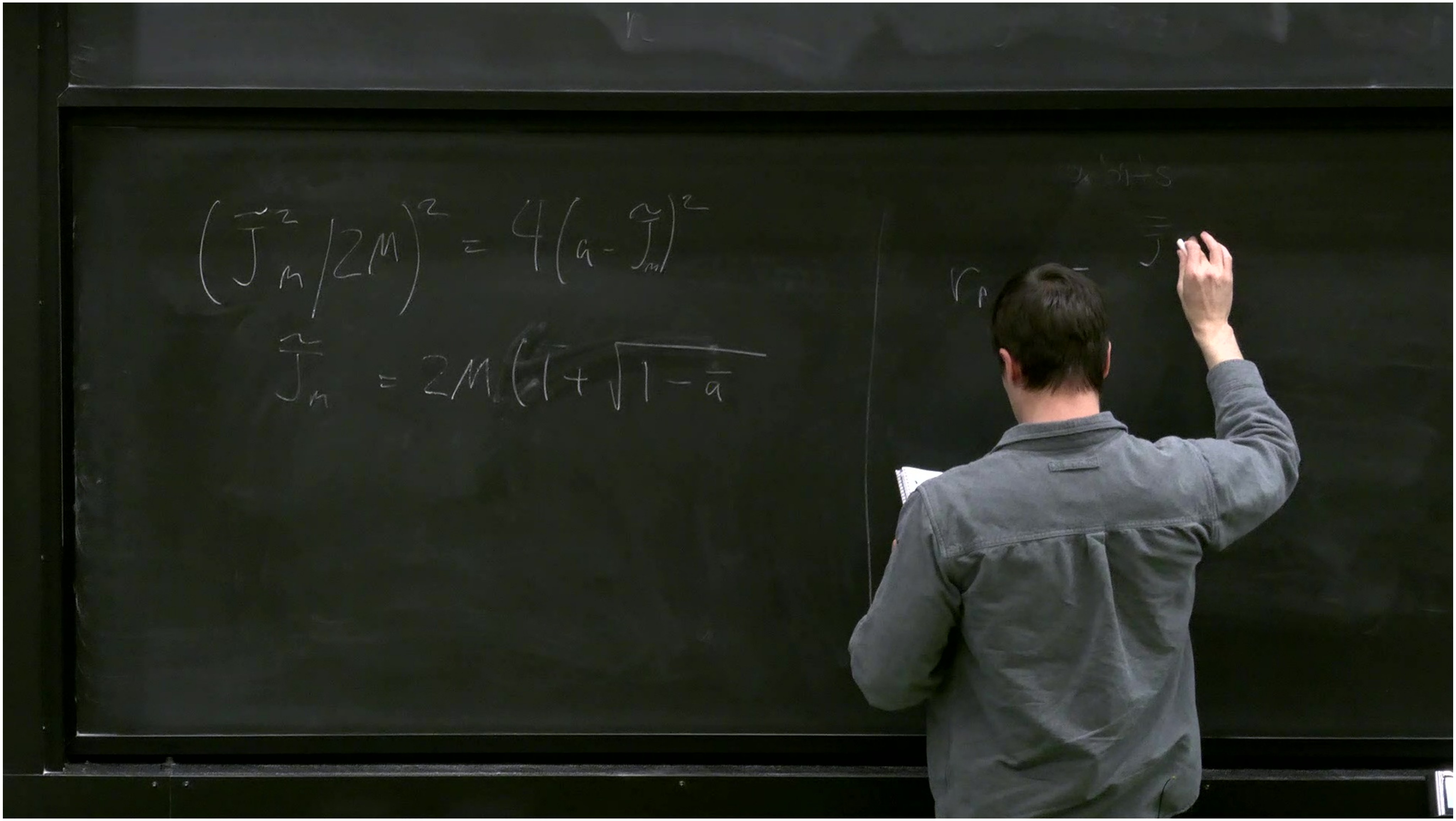
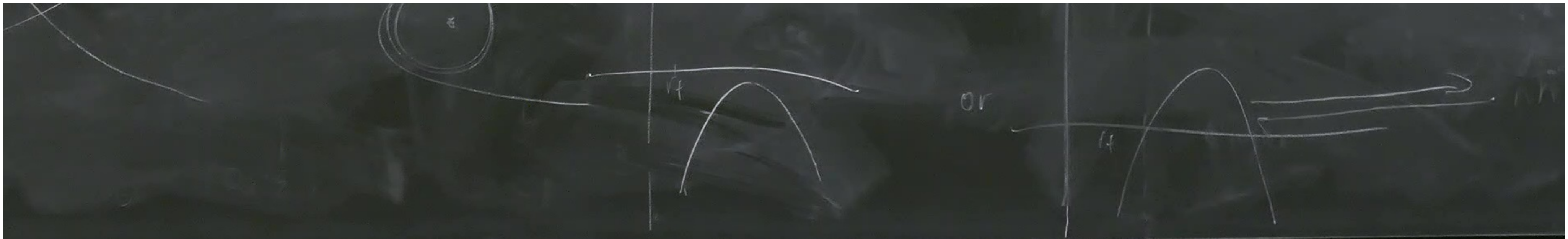
URL: https://pirsa.org/24020102


First problem set for SG - due April 13th
$\tilde{E} = 1$ Marginally bound (parabolic) orbits


$(\partial_\lambda r)^2 = 2V(\tilde{E}, \tilde{J}, r)$
for $\tilde{E}=1$, $V = -M\left(\frac{1}{r^3}\right)\left[r^2 - \frac{\tilde{J}^2}{2M} r + (a-\tilde{J})^2\right]$
Turning points: $r_p = \dfrac{\frac{\tilde{J}^2}{2M} \pm \sqrt{\left(\tilde{J}^2/2M\right)^2 - 4(a-\tilde{J})^2}}{2}$
or


$(\tilde{J}_m^2/2M)^2 = 4(a - \tilde{J}_m)^2$
$\tilde{J}_m = 2M(1 + \sqrt{1 - a}$



orbits

$$r_{p,min} = \frac{\tilde{J}^2}{4M} = M\left(2-\bar{a}+2\sqrt{1-\bar{a}}\right) < 2M \quad , \Rightarrow \quad \begin{array}{l} \bar{a} > 2\sqrt{1-\bar{a}} \\ \bar{a} > -\frac{1}{2}+\sqrt{2} \approx 0.91 \end{array}$$

For $\bar{a}=-1$, $r_p = M(3+2\sqrt{2}) \approx 5.8M$

$\bar{a}=0$, $r_p = 4M$

$\bar{a}=1$, $r_p = M$

$+2\sqrt{1-\bar{a}}) < 2M \quad , \quad \Rightarrow \quad \bar{a} > 2\sqrt{1-\bar{a}}$

$\bar{a} > -\frac{1}{2}+\sqrt{2} \approx 0.91$

$+2\sqrt{2}) \approx 5.8M$

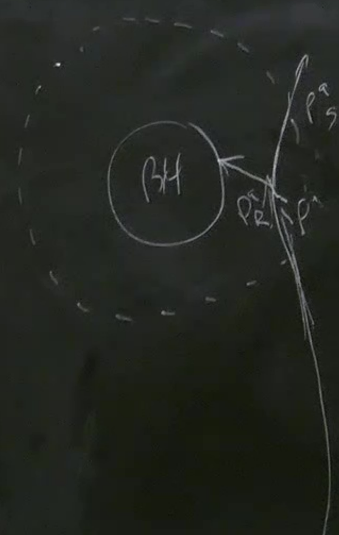
(A) Penrose Process

$p^a = m u^a$

n^a

$E_n = -n_a p^a > 0$

$E_k = -t_a p^a$



① Initial momentum: p^a

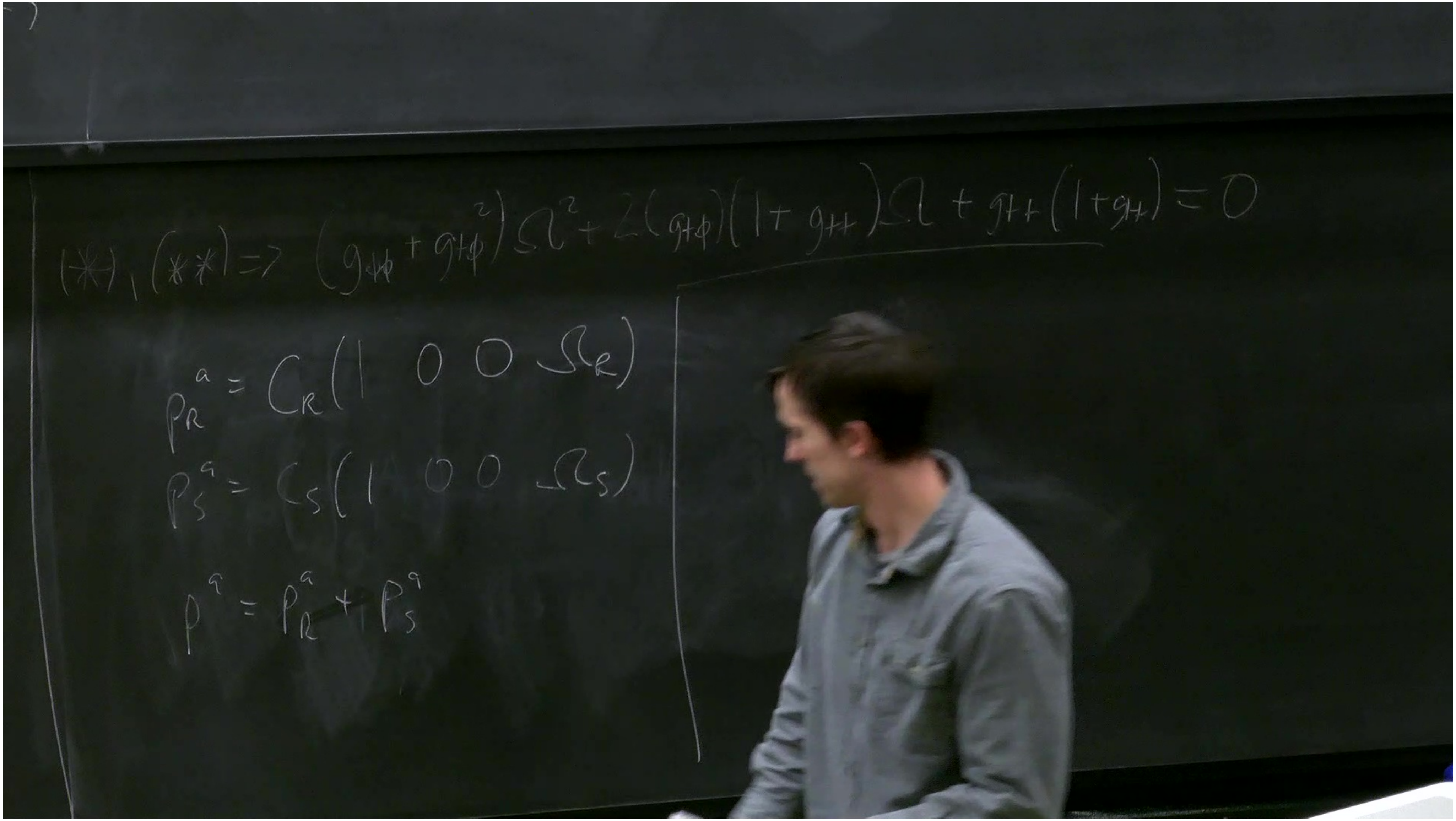
② Split into two:

$p^a = p_S^a + p_R^a$ falls into BH

escapes

$E_R = -t_a p_R^a < 0$

$E_S = E - E_R > E$


$(*), (**) \Rightarrow (g_{\phi\phi} + g_{t\phi}^2)\Omega^2 + 2(g_{t\phi})(1+g_{tt})\Omega + g_{tt}(1+g_{tt}) = 0$
$\rho_R^a = C_R(1 \quad 0 \quad 0 \quad \Omega_R)$
$\rho_S^a = C_S(1 \quad 0 \quad 0 \quad \Omega_S)$
$\rho^a = \rho_R^a + \rho_S^a$

$$\frac{dt}{d\tau} = C_R + C_S$$

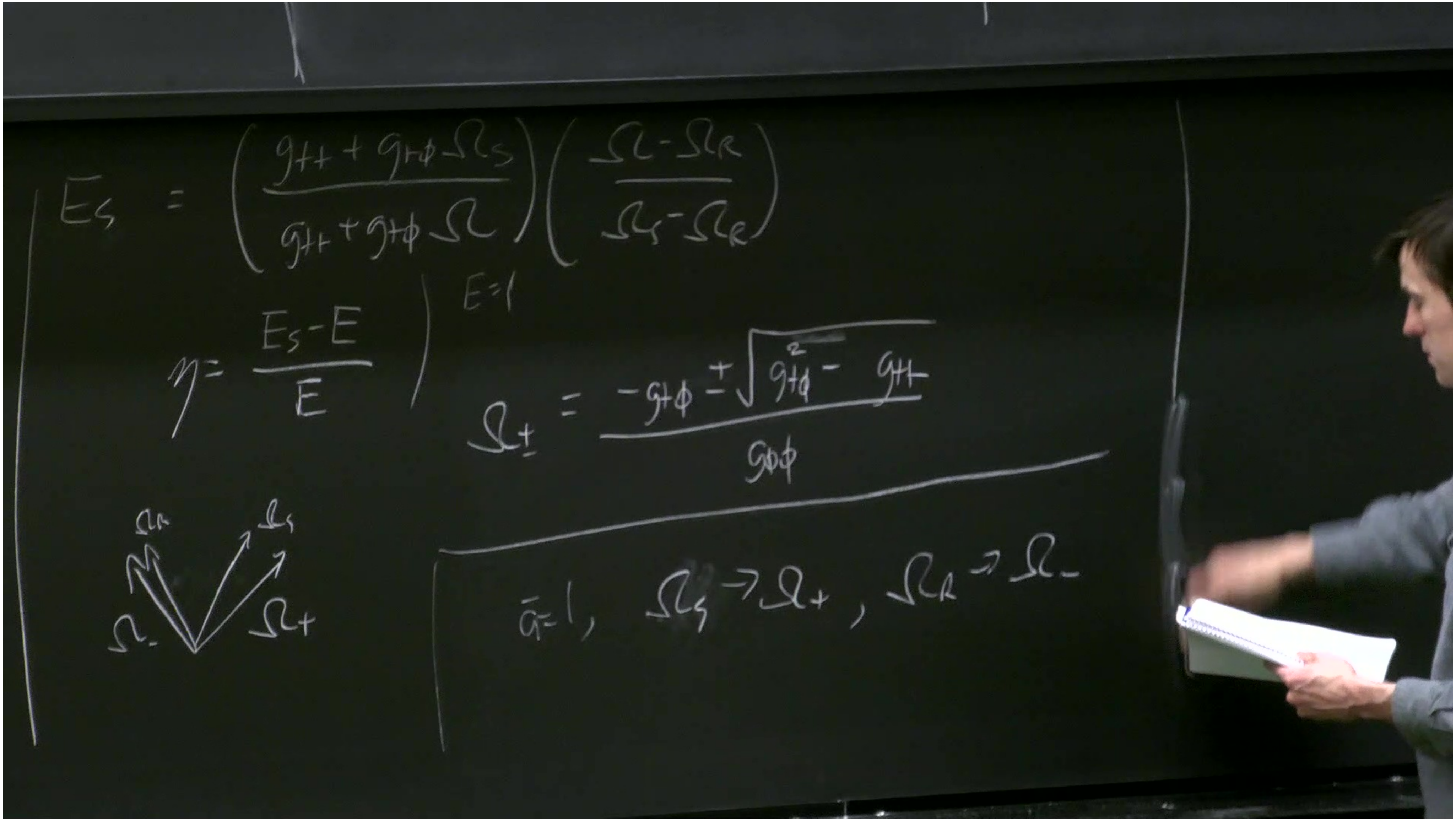
$$\frac{d\phi}{d\tau} = C_R \Omega_R + C_S \Omega_S$$

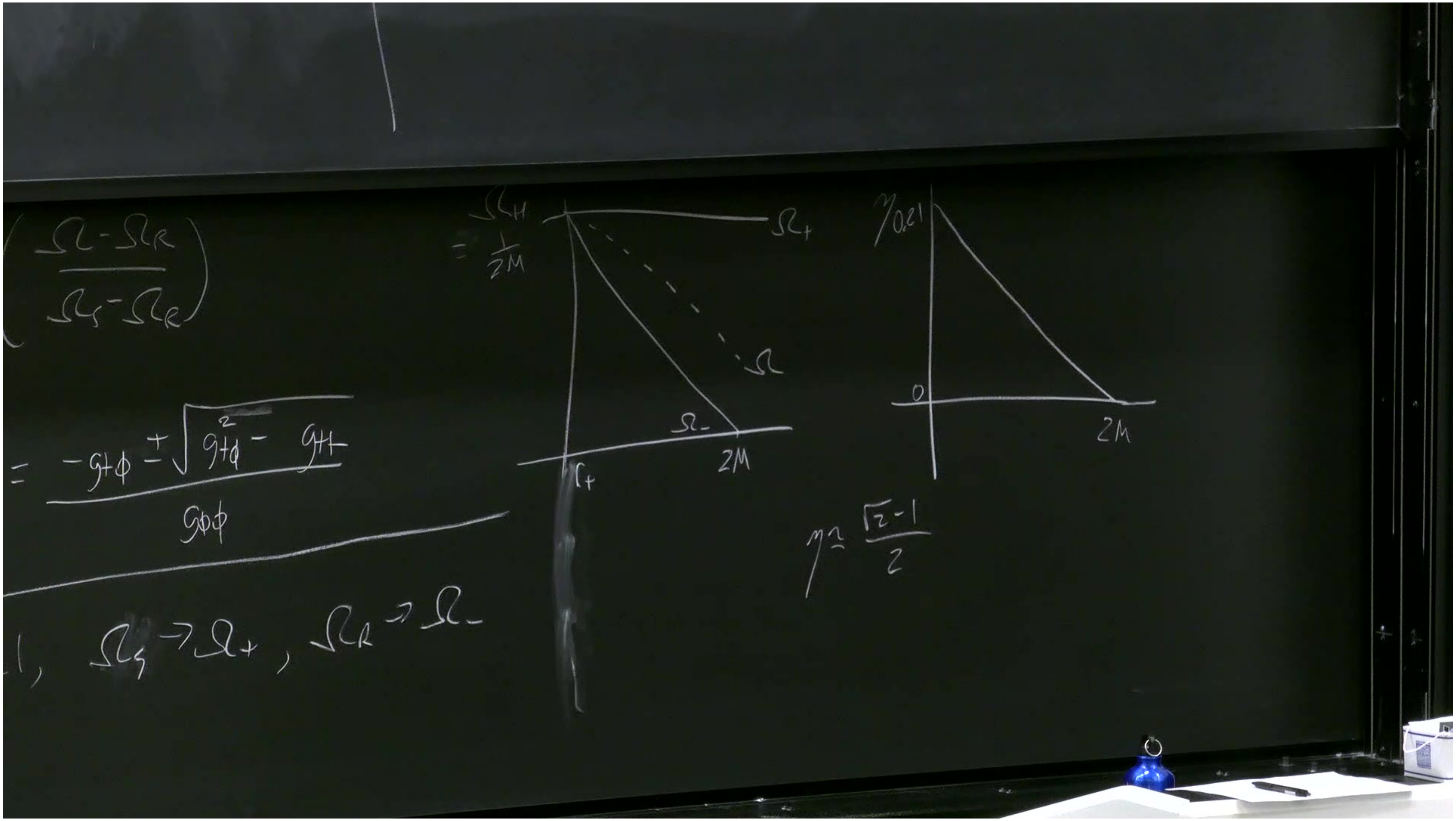
$$C_S = \frac{\frac{dt}{d\tau}(\Omega - \Omega_R)}{(\Omega_S - \Omega_R)}$$

$$E_S = -g_{ab} t^a p_S^b = -C_S\left(g_{tt} + g_{t\phi}\Omega_S\right)$$

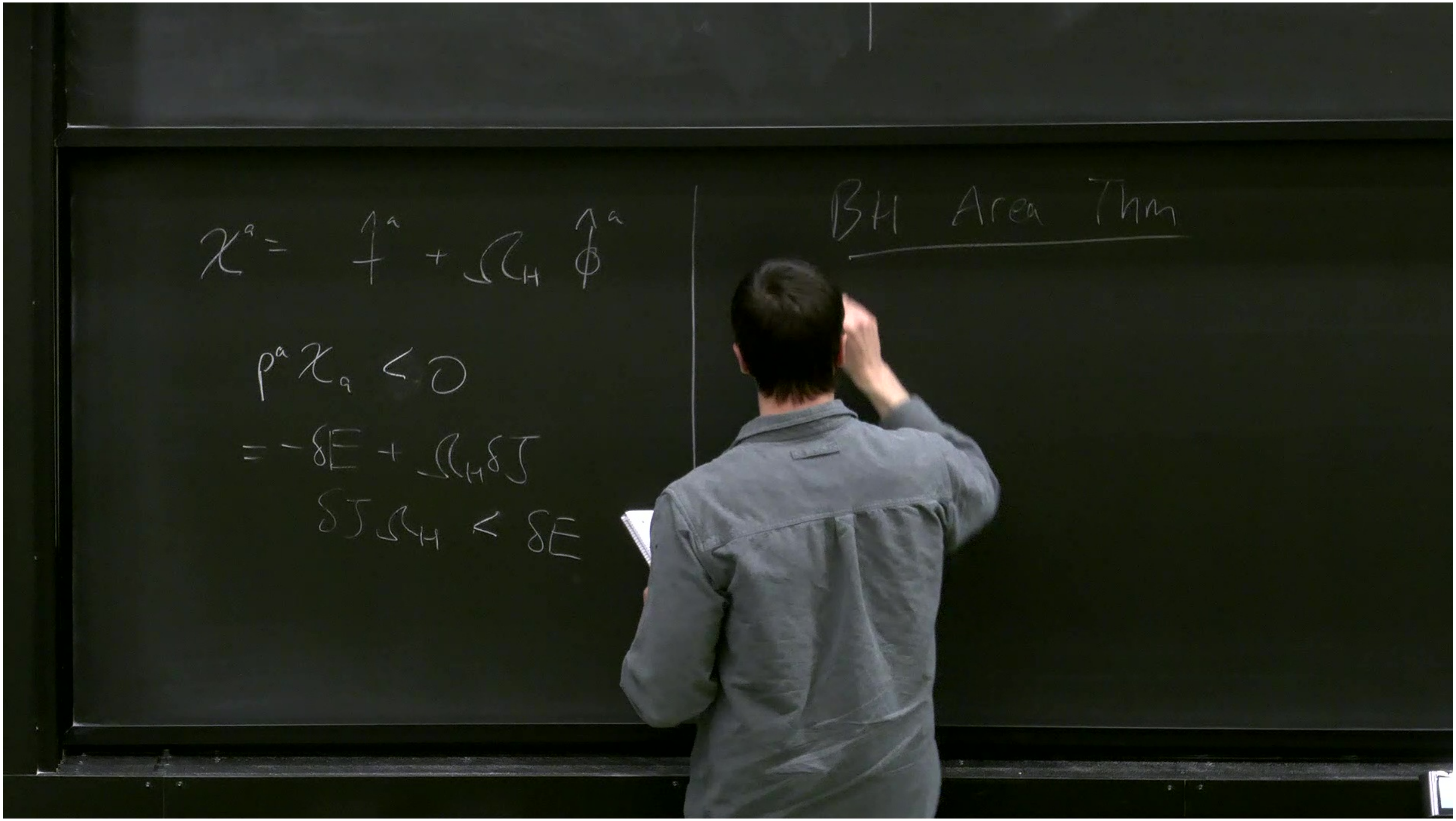
$$E_S = \left(\frac{g_{tt} + g_{t\phi}\Omega_S}{g_{tt} + g_{t\phi}\Omega}\right)\left(\frac{\Omega - \Omega_R}{\Omega_S - \Omega_R}\right)$$

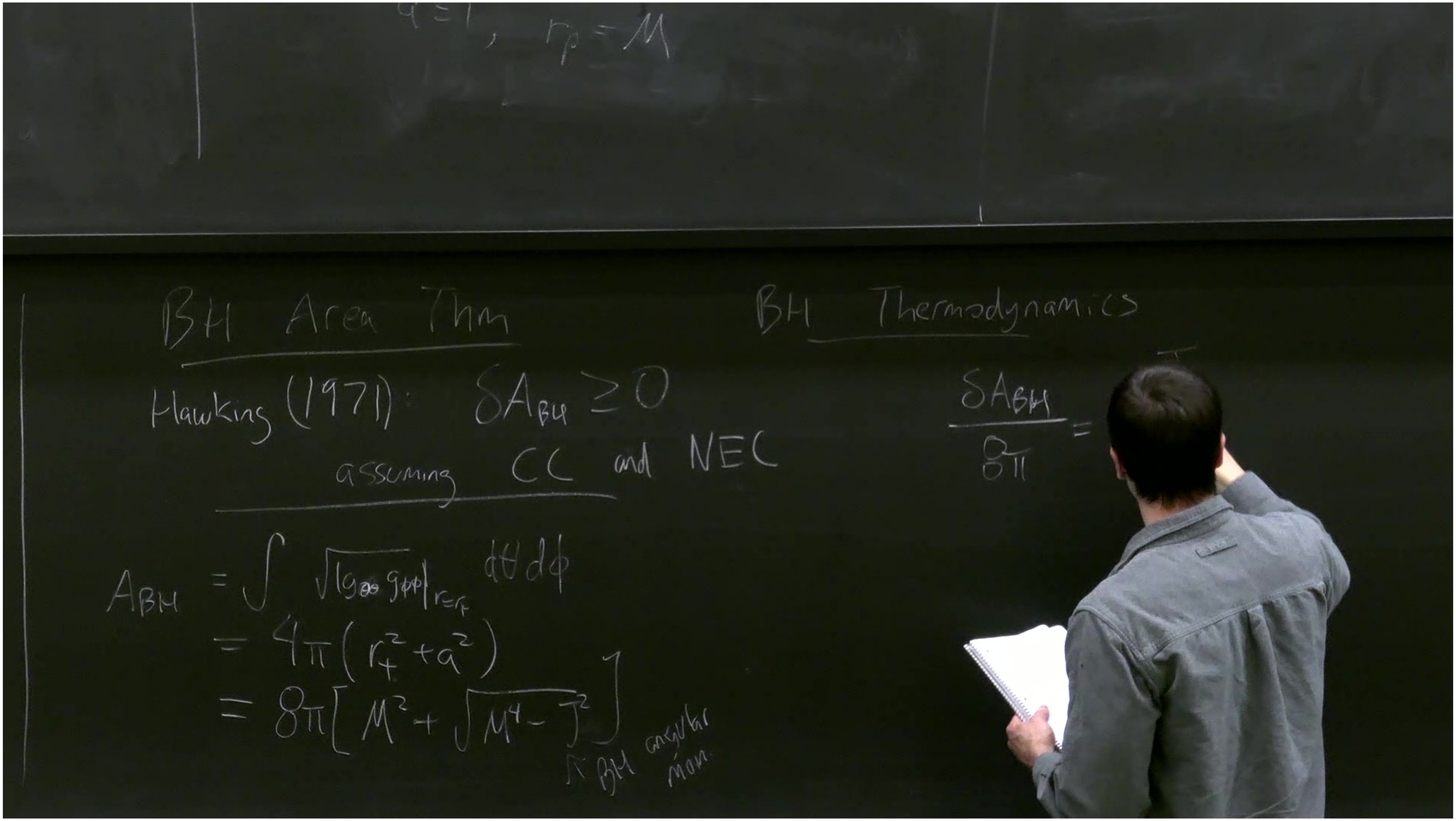
$$\eta = \frac{E_S - E}{E} \Big|^{E=1}$$


$E_s = \left(\frac{g_{tt} + g_{t\phi}\Omega_s}{g_{tt} + g_{t\phi}\Omega}\right)\left(\frac{\Omega - \Omega_R}{\Omega_s - \Omega_R}\right)$
$\eta = \frac{E_s - E}{E}$
$E=1$
$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}}}{g_{\phi\phi}}$
Ω_R
Ω_s
Ω_-
Ω_+
$\bar{q} = 1, \quad \Omega_s \to \Omega_+, \quad \Omega_R \to \Omega_-$



$\frac{\Omega - \Omega_R}{\Omega_S - \Omega_R}$
$= \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}}}{g_{\phi\phi}}$
$\Omega_S \to \Omega_+ , \Omega_R \to \Omega_-$
$\Omega_H = \frac{1}{2M}$
Ω_+
Ω
Ω_-
$2M$
r_+
$\eta_{0.21}$
0
$2M$
$\eta \simeq \frac{\sqrt{2}-1}{2}$


$\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$
$p^a \chi_a \leq 0$
$= -\delta E + \Omega_H \delta J$
$\delta J \, \Omega_H < \delta E$
BH Area Thm


$a = 1, \quad r_p = M$
BH Area Thm
Hawking (1971): $\delta A_{BH} \geq 0$
assuming CC and NEC
$A_{BH} = \int \sqrt{|g_{\theta\theta} g_{\phi\phi}|}\Big|_{r=r_+} d\theta\, d\phi$
$= 4\pi(r_+^2 + a^2)$
$= 8\pi\left[M^2 + \sqrt{M^4 - J^2}\right]$
BH angular mom.
BH Thermodynamics
$\frac{\delta A_{BH}}{8\pi} =$

BH Thermodynamics

EC

$$\frac{\delta A_{BH}}{8\pi} = \frac{J}{\sqrt{M^4 - J^2}}\left[\left(\frac{2M\sqrt{M^4-J^2}}{J} + \frac{2M^3}{J}\right)\delta M - \delta J\right]$$

$$\frac{2r_+}{\bar{a}} = \frac{1}{\Omega_H}$$

$$\frac{\delta A_{BH}}{8\pi} = \frac{\bar{a}}{\sqrt{1-\bar{a}^2}}\left(\Omega_H^{-1}\delta M - \delta J\right) \geq 0$$

$M_{ir} := \sqrt{\frac{A_{BH}}{16\pi}}$, $\bar{a} = 0$, $M_{ir} = M$; $\bar{a} = 1$, $M_{ir} = \frac{M}{\sqrt{2}}$, $E_{rot} = M - M_{ir} = M\left(1 - \frac{1}{\sqrt{2}}\right) \approx 0.29\, M_{BH}$

2nd Law: $\delta A_{BH} \geq 0 \iff \delta S \geq 0$

$\chi_a \chi^a = 0$ at $r = r_+$

$\nabla_a(\chi_b \chi^b) = -2\kappa\, \chi_a$

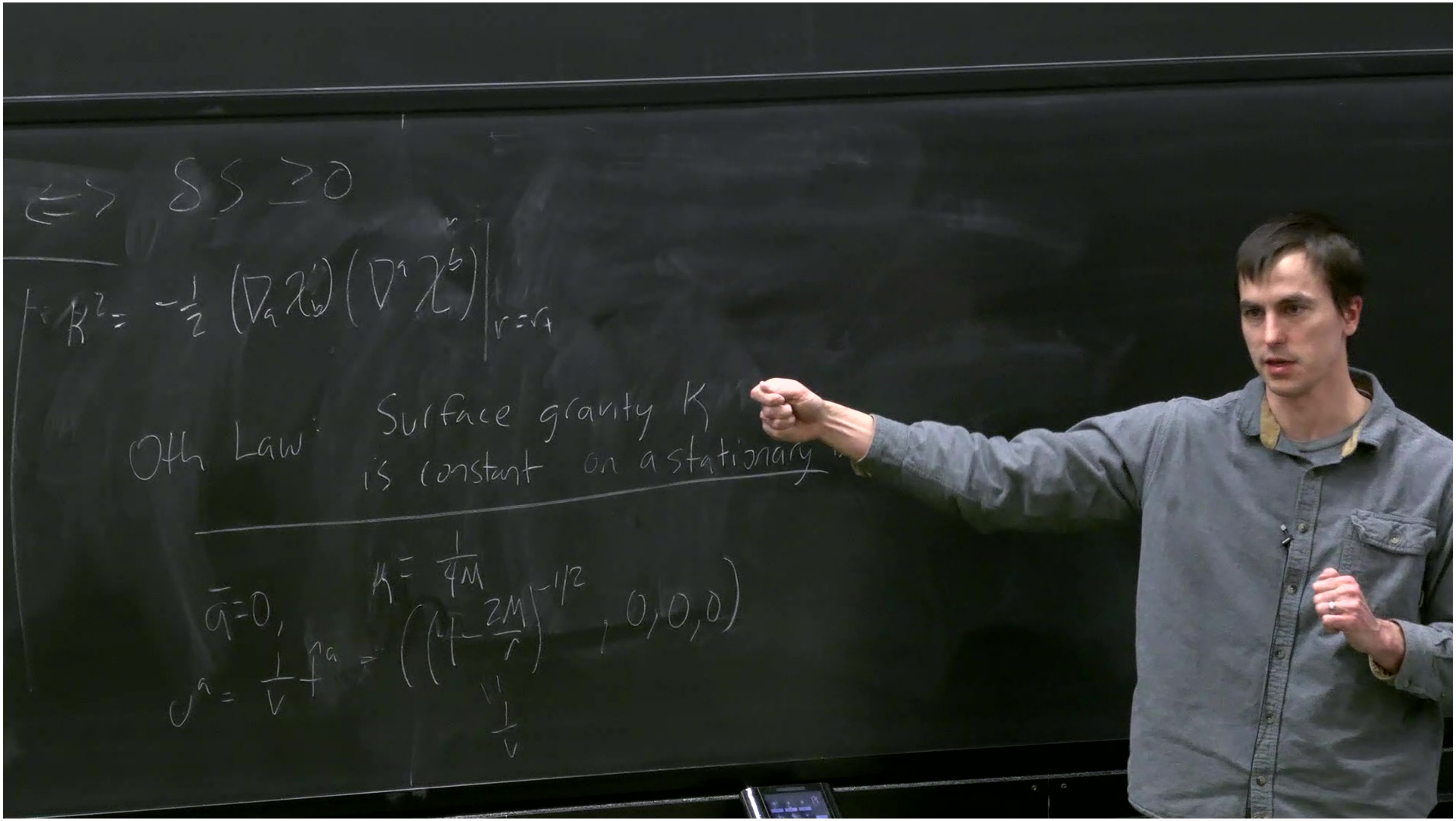
$$\kappa = \frac{\sqrt{M^2-a^2}}{2M(M+\sqrt{M^2-a^2})} = \frac{\sqrt{M^2-a^2}}{2Mr_+}$$

$$\kappa^2 = -\frac{1}{2}(\nabla_a \chi_b)(\nabla^a \chi^b)\Big|_{r=r_+}$$

0th Law: Surface gravity κ is constant on a stationary B

$a=0$, $\kappa = \frac{1}{4M}$

$$u^a = \frac{1}{V}\xi^a = \left(\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right)$$


⟺ δS ≥ 0
κ² = −½ (∇_a χ_b)(∇^a χ^b)|_{r=r+}
0th Law: Surface gravity κ
is constant on a stationary
κ = 1/4M
ā = 0,
u^a = 1/V t^a = ((1 − 2M/r)^{-1/2}, 0, 0, 0)
1/V

$$+2\Omega g_{t\phi}) \quad (*)$$

$$\phi \Omega) \quad (**)$$

$$(*), (**) \Rightarrow (g_{\phi\phi} + g_{t\phi}^2)\Omega^2 + 2(g_{t\phi})(1+g_{tt})\Omega + g_{tt}(1+g_{tt}$$

$$\rho_R^a = C_R(1 \quad 0 \quad 0 \quad \Omega_R)$$

$$\rho_S^a = C_S(1 \quad 0 \quad 0 \quad \Omega_S)$$

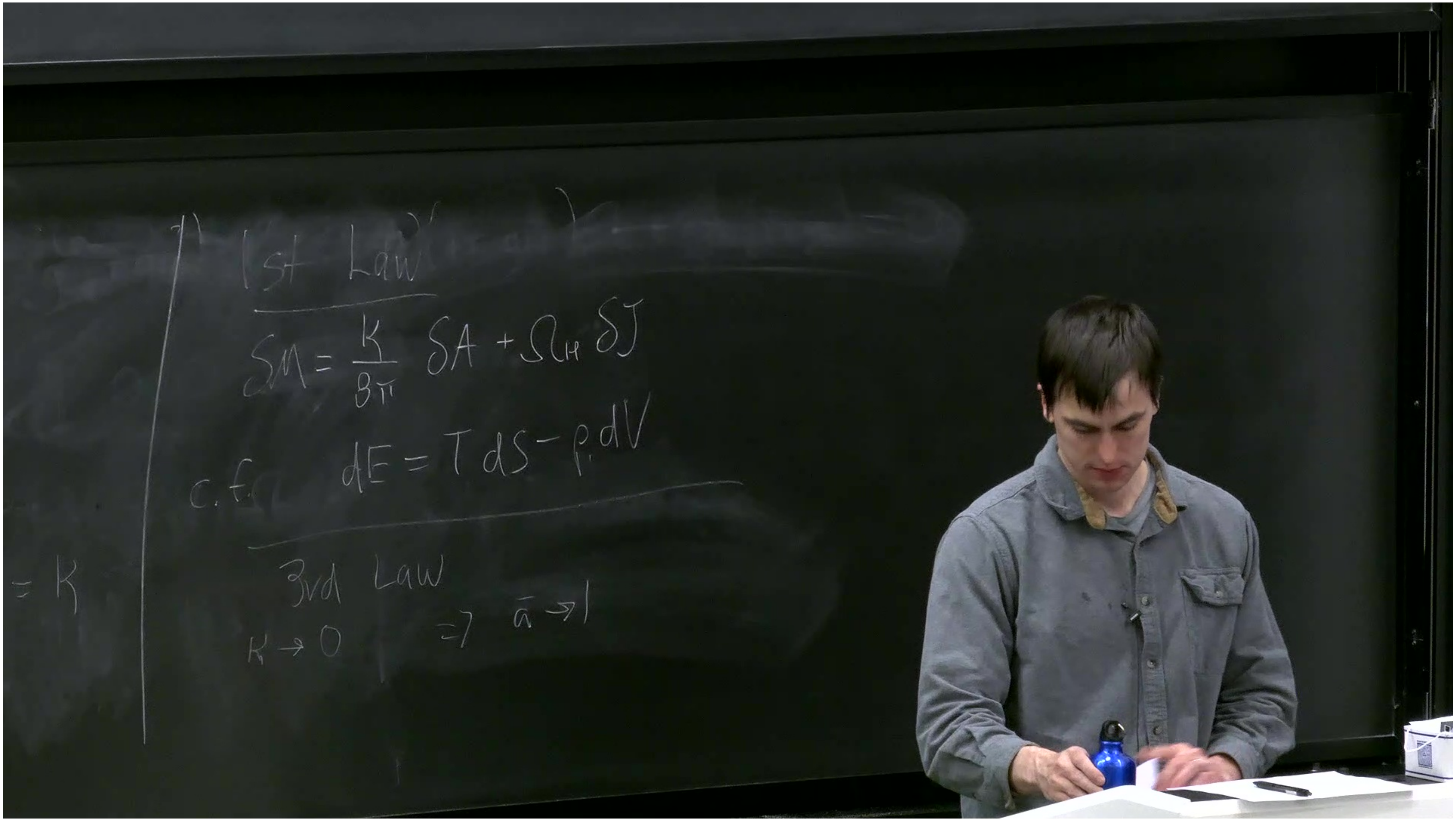
$$\rho^a = \rho_R^a + \rho_S^a$$

Acceleration $A_b = u^b \nabla_b u^a = u^t \cancel{\partial_t u_a}^{\,0} - \Gamma^t_{ta} u_t u^t$

$$= \frac{M}{r(r-2M)} \delta_a^r$$

$$A_b A^b = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}$$

$$\Rightarrow \quad \lambda V_a = \left(1-\frac{2M}{r}\right)^{1/2} \frac{M}{r^2\left(1-\frac{2M}{r}\right)^{1/2}} = \frac{M}{r^2}\bigg|_{r=2M} = \frac{1}{4M} = \kappa$$


1st Law
$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J$
c.f. $dE = T dS - p dV$
3rd Law
$\kappa \to 0 \Rightarrow \bar{a} \to 1$
$= \kappa$