Title: Implications of the additivity anomaly in large N field theories for holography

Speakers: Samuel Leutheusser

Series: Quantum Fields and Strings

Date: February 27, 2024 - 2:00 PM

URL: https://pirsa.org/24020097

Abstract: Holographic conformal field theories exhibit dramatic changes in the structure of their operator algebras in the limit where the number of local degrees of freedom (N) becomes infinite. An important example of such phenomena is the violation of the additivity property for algebras associated to local subregions. We investigate the consequences of this "additivity anomaly" in the context of holographic duality. We propose that the difference in volumes of bulk dual subregions can be used as a holographic measure for this additivity anomaly of large N boundary algebras. We demonstrate how the additivity anomaly underlies the success of quantum error correcting code models of holography. Finally, we argue that the connected wedge theorems (CWTs) of May, Penington, Sorce, and Yoshida can be re-phrased in terms of the additivity anomaly, allowing for the definition of a generalized scattering region for which a generalization of the CWTs can be formulated such that both the theorem and its converse should hold.

Zoom link TBA

IMPLICATIONS OF THE ADDITIVITY ANOMALY FOR HOLOGRAPHY

Perimeter Institute Quantum Fields & Strings Seminar

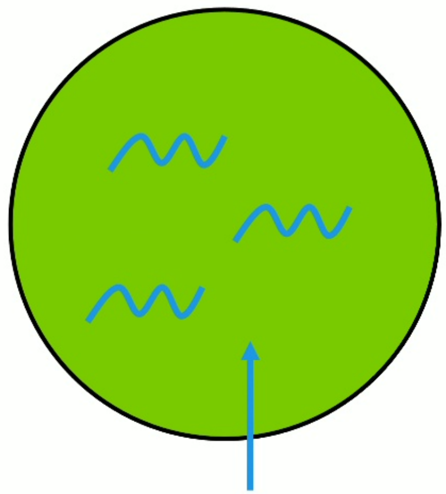
February 27, 2024

Sam Leutheusser
Princeton Gravity Initiative

Based on work with Hong Liu: 2403.xxxxx and 2212.13266

AdS/CFT Correspondence

- Quantum Gravity in AdS = CFT
- AdS = anti-de Sitter
- CFT = conformal field theory
- Concrete realization of the holographic principle
- Holographic duality allows us to treat gravity as an emergent QM phenomenon



Maldacena (1997)

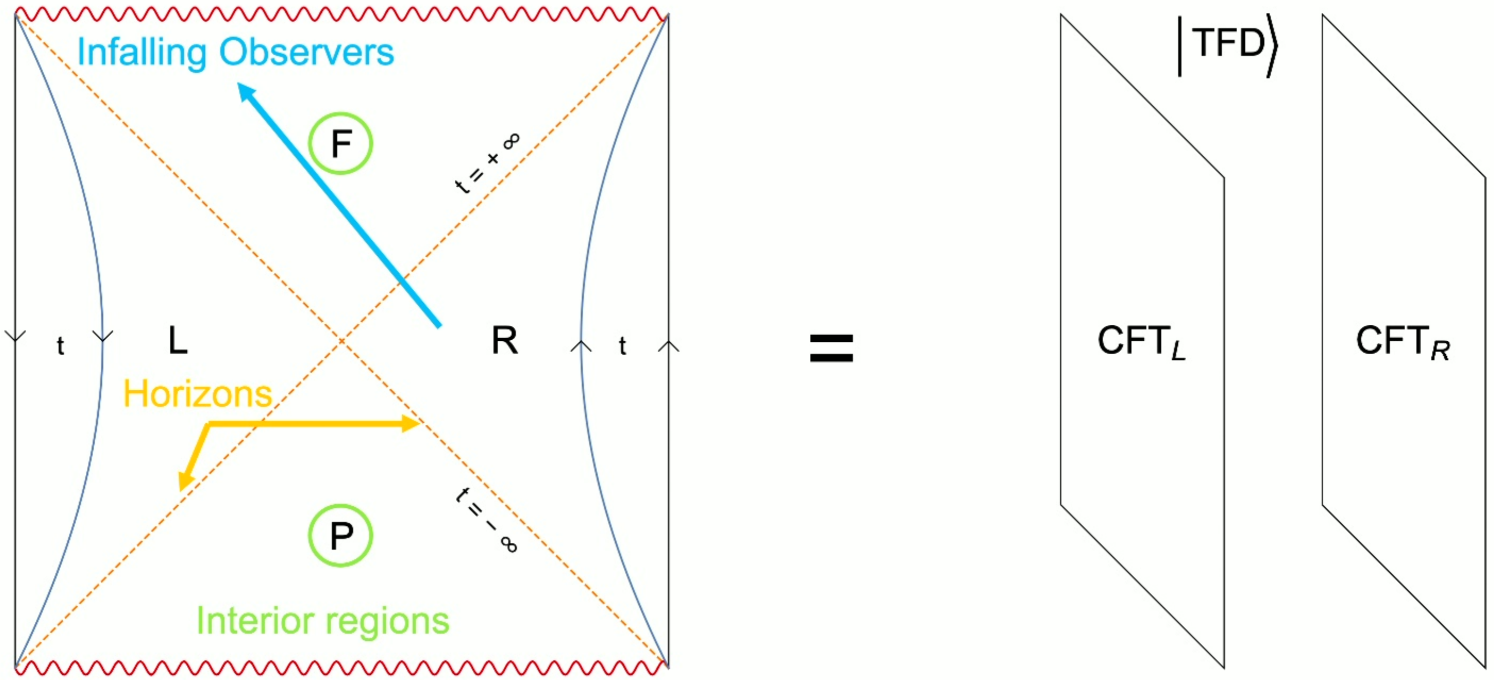
AdS/CFT Correspondence

Bulk Gravity Theory ⟷ Boundary CFT

Quantum Gravity State ⟷ CFT State

A Simple Example

- Eternal AdS black hole and thermofield double



Maldacena (2001)

AdS/CFT Correspondence Questions

Bulk Gravity Theory	$\longleftrightarrow$	Boundary CFT
Quantum Gravity State	$\longleftrightarrow$	CFT State
Semiclassical $G_N \to 0$	$\longleftrightarrow$	Many DOF $N \to \infty$
$\downarrow$		$\downarrow$
Classical Geometry	$\longleftrightarrow$	"$N = \infty$" CFT
Spacetime Subregions	$\longleftrightarrow$	???

How does the boundary theory probe *local* bulk physics?

Emergence of Bulk Spacetime

- What is the precise mathematical structure in the CFT that describes bulk geometric objects?
- We will argue that

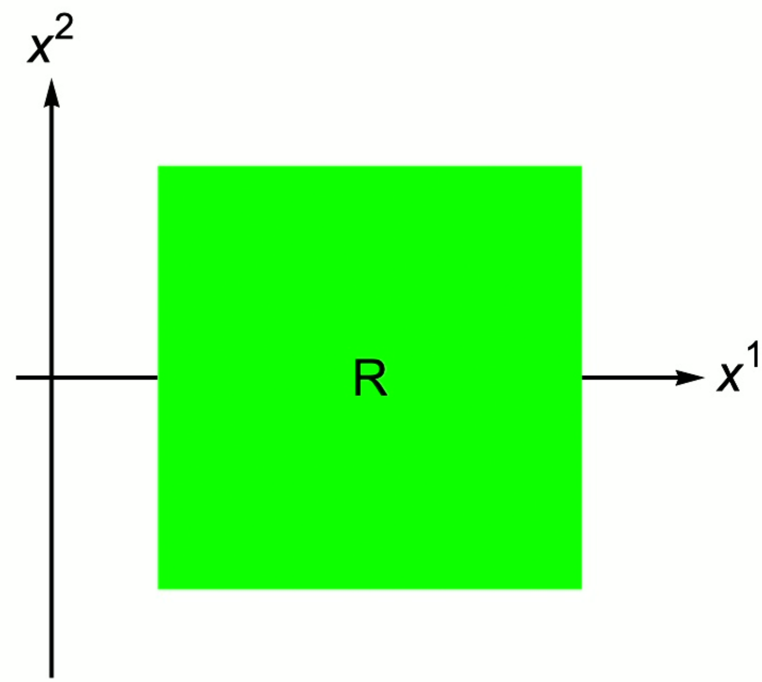
> Bulk spacetime is a geometrization of emergent type III_1 algebras on the boundary

- Bulk geometric notions are related to the properties of these emergent type III_1 algebras

Talk Outline

1. Algebraic description of relativistic QFT
2. Large N Limits and GNS Hilbert spaces
   - Definitions and properties of local algebras at $N = \infty$
3. Subalgebra-subregion duality
   - Connection to subregion-subregion duality
   - More general cases
4. Additivity anomaly in holography
   - Volume as a measure $\Rightarrow$ relation to complexity?
   - Relation to Quantum Error Correction (QEC)
   - Generalized two-to-two connected wedge theorem (CWT)

Physical meaning of additivity



Algebraic QFT

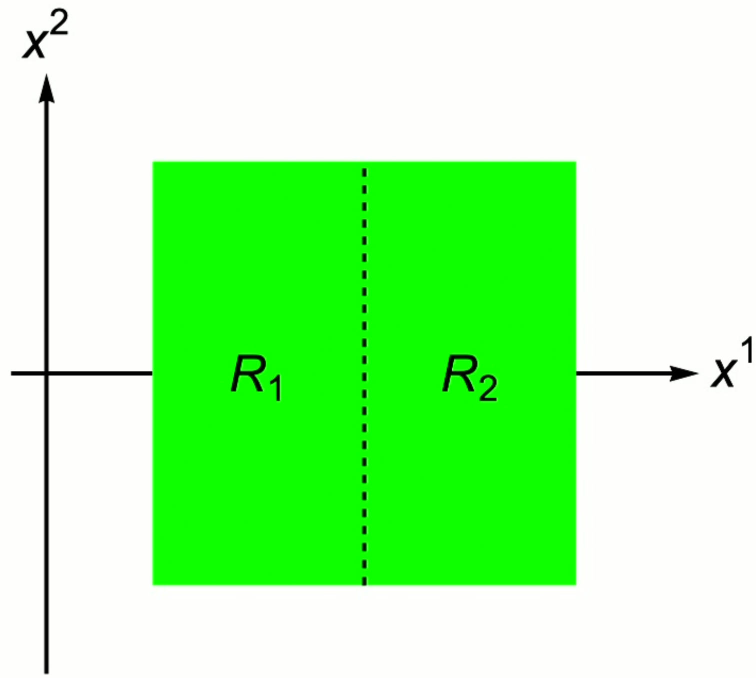
- Net of algebras: $\{R\} \to \{\mathcal{B}_R\}$ such that
  1. Isotony: $R_1 \subseteq R_2 \Longrightarrow \mathcal{B}_{R_1} \subseteq \mathcal{B}_{R_2}$
  2. Causality: $\mathcal{B}_{(R_1)'} \subseteq \left(\mathcal{B}_{R_1}\right)'$
- Possible additional properties
  3. Local equation of motion: $\mathcal{B}_R = \mathcal{B}_{\hat{R}}$, $\hat{R} \equiv R''$
  4. Haag duality: $\mathcal{B}_{(R_1)'} = \left(\mathcal{B}_{R_1}\right)'$
  5. Additivity:

$$\mathcal{B}_{(R_1 \cup R_2)''} = \mathcal{B}_{R_1} \vee \mathcal{B}_{R_2} \equiv \left(\mathcal{B}_{R_1} \cup \mathcal{B}_{R_2}\right)''$$

Haag, Kastler (1964)

Physical meaning of additivity

- Additivity is a strong form of locality
- Decompose a subregion into two parts
- Additivity implies $\mathcal{B}_R = \mathcal{B}_{R_1} \vee \mathcal{B}_{R_2}$



The Large N Limit

- Consider a CFT with $N \times N$ matrix degrees of freedom
- At finite N there is a well-defined Hilbert space, and the full algebra of operators is of type I
- Some states and operators in this theory may not survive the large N limit
- This can lead to dramatic changes in the structure of the Hilbert space and operator algebras

Single-Trace Algebra

- In a theory with $N \times N$ matrix degrees of freedom, commutators can be expanded in powers of $1/N$

$$[\mathcal{O}_1, \mathcal{O}_2] = \sum_{k=0}^{\infty} \frac{1}{N^k} c_{12}^{(k)} \mathcal{O}^{(k)}$$

- About a state $|\Psi\rangle$, the (subtracted) **single-trace** operators have a well-defined large N limit.
- If $|\Psi\rangle$ is a *semiclassical state* (obeying large N factorization) one has

$$\langle\Psi|\mathcal{O}_L[\mathcal{O}_1, \mathcal{O}_2]\mathcal{O}_R|\Psi\rangle = C_{12}^{(\Psi)}\langle\Psi|\mathcal{O}_L\mathcal{O}_R|\Psi\rangle + O\left(\frac{1}{N}\right)$$

(i.e. to leading order in $1/N$ commutators are equivalent to c-numbers when inserted in correlation functions in $|\Psi\rangle$)

The GNS Hilbert Space

- Thus in the large N limit, single-trace operators form a (state-dependent) algebra, $\mathcal{A}_\Psi$
- Acting $\mathcal{A}_\Psi$ on $|\Psi\rangle$ we may build a GNS Hilbert space $\mathcal{H}_\Psi^{\mathrm{GNS}}$ of "small" excitations around $|\Psi\rangle$
- The GNS vacuum state $|0\rangle_{\Psi,\mathrm{GNS}}$ reproduces the correlation functions of single-trace operators in $|\Psi\rangle$
- $\mathcal{H}_\Psi^{\mathrm{GNS}}$ furnishes a representation of $\mathcal{A}_\Psi$

$$\mathcal{M}_\Psi = \pi_\Psi(\mathcal{A}_\Psi)'' \subset \mathcal{B}\left(\mathcal{H}_\Psi^{\mathrm{GNS}}\right)$$

Semi-classical AdS/CFT Duality

- The $G_N \to 0$ limit of AdS/CFT then implies

$$\mathcal{H}_\Psi^{\text{GNS}} = \mathcal{H}_\Psi^{\text{Bulk}}$$

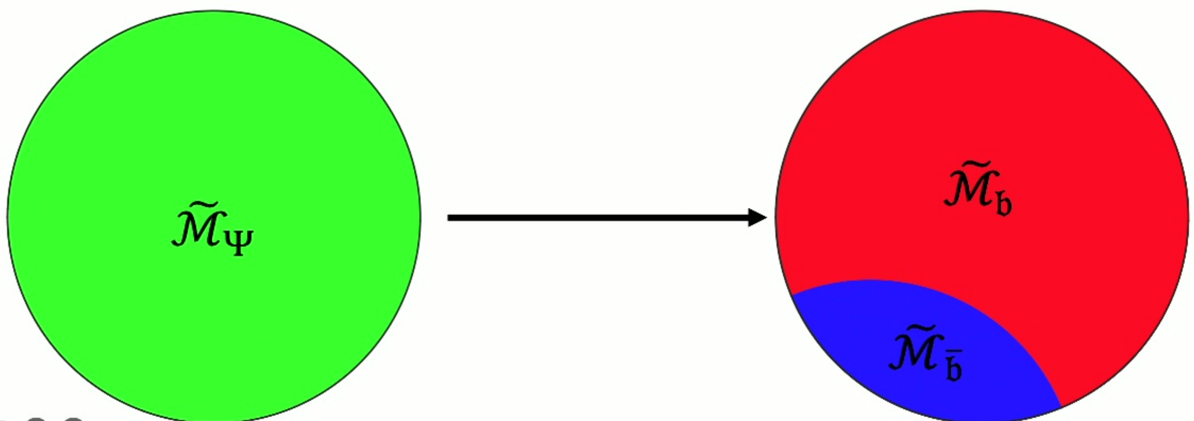
- We then must identify $\mathcal{M}_\Psi = \widetilde{\mathcal{M}}_\Psi$
- This is the statement of global reconstruction

Identification of subalgebras

- The identification $\mathcal{M}_\Psi = \widetilde{\mathcal{M}}_\Psi$ implies that there must also be an identification of **subalgebras**
- In the semiclassical limit, the bulk theory is that of quantum fields on a fixed curved spacetime background
- The associated subalgebras have a special structure that arises from causality and the equation of motion of the bulk theory
- Want to understand how the boundary theory reproduces this special structure of the bulk subalgebras

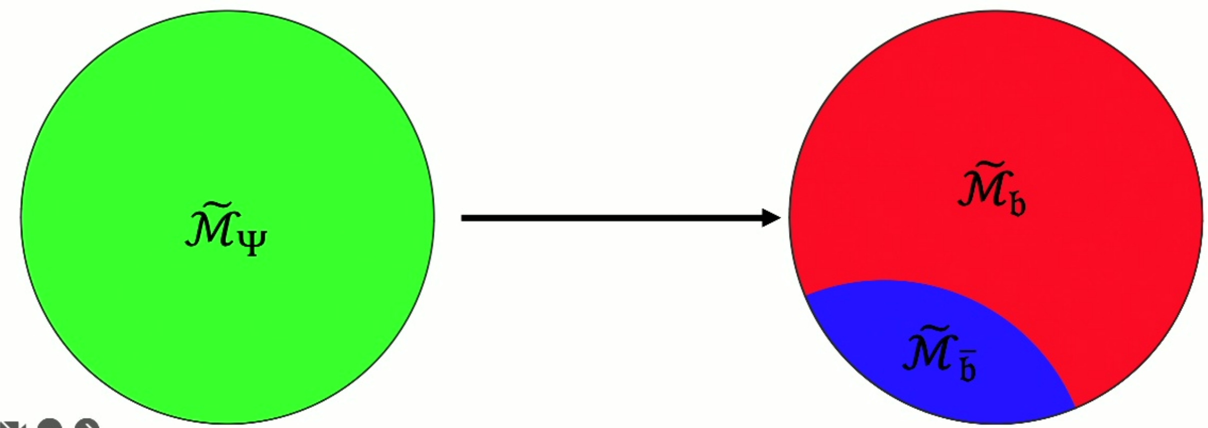
Local bulk EFT

- The full algebra of bulk fields is defined on a single Cauchy slice
- It can be decomposed into subalgebras associated to spatial subregions



Local bulk EFT

- The full algebra of bulk fields is defined on a single Cauchy slice
- It can be decomposed into subalgebras associated to spatial subregions
- A type III_1 algebra is associated to each subregion



Single-traces at large N

- Consider a single-trace operator, e.g. $V = \mathrm{Tr} F_{\mu\upsilon} F^{\mu\upsilon}$
- At finite N we have

$$V(t) = e^{iHt} V(0) e^{-iHt} = \sum_i c_i(t) \mathcal{O}_i(0)$$

for some complete set of operators $\{\mathcal{O}_i\}$ at $t = 0$

- This includes multi-trace CFT operators
- As $N \to \infty$, $V(t)$ remains well-defined, but the equations above no longer hold
- The single-trace algebra is described by a Generalized Free Field (GFF) theory

Single-traces at large N

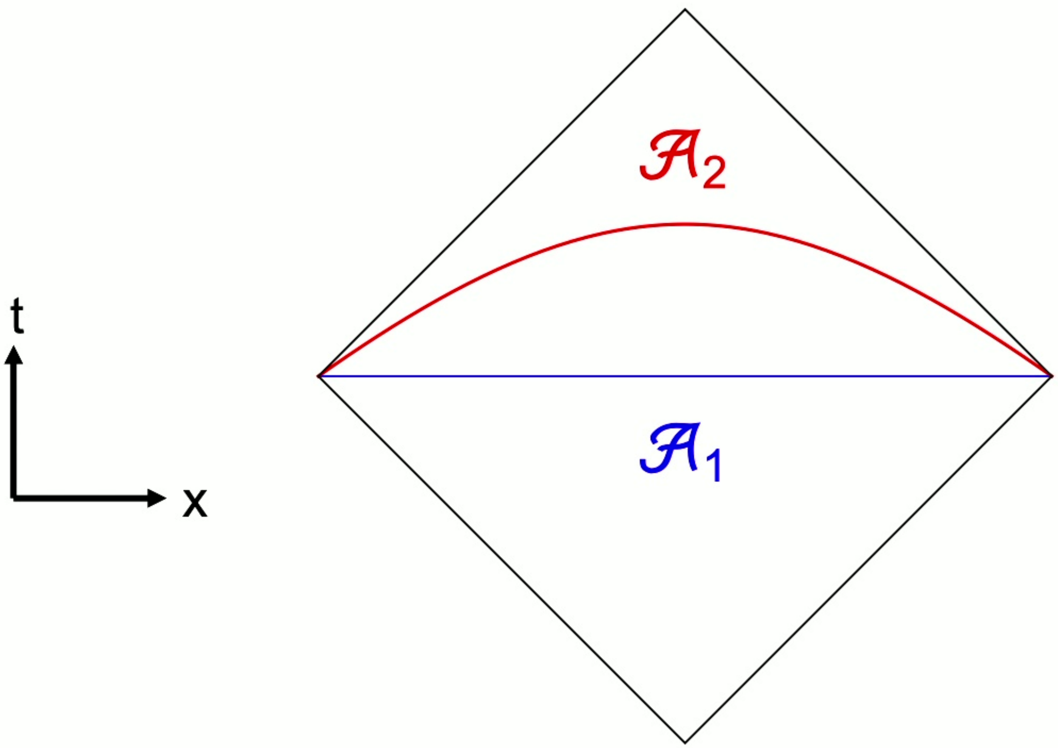
- Consider a single-trace operator, e.g. $V = \mathrm{Tr} F_{\mu\upsilon} F^{\mu\upsilon}$
- At finite N we have

$$V(t) = e^{iHt} V(0) e^{-iHt} = \sum_i c_i(t) \mathcal{O}_i(0)$$

for some complete set of operators $\{\mathcal{O}_i\}$ at $t = 0$

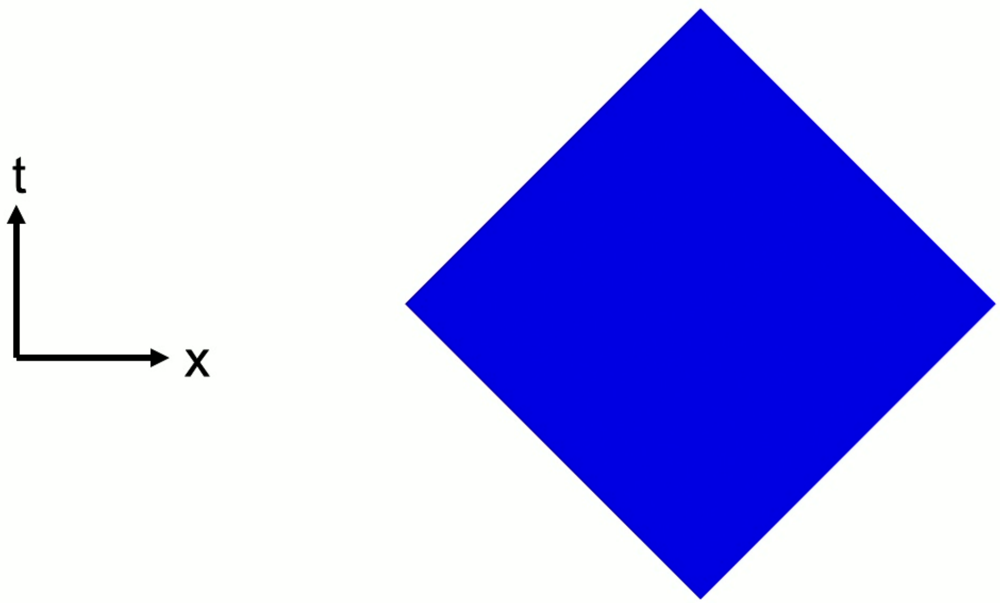
- This includes multi-trace CFT operators
- As $N \to \infty$, $V(t)$ remains well-defined, but the equations above no longer hold
- The single-trace algebra is described by a Generalized Free Field (GFF) theory
- Matches boundary limit of bulk fields [Duetsch, Rehren `02]

GFF peculiarities



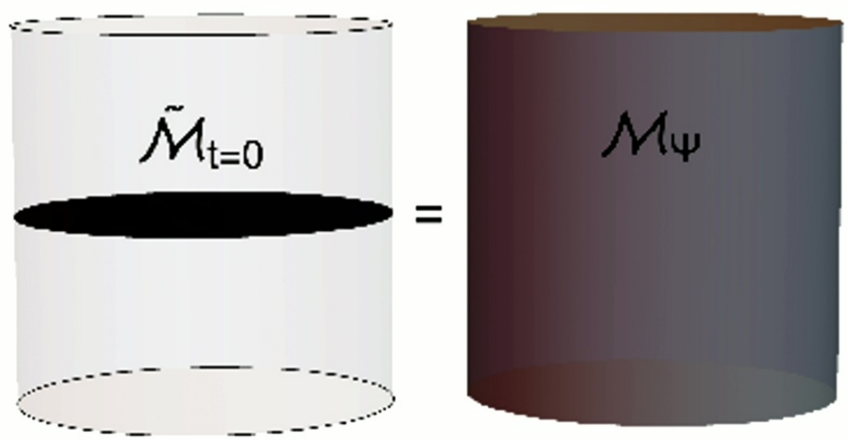
- Usually in QFT $\mathcal{A}_1 = \mathcal{A}_2$, but for GFFs $\mathcal{A}_1 \neq \mathcal{A}_2$

GFF peculiarities



- The algebra of a causally incomplete subregion is not equal to that of its causal completion

Global Reconstruction Revisited



- Bulk fields in AdS are dual to GFF operators supported on the entire boundary *spacetime*

Subalgebras for bulk subregions

- In the large N limit of the CFT, many additional type III_1 subalgebras appear
- On the gravity side, we expect a type III_1 subalgebra for each causally complete subregion
- Proposal: The algebra of local bulk operators in each causally complete bulk subregion, $\mathfrak{b}$, is dual to an emergent type III_1 subalgebra in the large N limit of the CFT, i.e. for some $\mathcal{Y} \subset \mathcal{M}_\Psi$

$$\mathcal{Y} = \tilde{\mathcal{Y}}_{\mathfrak{b}} \subset \widetilde{\mathcal{M}}_\Psi$$

- This generalizes the RT surface formulation of subregion-subregion duality

Assigning Algebras

- Two natural subalgebras to associate with a boundary *spacetime* subregion R

$$\mathcal{X}_R = \pi_\Psi\left(\lim_{N\to\infty,|\Psi\rangle} \mathcal{B}_R^{(N)}\right), \qquad \mathcal{Y}_R = \mathcal{M}_{\hat{R}}$$

($\hat{R}$ is the boundary causal completion of R)

- Explicitly

$$\mathcal{X}_R = \left\langle\left\{\Delta_\Psi^{-is}\pi_\Psi(\mathcal{O}(t=0, x \in R))\Delta_\Psi^{is}, s \in \mathbb{R}\right\}\right\rangle,$$
$$\mathcal{Y}_R = \left\langle\left\{\pi_\Psi\big(\mathcal{O}(t,x)\big), (t,x) \in \hat{R}\right\}\right\rangle$$

- By definition $\mathcal{X}_R \supseteq \mathcal{Y}_R$, but in general not equal

Assigning Algebras

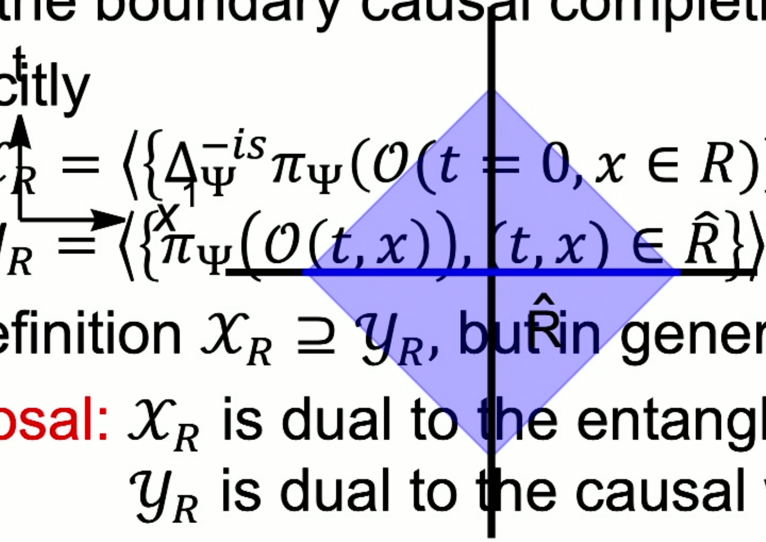
- Two natural subalgebras to associate with a boundary *spacetime* subregion R

$$\mathcal{X}_R = \pi_\Psi\left(\lim_{N\to\infty,|\Psi\rangle} \mathcal{B}_R^{(N)}\right), \qquad \mathcal{Y}_R = \mathcal{M}_{\hat{R}}$$

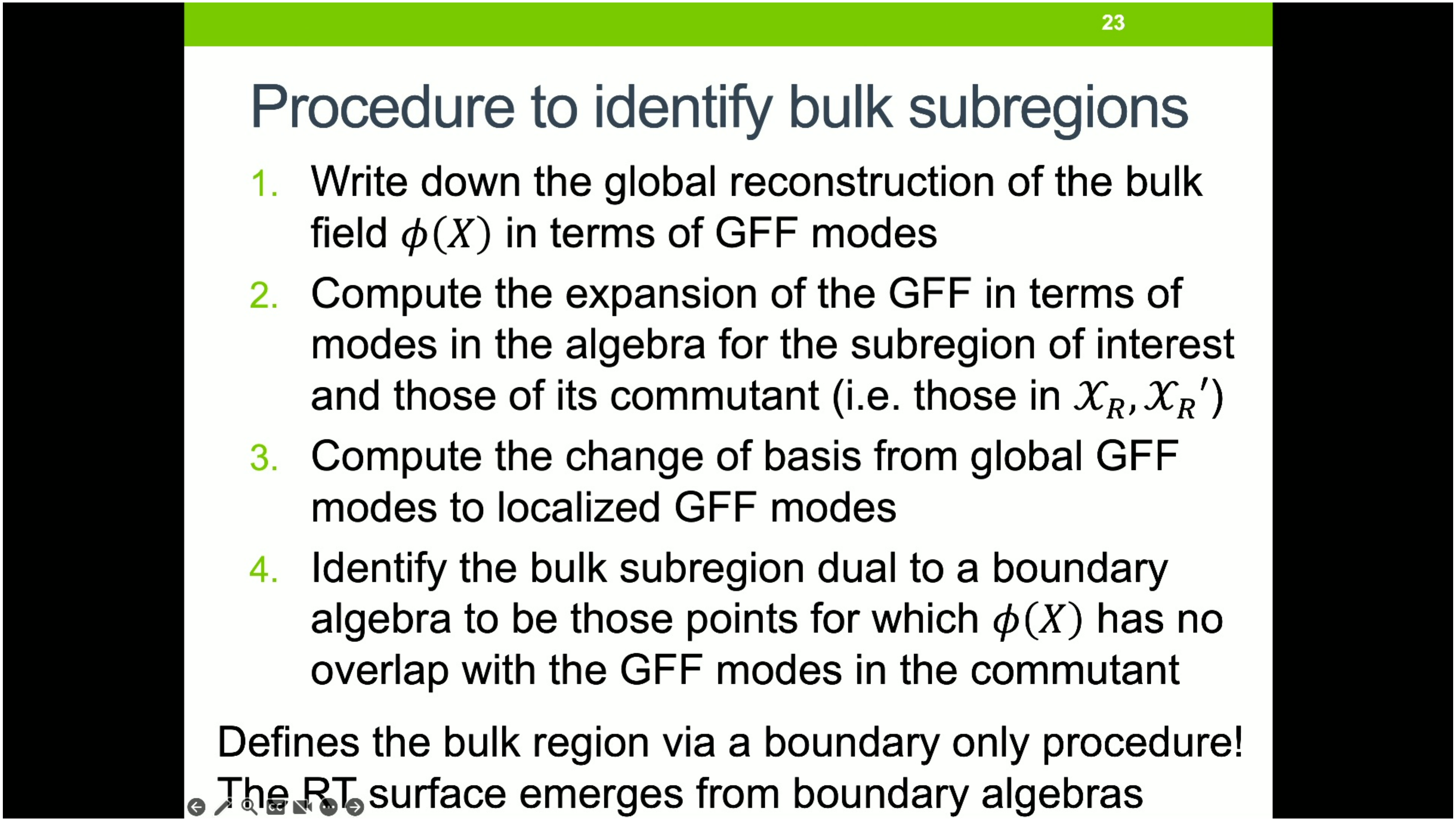
($\hat{R}$ is the boundary causal completion of R)

- Explicitly

$$\mathcal{X}_R = \langle\{\Delta_\Psi^{-is}\pi_\Psi(\mathcal{O}(t=0, x\in R))\Delta_\Psi^{is}, s\in\mathbb{R}\}\rangle,$$
$$\mathcal{Y}_R = \langle\{\pi_\Psi(\mathcal{O}(t,x)), (t,x)\in\hat{R}\}\rangle$$



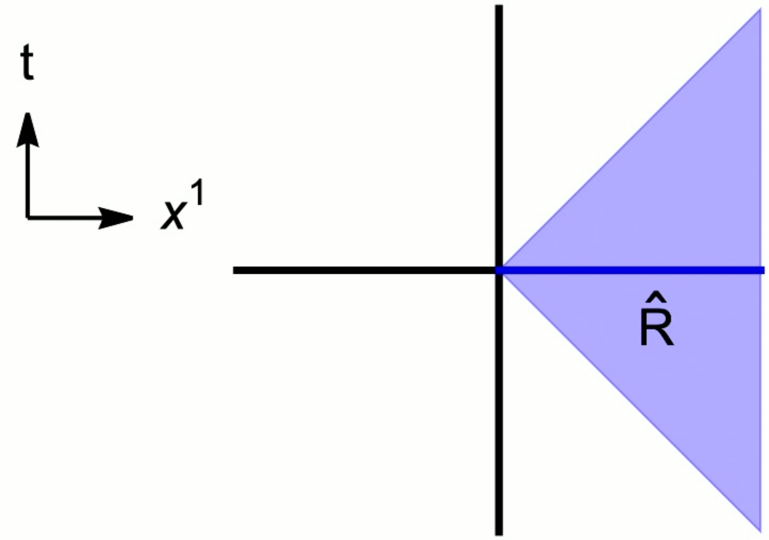
- By definition $\mathcal{X}_R \supseteq \mathcal{Y}_R$, but in general not equal
- Proposal: $\mathcal{X}_R$ is dual to the entanglement wedge
  $\mathcal{Y}_R$ is dual to the causal wedge



23
Procedure to identify bulk subregions
1. Write down the global reconstruction of the bulk field $\phi(X)$ in terms of GFF modes
2. Compute the expansion of the GFF in terms of modes in the algebra for the subregion of interest and those of its commutant (i.e. those in $\mathcal{X}_R, \mathcal{X}_R{}'$)
3. Compute the change of basis from global GFF modes to localized GFF modes
4. Identify the bulk subregion dual to a boundary algebra to be those points for which $\phi(X)$ has no overlap with the GFF modes in the commutant
Defines the bulk region via a boundary only procedure!
The RT surface emerges from boundary algebras

Subregion-subregion duality

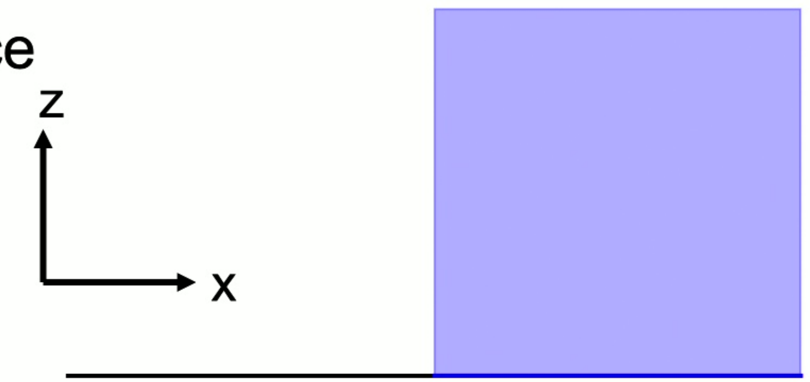
- Minkowski boundary & time-slice of Poincare AdS
- Example: Half-space



Subregion-subregion duality

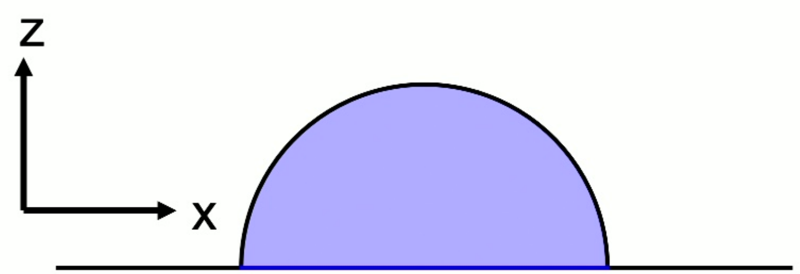
- Minkowski boundary & time-slice of Poincare AdS
- Example: Half-space

$$\mathcal{X}_R = \mathcal{Y}_{\hat{R}}$$



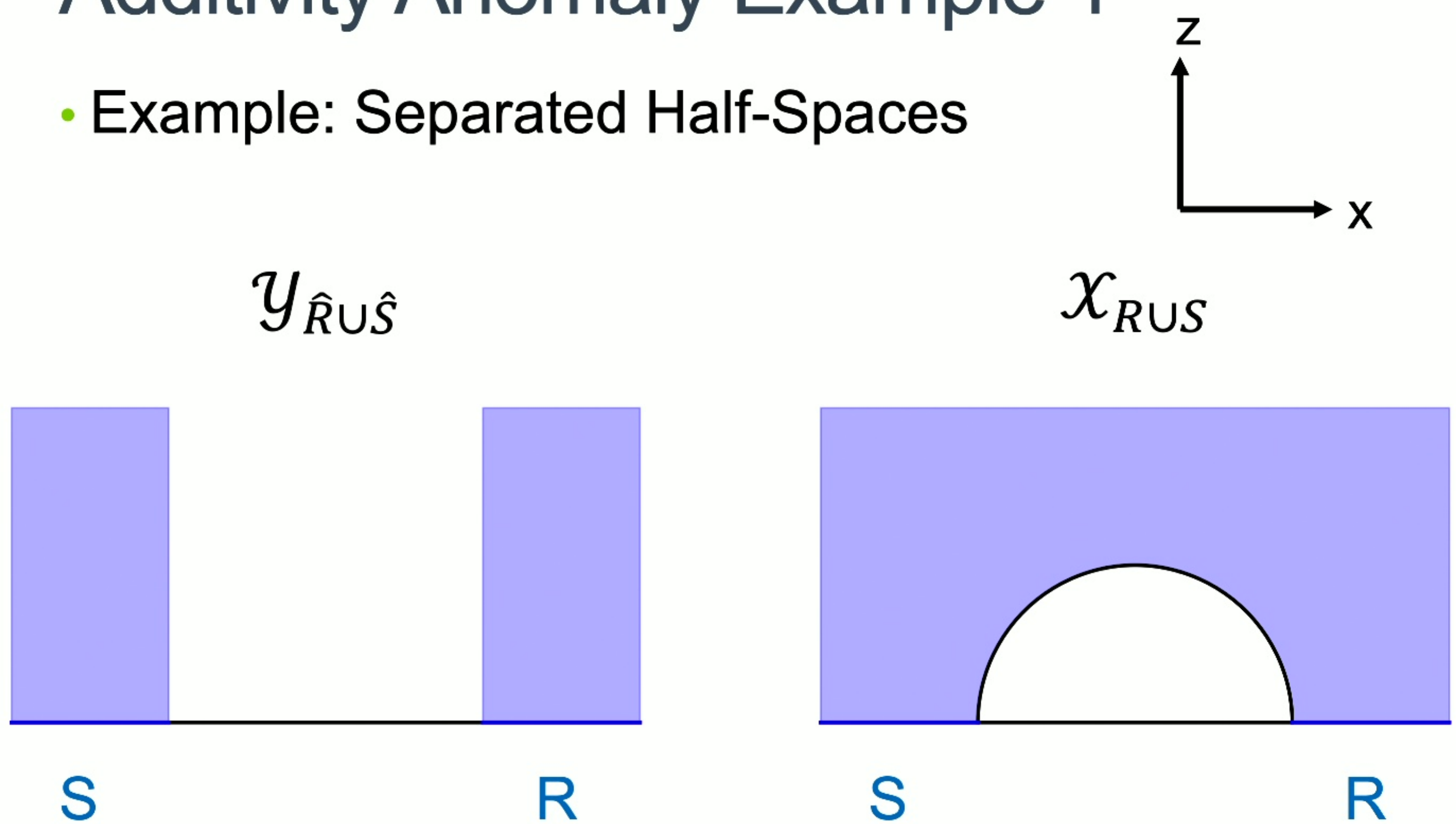
- Example: Interval

$$\mathcal{X}_R = \mathcal{Y}_{\hat{R}}$$



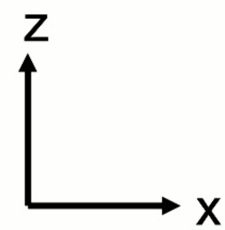
Additivity Anomaly Example 1

- Example: Separated Half-Spaces



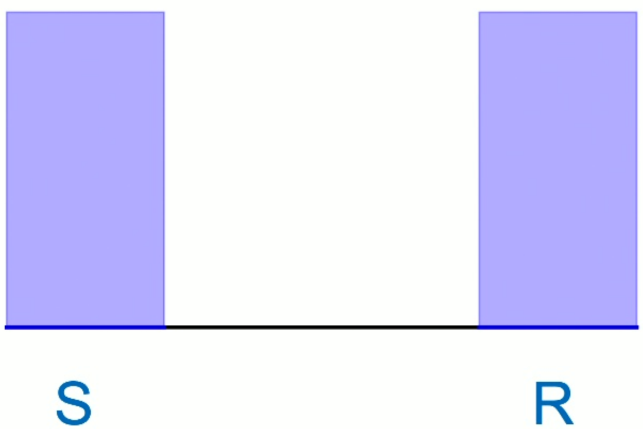
- Entanglement wedge exceeds the causal wedge!

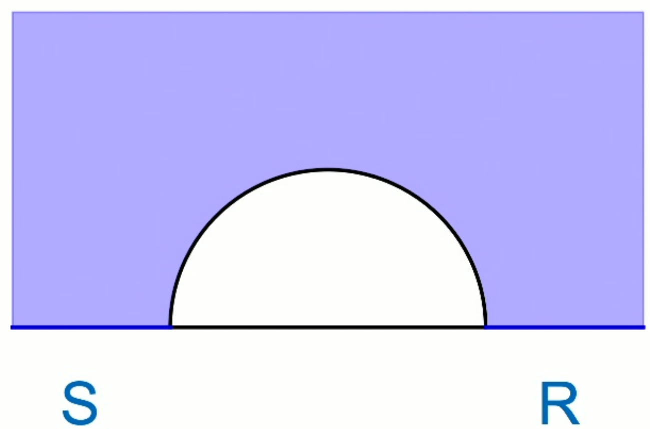
Additivity Anomaly Example 1



- Example: Separated Half-Spaces

$$\mathcal{Y}_{R \cup S} = \mathcal{X}_R \vee \mathcal{X}_S \qquad \mathcal{X}_{R \cup S}$$

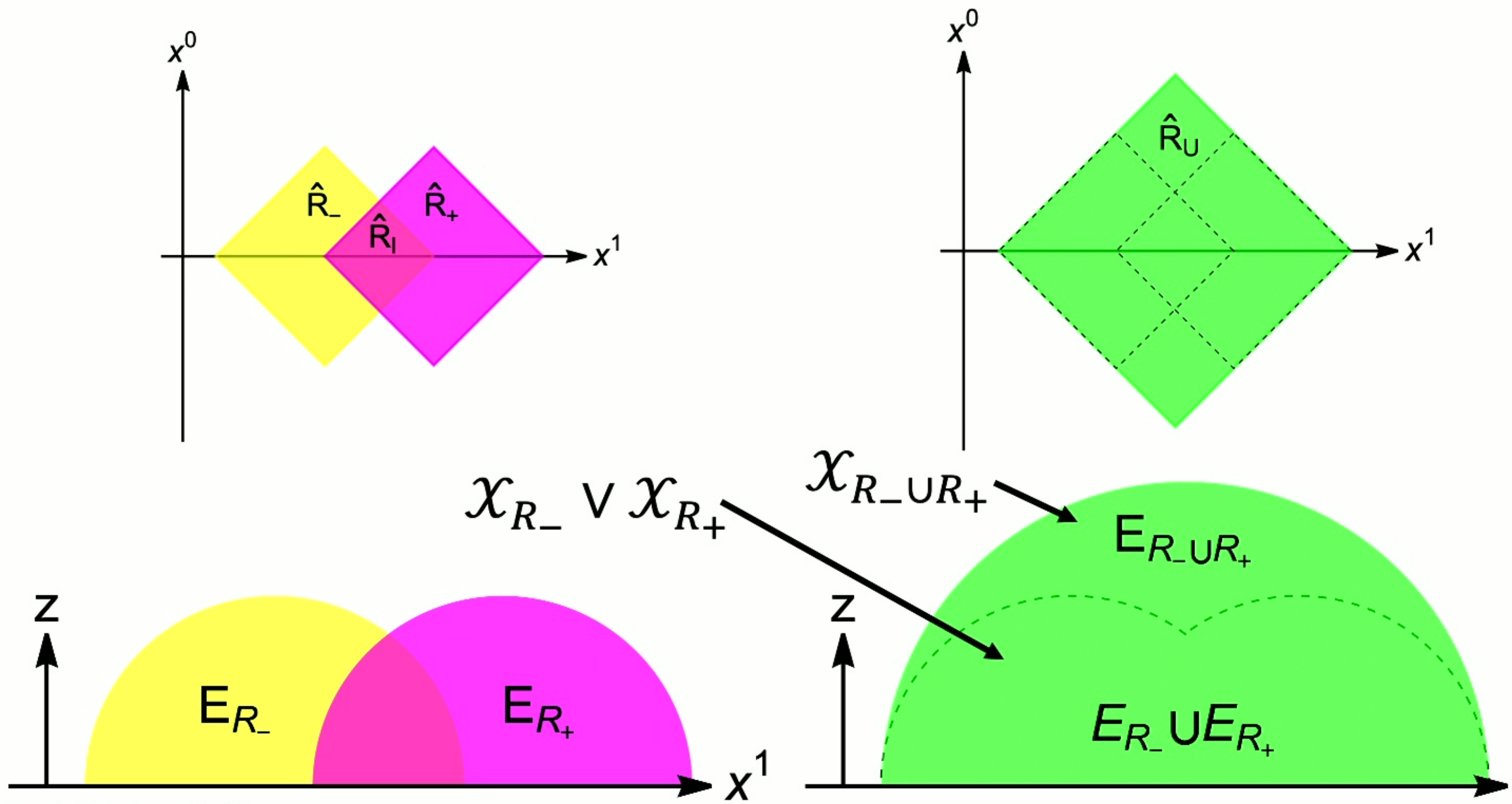




- Example of the additivity anomaly!

Additivity Anomaly Example 2

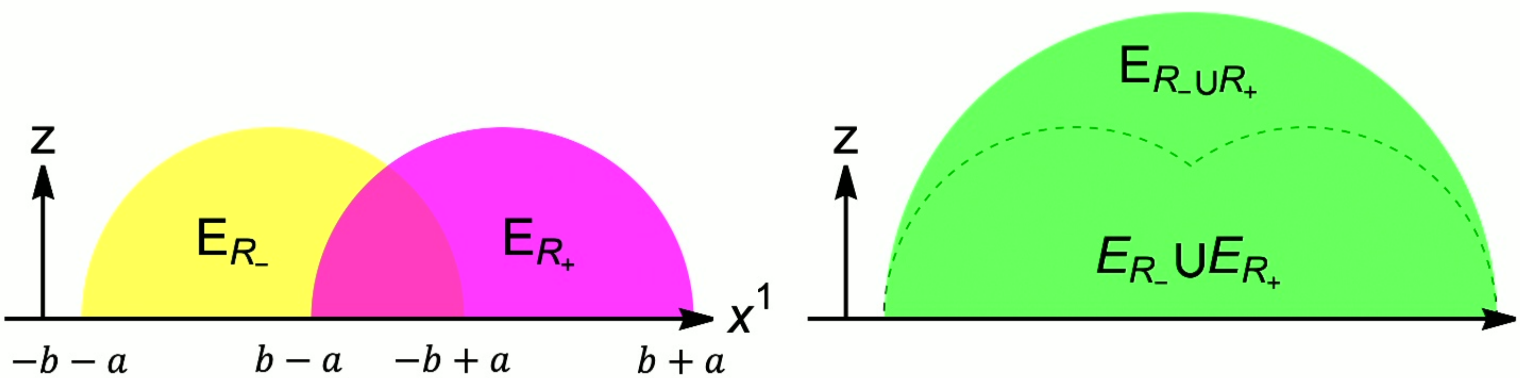
- Consider two overlapping intervals



Additivity Anomaly Example 2

- The difference in volumes of bulk duals is

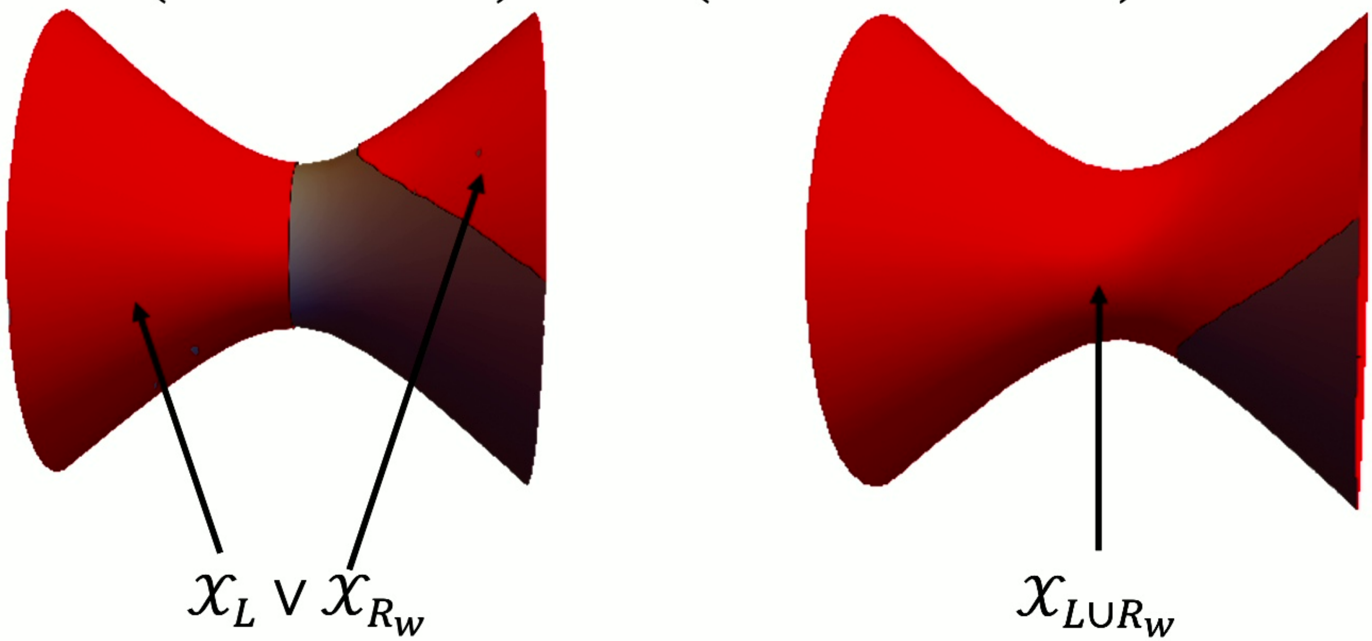
$$\mathrm{vol}\left(\mathrm{dual}\left(\mathcal{X}_{R_- \cup R_+}\right)\right) - \mathrm{vol}\left(\mathrm{dual}\left(\mathcal{X}_{R_-} \vee \mathcal{X}_{R_+}\right)\right)$$
$$= \mathrm{vol}\left(E_{R_- \cup R_+}\right) - \mathrm{vol}\left(E_{R_-} \cup E_{R_+}\right)$$
$$= 2 l_{AdS}^2 \tan^{-1} \frac{b}{\sqrt{a^2 - b^2}}$$



Additivity Anomaly Example 3 - BTZ

- The difference in volumes of bulk duals is

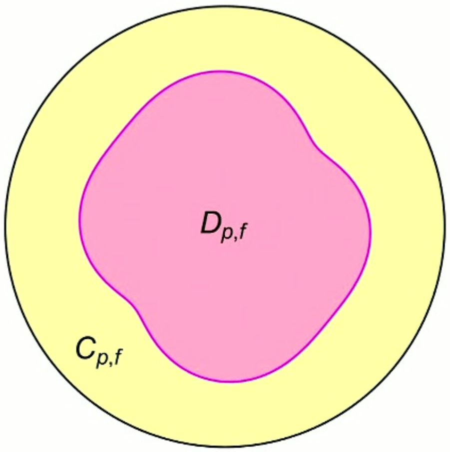
$$\mathrm{vol}\left(\mathrm{dual}\left(\mathcal{X}_{L\cup R_w}\right)\right) - \mathrm{vol}\left(\mathrm{dual}\left(\mathcal{X}_L \vee \mathcal{X}_{R_w}\right)\right) = 2\pi l_{AdS}^2$$



[Volumes computed in Ben-Ami, Carmi (2016); Abt *et. al.* (2018)]

General Bulk Subregions

- Bulk subregions that are not bounded by a boundary-anchored extremal surface are not described by subregion-subregion duality
- With boundary subalgebras we can describe more general bulk subregions

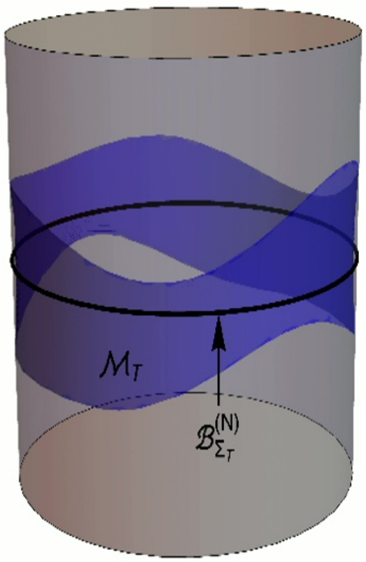


General Bulk Subregions

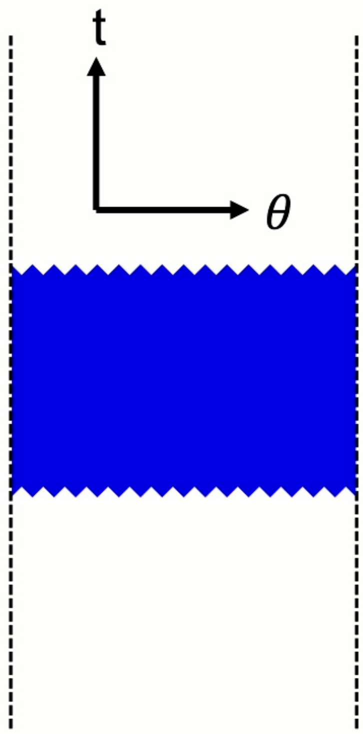
- Bulk subregions that are not bounded by a boundary-anchored extremal surface are not described by subregion-subregion duality
- With boundary subalgebras we can describe more general bulk subregions
- For non-causally complete boundary *spacetime* subregions R, we can still assign the algebra $\mathcal{M}_R$
- These are dual to the causal domain of R

Time-Band Algebras

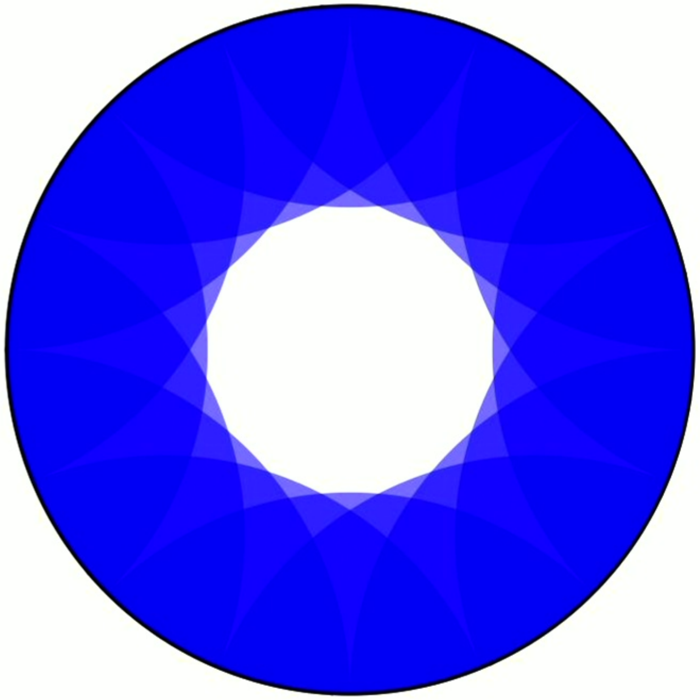
- T: time-band on the boundary
  $\mathcal{M}_T$ is a non-trivial subalgebra
- We can build up the algebra for T by combining the algebras of all small diamond-shaped regions in T
- Assuming bulk additivity, this allows us to deduce the bulk dual, which contains all near-boundary points
- Similar to the approach to the duality of differential entropy with the area of general bulk surface [Balasubramanian et. al. `13] [Headrick, Myers, Wien `14]



Example: Vacuum AdS_3, Many Diamonds



Boundary



Bulk time slice

Application: Subregion complexity?

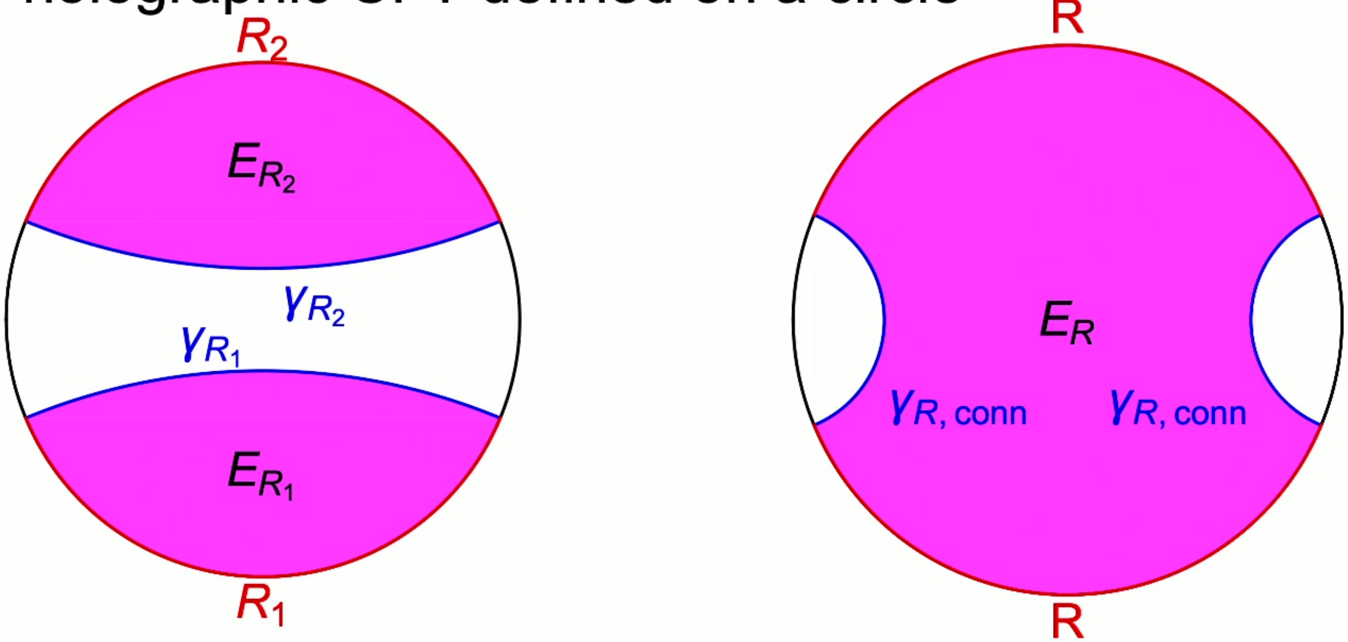
- Differences of volumes of bulk dual subregions gives a measure of the additivity anomaly, i.e.

$$\mathrm{vol}\left(E_{R_1 \cup R_2}\right) - \mathrm{vol}\left(E_{R_1} \cup E_{R_2}\right) \geq 0$$
$$\Updownarrow$$
$$\mathcal{X}_{R_1} \vee \mathcal{X}_{R_2} \subseteq \mathcal{X}_{R_1 \cup R_2}$$

  with equality of volumes only when the algebras agree
- Bulk volumes have been argued to be dual to complexity in the boundary theory [Susskind,…]

A suggestive example

- Consider two disjoint angular intervals for a holographic CFT defined on a circle



- Set-up has a "Python's lunch" even at $G_N = 0$
- Operators in the "lunch" are not in additive algebra

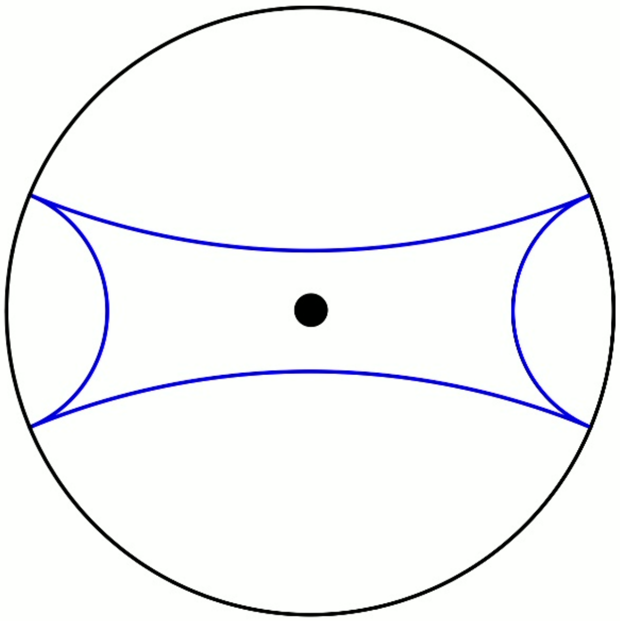
Application: Quantum Error Correction

- Subregion duality has been fruitfully understood from the language of QEC [Almheiri, Dong, Harlow `14,…]
- The embedding of the bulk Hilbert space in the CFT Hilbert space does not exist at $N = \infty$
- Semi-classical bulk physics is dual to the boundary GNS Hilbert space (not a subspace of $\mathcal{H}_{CFT}$)
- The delocalization of quantum information in holographic QEC can be understood as arising from violations of additivity in the large N limit

$$\mathcal{X}_R \vee \mathcal{X}_S \subseteq \mathcal{X}_{R \cup S}$$

Application: Global symmetries in QG

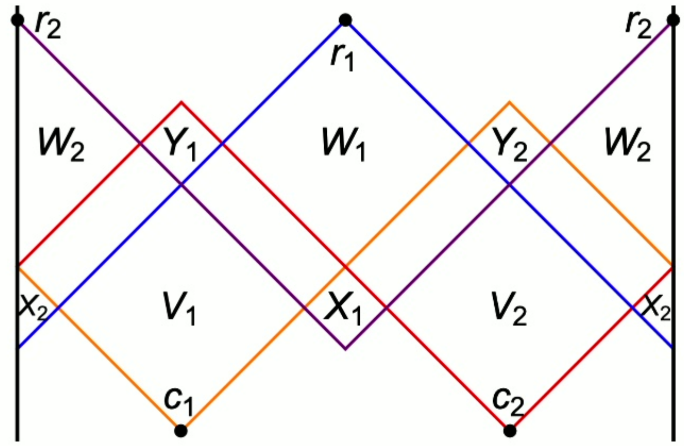
- The additivity anomaly provides an avenue for emergent global symmetry in the $G_N \to 0$ limit of quantum gravity, as it allows bulk symmetry operators to be non-additively generated, thereby evading the arguments of Harlow-Ooguri



[Harlow, Ooguri '18]

Application: Connected Wedge Theorem

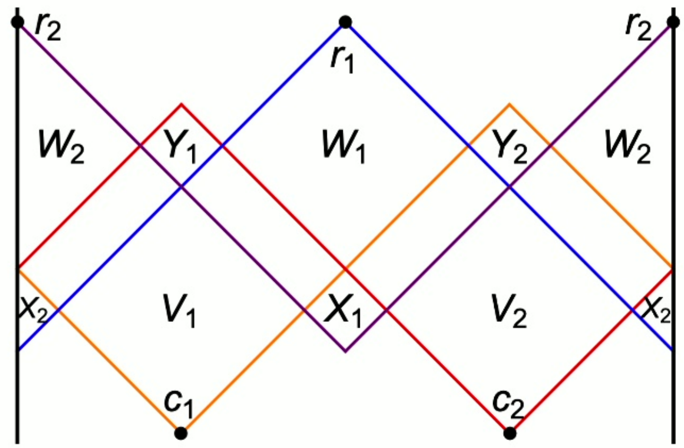
- The CWTs concern the relationship between boundary entanglement structure and bulk causal structure
- Bulk scattering may be possible when boundary scattering is impossible



[May '19, '21; May, Penington, Sorce '20; May, Sorce, Yoshida '22]

Application: Connected Wedge Theorem

- The CWTs concern the relationship between boundary entanglement structure and bulk causal structure
- Bulk scattering may be possible when boundary scattering is impossible



- Bulk scattering ⇒ large boundary mutual information
- ⇒ $E_{V_1 \cup V_2}$ is connected ⇒ $\mathcal{X}_{V_1} \vee \mathcal{X}_{V_2} \subset \mathcal{X}_{V_1 \cup V_2}$

[May '19, '21; May, Penington, Sorce '20; May, Sorce, Yoshida '22]

Application: CWT in Vacuum

- In vacuum, scattering region and EWs can be computed explicitly to verify the CWT
- In this case the scattering region is equal to the intersection of EWs: $S = S_E \equiv \mathfrak{b}_{V_1 \cup V_2} \cap \mathfrak{b}_{W_1 \cup W_2}$
- The vacuum CWT can then be phrased as

$$S_E \neq \emptyset \Leftrightarrow \mathcal{X}_{V_1} \vee \mathcal{X}_{V_2} \subset \mathcal{X}_{V_1 \cup V_2}$$

- Can be turned into a pure boundary statement

$$\mathcal{X}_{V_1 \cup V_2} \wedge \mathcal{X}_{W_1 \cup W_2} \neq \emptyset \Leftrightarrow \mathcal{X}_{V_1} \vee \mathcal{X}_{V_2} \subset \mathcal{X}_{V_1 \cup V_2}$$

- Relates an intersection anomaly and an additivity anomaly

Application: Generalized CWT?

- **Conjecture:** (outside of vacuum)
  $S_E \equiv \mathfrak{b}_{V_1 \cup V_2} \cap \mathfrak{b}_{W_1 \cup W_2} \neq \emptyset \Leftrightarrow \mathcal{X}_{V_1} \vee \mathcal{X}_{V_2} \subset \mathcal{X}_{V_1 \cup V_2}$
- Stronger version of CWT in the forward direction since $S_E \supseteq S$ when obeying NCC
- Weaker in the backwards direction, which allows it to evade the counterexample to the converse of the CWT of May, Penington, and Sorce
  - (Positive energy delays null rays from $\left(\mathfrak{b}_{V_1 \cup V_2}\right)'$ and $\left(\mathfrak{b}_{V_1 \cup V_2}\right)'$ which causes $\mathfrak{b}_{V_1 \cup V_2}$ and $\mathfrak{b}_{W_1 \cup W_2}$ to grow, unlike the scattering region which shrinks)

Future directions

- Obtain quantum extremal surfaces and islands from the algebraic definition of the dual
- Construct the differential entropy for algebras associated to time bands
- Explore the potential relationship between the additivity anomaly and complexity
- Define a generalized notion of complexity in terms of algebra inclusions at $N = \infty$
- Prove the conjectured generalization of the connected wedge theorem?

Thank you!