Title: Algebraic geometry at the limits of perturbative QFT and dark matter in moduli spaces

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Series: Colloquium

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Abstract: Feynman diagrams and the associated integrals are crucial in perturbative quantum field theory for translating theory input into concrete, observable quantities. I will talk about the fascinating interplay of physics and mathematics that emerges from the ensemble of these diagrams. From linear algebra to quantum mechanics or wave phenomena to Fourier analysis - physics and mathematics tend to regularly exchange ideas that often lead to breakthroughs on the other side. I will illustrate two new examples of such exchanges I contributed to. One exchange is from the mathematical theory of tropical geometry to evaluating intricate, physically relevant Feynman integrals that have been inaccessible before. In the second exchange, we used ensembles of Feynman diagrams and their renormalization to prove a long-standing conjecture in geometric group theory. The results give new insights into the 'dark matter'-problem in the moduli spaces of graphs and curves' cohomology. Both these moduli spaces' cohomologies are of fundamental interest in algebraic geometry, topology and geometric group theory.

Zoom link

Algebraic geometry at the limits of perturbative QFT and dark matter in moduli spaces

Perimeter Institute — February 20, 2024

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Quantum field theory

Language of physics at the fundamental level

Moduli Spaces

Collect geometric objects with varying parameters

Central structures in *(algebraic) geometry*, *topology* and *group theory*

\+ Interaction between both

Part I

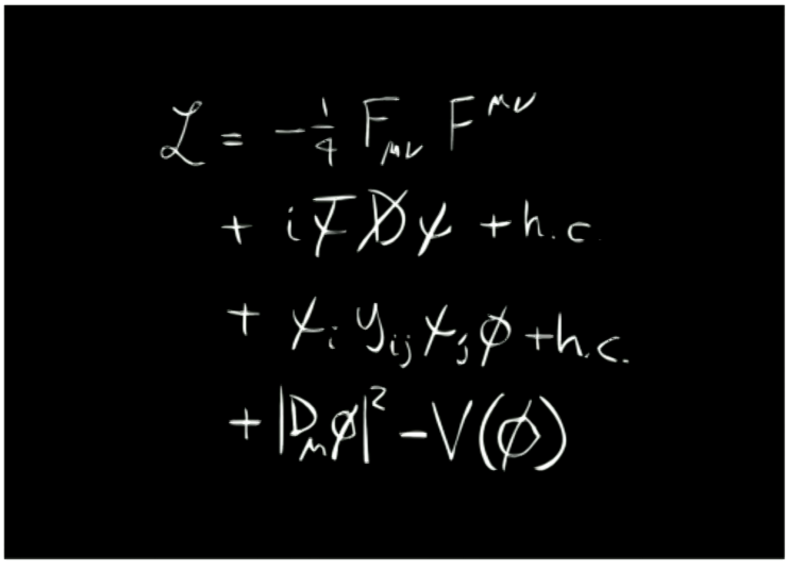
(Tropical) Geometry → Quantum Field Theory

applied to

Quantum Field Theory

Input

Lagrangian



Output

Perturbative expansions

$$\mathcal{O}(\lambda) = \sum_{n\geq 0} A_n \lambda^n = \sum_{\text{graphs } G} \frac{I_G}{|\mathrm{Aut}(G)|} \lambda^{|G|}$$

$$I_G = \int \frac{N}{\prod_e D_e} \mathrm{d}^D \mathbf{k}_1 \cdots \mathrm{d}^D \mathbf{k}_L$$

Feynman integral

Perturbative QFT work-flow used to describe

- Particle physics (e.g. collider phenomenology)
- Condensed matter physics (e.g. percolation theory (e.g MB-Gracey-Kompaniets-Schnetz ’21))
- Classical gravity/field theory
- …

- Various structures in mathematics (→ more about that in Part II)

Questions

$$\mathcal{O}(\lambda) = \sum_{n=0}^{\infty} A_n \lambda^n$$

Lower orders $A_0, A_1, A_2, \ldots$ needed to interpret experimental data.

Practical questions

What is the value of $A_0, A_1, A_2, A_3, \ldots$?

How can we calculate them effectively?

→

Abstract questions

Is there an algorithm to compute A_n?

What is the *fastest* algorithm?

What is its computational complexity?

Theoretical motivation

$$\mathcal{O}(\lambda) = \sum_{n=0}^{\infty} A_n \lambda^n$$

Computational complexity of A_n gives an explicit limit of our understanding of nature

Example

The Muon $g-2$ value is one of the most accurate predictions of theoretical physics.

Q: How many digits would we get using all of earth's computers after *1 year/10 years/a lifetime*?

Q: What is cheaper eventually (i.e. scales better): theoretical prediction or experimental measurement?

Runtime of perturbative QFT predictions

$$\mathcal{O}(\lambda) = \sum_{n=0}^{\infty} A_n \lambda^n \qquad\qquad A_n = \sum_{\substack{\text{graphs } G \\ |G| = n}} \frac{I_G}{|\text{Aut}(G)|}$$

Runtime to compute A_n

$$\mathcal{O}\left(\underbrace{c^n\,(n+k)!}\; \times \; \underbrace{F(n)} \right)$$

# of Feynman graphs with n loops

Runtime to evaluate a single Feynman integral

Q: What is $F(n)$?

Feynman integral evaluation

$$I_G = \int \frac{\mathrm{d}^D\mathbf{k}_1 \cdots \mathrm{d}^D\mathbf{k}_L}{\prod_e D_e}$$

$$\texttt{NIntegrate}\left[\int \frac{\mathrm{d}^D\mathbf{k}_1 \cdots \mathrm{d}^D\mathbf{k}_L}{\prod_e D_e}\right]?$$

…unfortunately, does not work. Evaluating I_G turns out to be a complicated & interesting algebraic-geometric problem.

Feynman integral evaluation

$$I_G = \int \frac{\mathrm{d}^D\mathbf{k}_1 \cdots \mathrm{d}^D\mathbf{k}_L}{\prod_e D_e} = C \cdot \int_{\sigma_E} \frac{\mathcal{N}}{\mathcal{U}^a \mathcal{F}^b}\, \Omega$$ *(Parametric representation)*

$$\sigma_E = \{\alpha \in \mathbb{R}^{|E|}_{>0} : \sum_e \alpha_e = 1\} \quad \Omega = \sum_k (-1)^k \alpha_k d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_k} \wedge \ldots \wedge d\alpha_{|E_\Gamma|}$$

$\mathcal{N}, \mathcal{U}, \mathcal{F}$ are homogeneous polynomials in $\alpha_1, \ldots, \alpha_{|E|}$

Vanishing loci of $\mathcal{N}, \mathcal{U}, \mathcal{F}$ meet boundary of $\sigma_E \Rightarrow$ many singularities

Fast evaluation algorithm for rational integrals

$$I = \int_{\sigma_n} \frac{p}{q}\, \Omega$$

with p, q polynomials in $\alpha_1, \ldots, \alpha_n$

Theorem (MB 2020)

If the Newton polytopes of p and q are generalized permutohedra*, then there is an $\mathcal{O}(n2^n + M_n\delta^{-2})$ evaluation algorithm for I.

*: **fulfilled by Feynman integrals**

n: **dimension of the integral;** M_n: **runtime to evaluate** p/q; δ: **rel. accuracy**

Critical technique: Tropical geometry

MB 2020

Tropical geometry is a degeneration technique in algebraic geometry in which the limiting object is combinatorial.

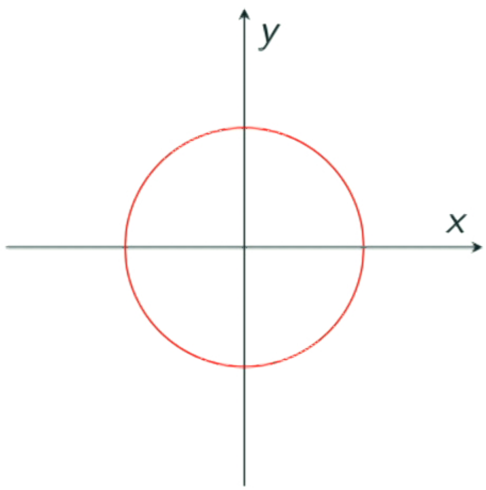
In the tropical limit, the rational integral $I = \int_{\sigma_n} \frac{p}{q}\,\Omega$ becomes the **volume of a polytope**.

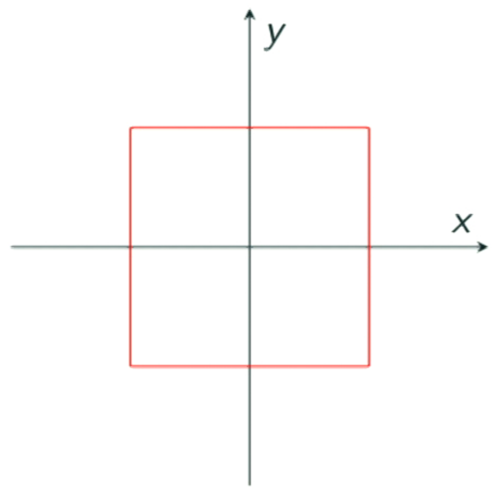
Tapping properties of special polytopes, **generalized permutohedra**, enabled the fast algorithm.

E.g. from: Postnikov 08; Aguiar-Ardila 17

Cartoon ‘Example’

$$1 = x^2 + y^2 \qquad \to \qquad 1 = (x^2 + y^2)^{\mathrm{tr}} = \max\{x^2, y^2\}$$





$$V = \pi \xrightarrow{\text{Rationalisation}} V = 4$$

Numerical Integration of Feynman integrals is a combinatorial task on certain polytopes

Tropical sampling algorithm MB 2020

Algorithm 4. Algorithm to generate a sample from μ^{tr} for generalized permutahedra

Set $A = [n]$ and $\kappa = 1$.
while $A \neq \emptyset$ **do**
  Pick a random $e \in A$ with probability $p_e = \frac{1}{J_r(A)} \frac{J_r(A \setminus e)}{r(A \setminus e)}$.
  Remove e from A, i.e., set $A \leftarrow A \setminus e$.
  Set $\sigma(|A|) = e$.
  Set $x_e = \kappa$.
  Pick a uniformly distributed random number $\xi \in [0, 1]$.
  Set $\kappa \leftarrow \kappa \xi^{1/r(A)}$.
end while
Return $\boldsymbol{x} = [x_1, \ldots, x_n] \in \text{Exp}(\mathcal{C}_\sigma) \subset \mathbb{P}^{n-1}_{>0}$ and $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$.

Improvements to traditional methods

- Fastest evaluation algorithms for Feynman integrals to date. Previous naive methods worked up to ≈ 3 loops, tropical based methods up to 20.
- Runtime question can be answered.
- Easy to implement.

⇒ Lots of interest from various communities (physics + mathematics).

⇒ New regimes in perturbation theory become accessible.

(Possible to study large-order behavior of pQFT → resurgence/non-pert. phenomena MB-Broadhurst-Dunne-Meynig ’20-’22)

✷Various possible extensions…

Extension to statistics on toric varieties

(MB-Sattelberger-Sturmfels-Telen '22)

Extension to integrals with Minkowski space-type singularities

(MB-Munch-Tellander '23)

+many other applications:
Heinrich '20
Arkani-Hamed, Hillman, Mizera '22
Dunne-Meynig '22
...
Brown-Schnetz '24

Open question

Is there a polynomial-time algorithm for Feynman integral evaluation?

State-of-art scaling: $\mathcal{O}(n \cdot 2^n + \delta^{-2} n^3)$

n = #edges/propagators; δ = rel. accuracy

$\delta^{-2} n^3$ can be improved to roughly $\delta^{-2} n$ using **Spielmann-Teng '04** Laplacian solver algorithm.

Open question

Is there a polynomial-time algorithm for Feynman integral evaluation?

State-of-art scaling: $\mathcal{O}(n \cdot 2^n + \delta^{-2} n^3)$

$n =$ #edges/propagators; $\delta =$ rel. accuracy

More ambitious open question

Is there a polynomial-time algorithm for amplitude evaluation in QFTs?

State-of-art scaling: $\mathcal{O}(\alpha^n \cdot (n+k)! \cdot (n \cdot 2^n + \delta^{-2} n^3))$

$n =$ perturbative order; $\delta =$ rel. accuracy

A promising approach: QFT tropicalization

Part II

Quantum Field Theory → (applied to) **Moduli space topology**

The moduli space of graphs

A motivation from physics

$$A_n = \sum_G \frac{1}{|\mathrm{Aut}(G)|} \int_{\sigma_{E_G}} \frac{\mathcal{N}}{\mathcal{U}^a \mathcal{F}^b} \Omega$$

Sum over graphs of order n

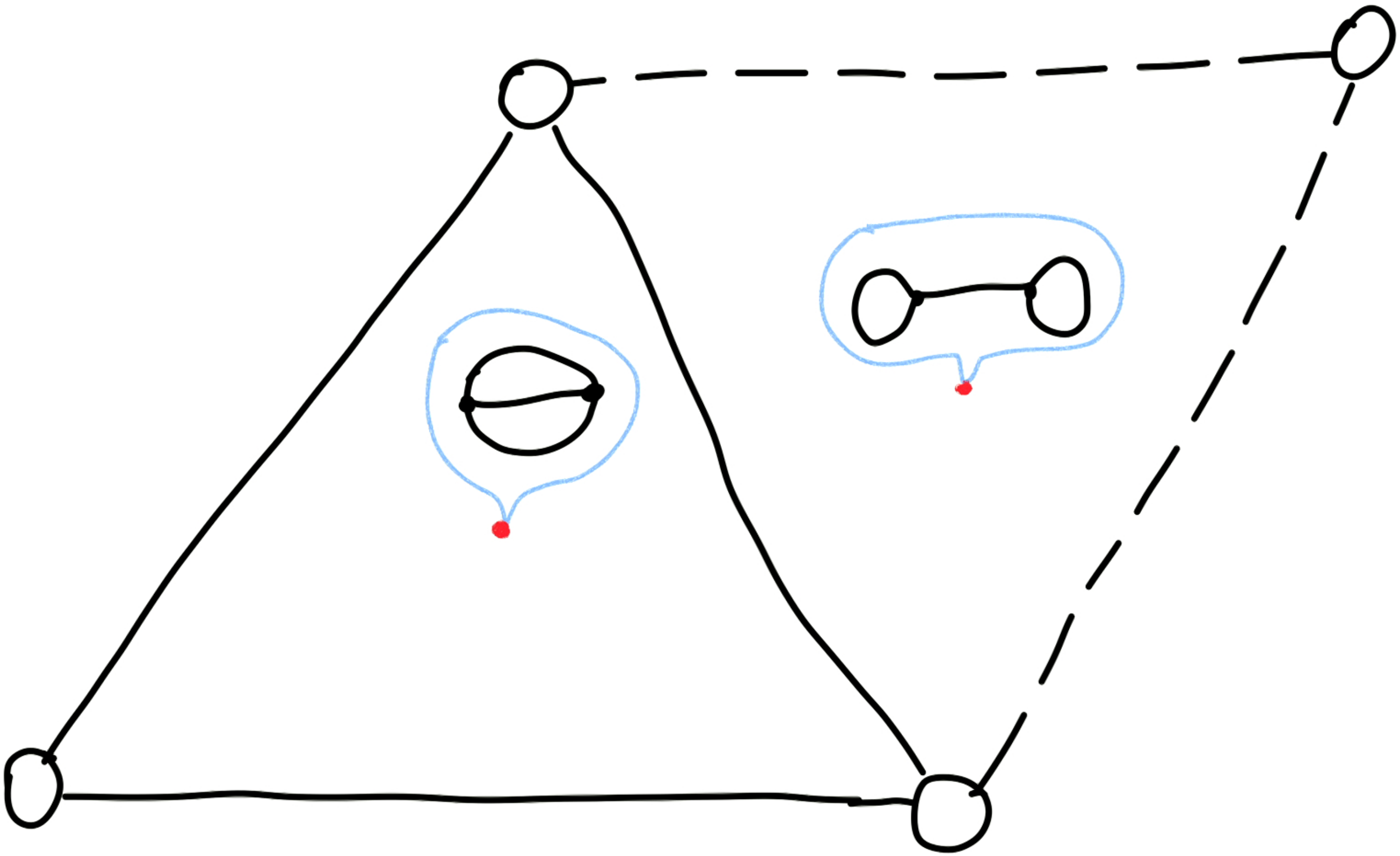
Can be interpreted as integral over the ***moduli space of graphs***, $\mathcal{MG}_n$

$$A_n = \int_{\mathcal{MG}_n} \mu_n$$

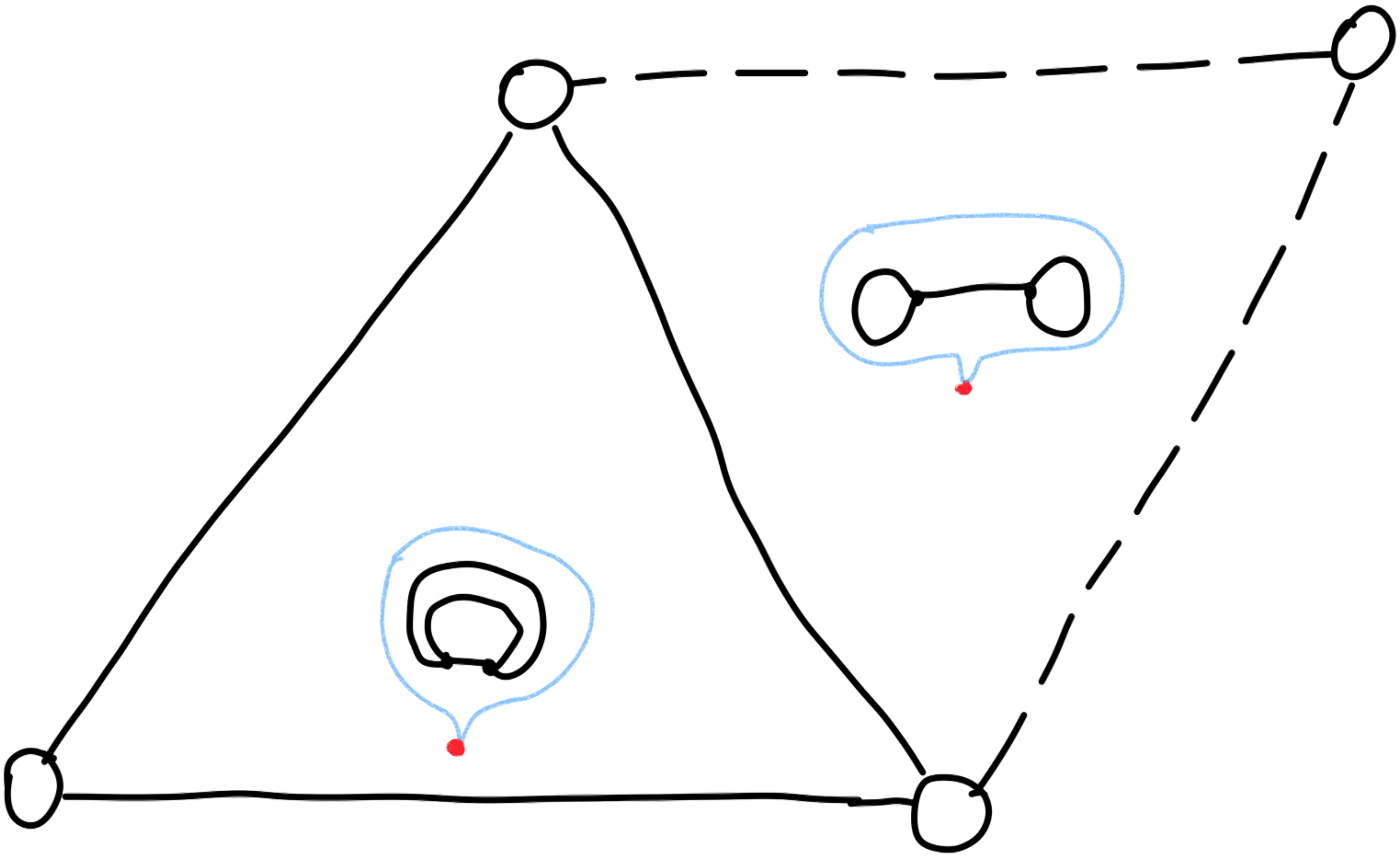
Moduli space of graphs $\mathcal{MG}_g$

- Each point in $\mathcal{MG}_g$ is metric graph i.e. a pair (G, ℓ) where
  - G is a graph with g loops and vertex degree ≥ 3
  - $\ell : E \to \mathbb{R}_{\geq 0}$ assigns a positive length to each edge, such that $\sum_e \ell_e = 1$
  - and such that there are no cycles of length 0.
- For $e \in G$, the point (G, ℓ) where $\ell(e) = 0$ is identified with $(G/e, \ell)$.
- Isomorphic metric graphs are identified.

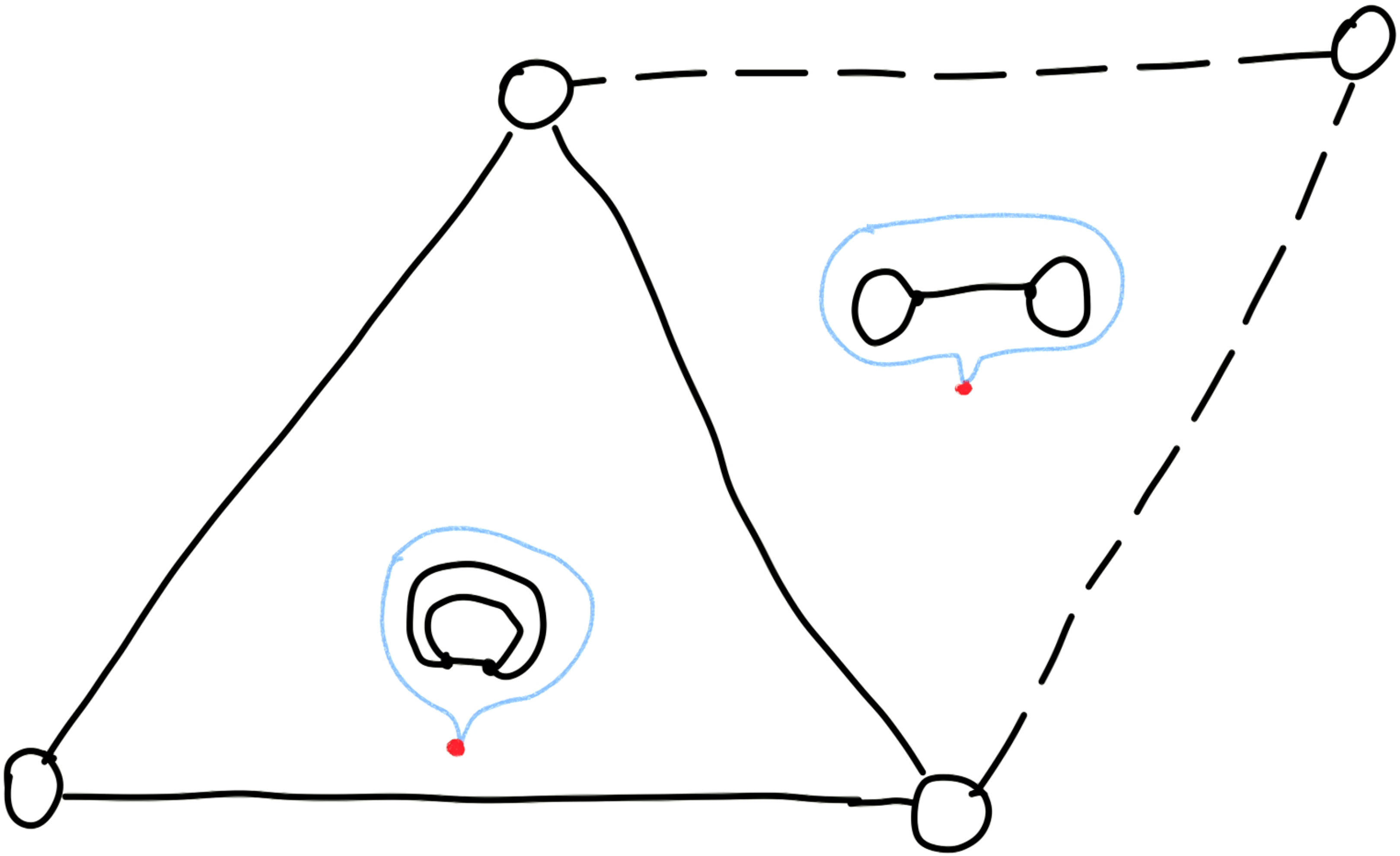
$\mathscr{MG}_2$



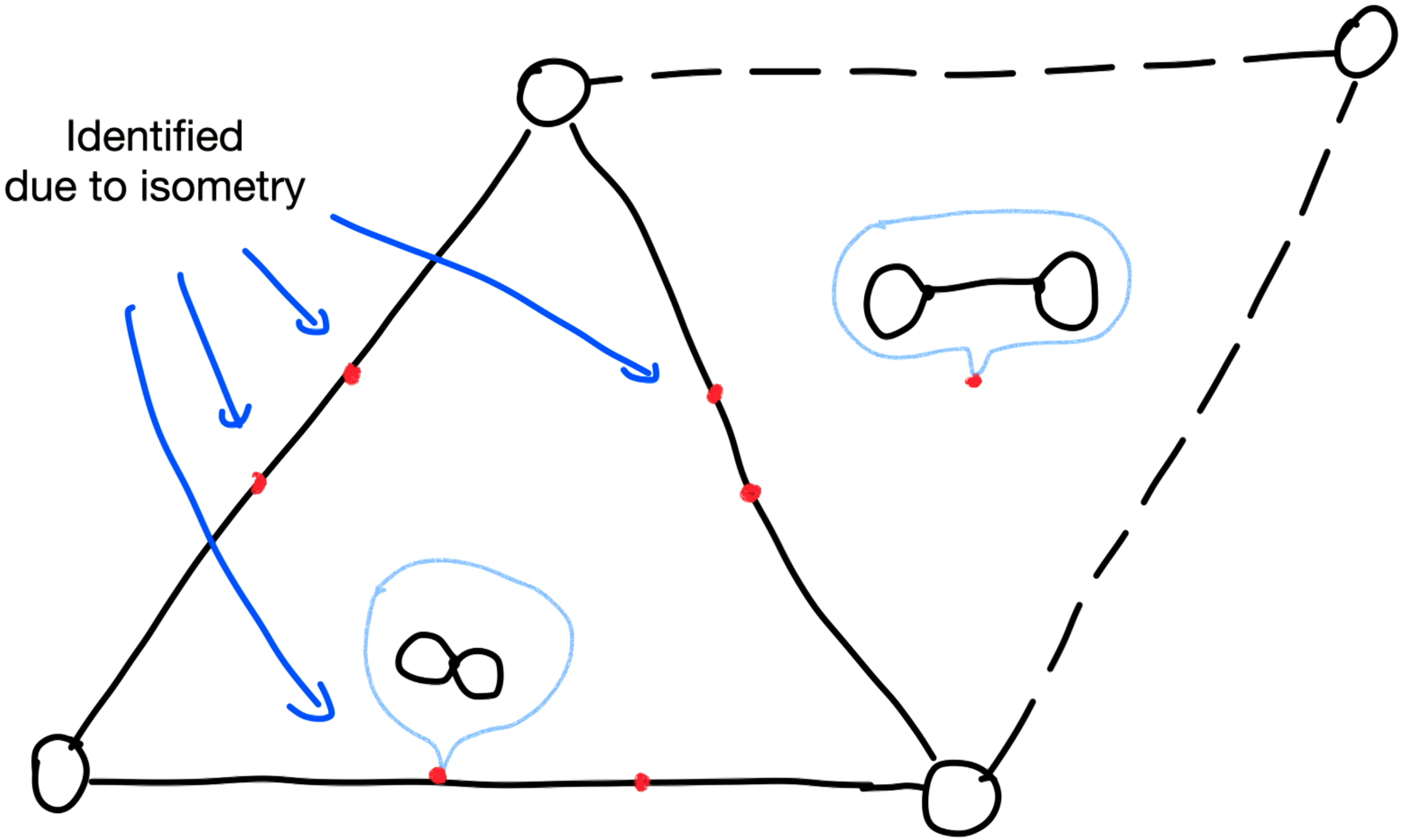
$\mathscr{MG}_2$



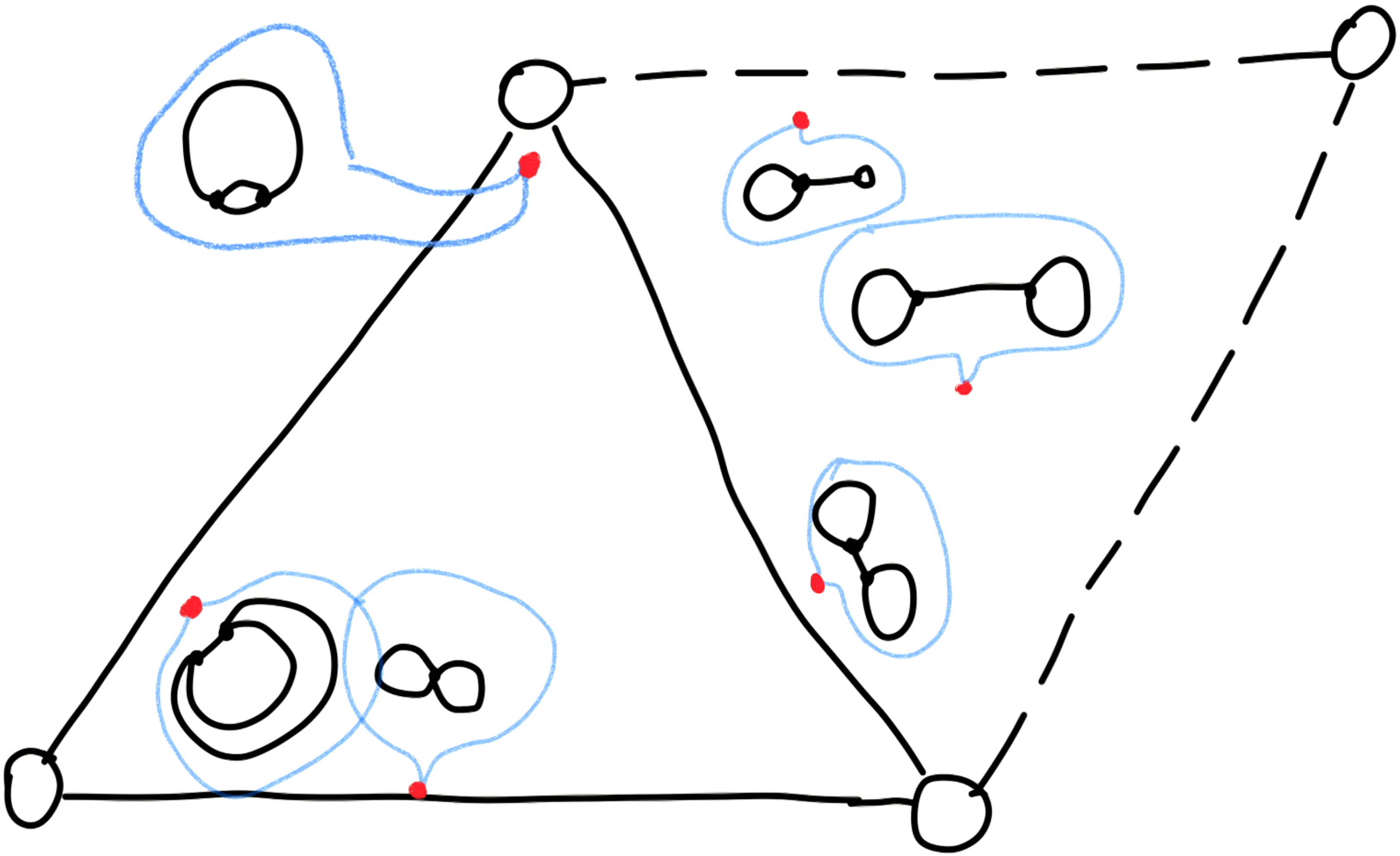
$\mathscr{MG}_2$



$\mathscr{M}\mathscr{G}_2$



$\mathscr{MG}_2$



Physics analogy

Moduli space of curves $\mathcal{M}_g$	$\Leftrightarrow$	**2D quantum gravity**
Moduli space of graphs $\mathcal{MG}_g$	$\Leftrightarrow$	**1D quantum gravity**

Moduli space of graphs $\mathcal{MG}_g$

Why is it interesting?

- $\mathcal{MG}_g$ is a (rational) classifying space for $\mathrm{Out}(F_g)$ **(Culler-Vogtmann 1986)**
- Topology of $\mathcal{MG}_g$ related to many other objects **(Kontsevich 1992; …)**
- Non-pure-math relevance: phylogenetic trees, QFT (e.g. CHY formalism), …
- *Tropical geometric* version of the moduli space of curves $\mathcal{M}_g$ **(Abramovich-Caporaso-Payne 2015; Chan-Galatius-Payne 2021)**

Cohomology of $\mathscr{M}\mathscr{G}_g$

Roughly, $H^k(\mathscr{M}\mathscr{G}_g)$ measures k-dimensional holes in $\mathscr{M}\mathscr{G}_g$.

Low rank computations

Ohashi '08, Bartholdi '16

$\dim H^k(\mathscr{M}\mathscr{G}_g; \mathbb{Q})$

(Similar in character to particle physics IBP reductions)

k\g	2	3	4	5	6	7	8	9	10	11	12
11						1					
10						0					
9					0	0					
8					1	1					
7				0	0	0					
6				0	0	0					
5			0	0	0	0					
4			1	0	0	0					
3		0	0	0	0	0					
2		0	0	0	0	0					
1	0	0	0	0	0	0					
0	1	1	1	1	1	1					

?

Low rank computations

Ohashi '08, Bartholdi '16

$\dim H^k(\mathscr{M}\mathscr{G}_g;\mathbb{Q})$

k\g	2	3	4	5	6	7	8	9	10	11	12
11						1					
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7				0	0	0					
6				0	0	0					
5			0	0	0	0					
4			1	0	0	0					
3		0	0	0	0	0					
2		0	0	0	0	0					
1	0	0	0	0	0	0					
0	1	1	1	1	1	1					
	1	1	2	1	2	1					

?

Euler characteristic

$$\chi(\mathscr{M}\mathscr{G}_g) = \sum_k (-1)^k \dim H^k(\mathscr{M}\mathscr{G}_g)$$

Lots of unexplained cohomology

Theorem (MB-Vogtmann, Adv. Math '23)

$$\chi(\mathscr{MG}_g) \sim -\, e^{-\frac{1}{4}} \left(\frac{g}{e}\right)^g / (g \log g)^2 \text{ for large } g.$$

$\Rightarrow$ The dimension of $H^\bullet(\mathscr{MG}_g)$ grows rapidly with g.

- Also, gives an effective generating function for $\chi(\mathscr{MG}_g)$,
- and answers open questions on graph complexes
  Kontsevich '92, Morita-Sakasai-Suzuki '15

Low rank computations

$\dim H^k(\mathcal{MG}_g; \mathbb{Q})$

Ohashi '08, Bartholdi '16

⇒ Lots of 'Dark matter'

k\g	2	3	4	5	6	7	8	9	10	11	12
11						1					
10						0					
9					0	0					
8					1	1					
7				0	0	0					
6				0	0	0					
5			0	0	0	0					
4			1	0	0	0					
3		0	0	0	0	0					
2		0	0	0	0	0					
1	0	0	0	0	0	0					
0	1	1	1	1	1	1					
	1	1	2	1	2	1	1	-21	-124	-1202	

?

Euler characteristic

$$\chi(\mathcal{MG}_g) = \sum_k (-1)^k \dim H^k(\mathcal{MG}_g)$$

Morita-Sakasai-Suzuki '14

Lots of unexplained cohomology

MB-Vogtmann, Adv. Math '23

Used combinatorial, analytic arguments and earlier results on the virtual Euler characteristic, $\chi^{\mathrm{virt}}(\mathcal{M}\mathcal{G}_g)$, of $\mathcal{M}\mathcal{G}_g$, (MB-Vogtmann, Comment. Math. Helv. '19)

Effectively we showed that:

$$\lim_{g\to\infty} \chi(\mathcal{M}\mathcal{G}_g)/\chi^{\mathrm{virt}}(\mathcal{M}\mathcal{G}_g) = e^{-1/4},$$

which contradicts the expectation that $\chi/\chi^{\mathrm{virt}} \to 1$.

Quantum field theoretical guidance

MB-Vogtmann, Comment. Math. Helv. '19

The generating function $T(\hbar) = \sum_{g \geq 2} \chi^{\mathrm{virt}}(\mathcal{M}\mathcal{G}_g)\hbar^{g-1}$ acts as ***counter-term***

for a **zero-dimensional quantum field theory**

$$1 = \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{1}{\hbar}\left(1 - x - e^x\right) + \frac{x}{2} + T(-\hbar e^{-x})\right) dx$$

'Semi-classical' analysis gives asymptotic results

Extension to $\mathscr{MG}_g$ with legs

Extension to moduli space of graphs with legs gives new **stabilisation result:**

Theorem (MB-Vermaseren, preprint '23)

If $n \geq g$, then $H^k(\mathscr{MG}_{g,n}; \mathbb{Q}) \simeq H^k(\mathscr{MG}_{g,\infty}; \mathbb{Q})$ for all k.

This large-n stable cohomology of $\mathscr{MG}_{g,n}$ is nontrivial.

(Guided by computations using particle physics software: **FORM**)

More applications

Theorem (MB-Brück-Willwacher, preprint '23)

$$H^\bullet(\mathscr{H}_g;\mathbb{Q}) \simeq H^\bullet(\mathscr{M}\mathscr{G}_g;\mathbb{Q}) \oplus H^\bullet(\overline{\mathscr{M}\mathscr{G}}_g;\mathbb{Q}) \oplus \cdots$$

Rational cohomology of the handlebody group $\mathscr{H}_g$ is also large.

Additional techniques used: (modular) operads and Feynman transform

(Getzler-Kapranov '98, Giansiracusa '11, Hainaut-Petersen '23)

Open question

What are all these classes in $H^k(\mathcal{MG}_g)$?

The 'dark matter problem' of moduli space cohomology

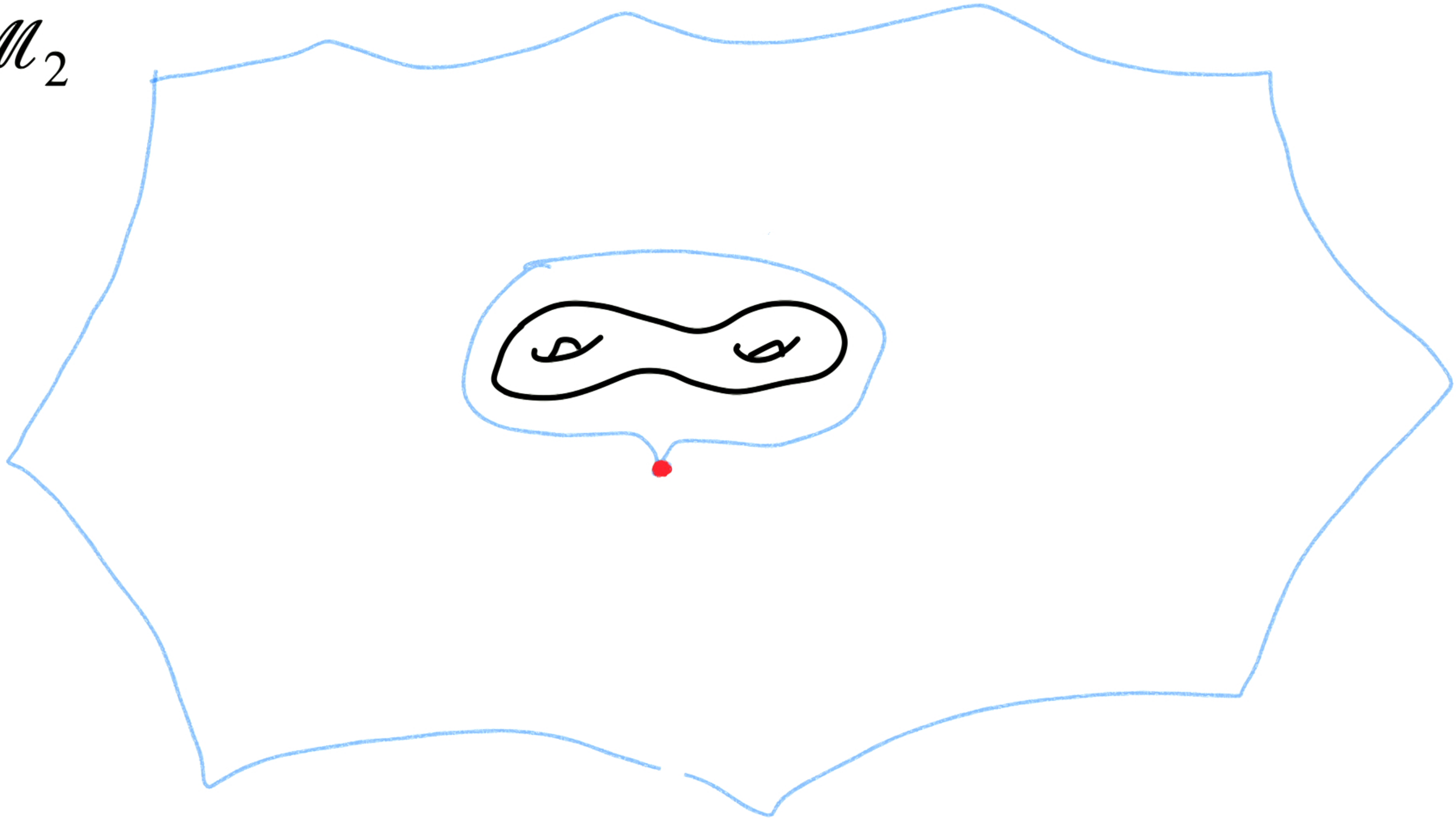
Moduli space of curves $\mathscr{M}_g$

$$\mathscr{M}_g =$$

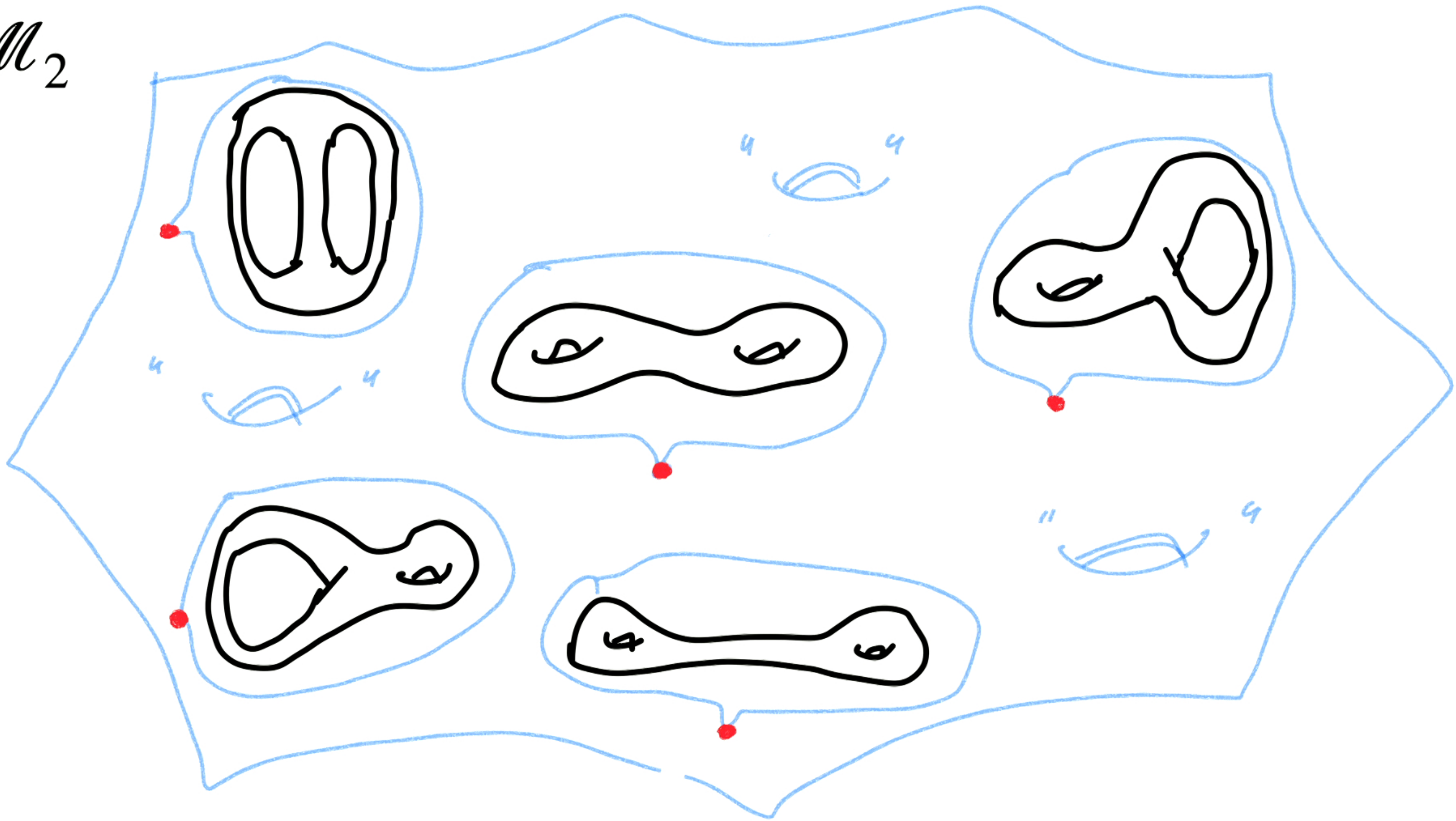
Space of all *hyperbolic metrics* on a compact, *connected and orientable surface* of genus g modulo isometric diffeomorphisms

i.e. space of compact (1+1)-dim spacetimes.

$\mathscr{M}_2$



$\mathscr{M}_2$



‘Boundary’ of $\mathscr{M}_g$ has the same topology as $\overline{\mathscr{M}\mathscr{G}}_g$, the compactification of $\mathscr{M}\mathscr{G}_g$

$\overline{\mathscr{MG}}_g$ gives information on the topology of $\mathscr{M}_g$

Theorem (Chan-Galatius-Payne '21):

The homology of $\overline{\mathscr{MG}}_g$ injects into the cohomology of $\mathscr{M}_g$.

$$H_{k-1}(\overline{\mathscr{MG}}_g) \hookrightarrow H^{6g-6-k}(\mathscr{M}_g)$$

(Image is the **top-weight cohomology** of $\mathscr{M}_g$)

Q: How much cohomology lives in the top-weight piece?

Euler characteristic of $\overline{\mathscr{M}\mathscr{G}}_g$ computed by infinite-field 0-dim QFT

Theorem MB '24 (upcoming)

$$\sum_{g\geq 2} \chi(\overline{\mathscr{M}\mathscr{G}}_g)\hbar^{g-1} = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \Psi(\hbar^k)$$

$$\Psi(\hbar) = \int \left(\prod_{k=1}^{\infty} \frac{dx_k}{\sqrt{2\pi\hbar^k/k}} \right) \exp\left(\sum_{k\geq 1} \frac{1}{k\hbar^k} \left(\exp\left(\sum_{\ell\geq 1} \frac{x_{k\cdot\ell}}{\ell} \right) - 1 - x_k \right) \right)$$

…which turns out to be integrable

$$\Phi(z) = \sum_{g\geq 1} \chi(\overline{\mathscr{M}\mathscr{G}}_{g+1})\, z^{-g}$$

$$\sum_{k\geq 1} \frac{\Phi(z^k)}{k} = \sum_{j\geq 1} \left(-B\left(\frac{j}{S_j(z)}\right) - \frac{1}{2}\log\frac{S_j(z)}{z^j} - \frac{z^j}{j} A\left(\frac{S_j(z)}{z^j}\right) + \frac{1+(-1)^j}{4j} \right)$$

$$A(x) = 1 - x + x\log x$$

$$B\,(x) = \sum_{k\geq 1} \frac{B_{k+1}}{k(k+1)} x^k$$

$$S_j(z) = \sum_{d|j} \mu(d) z^{j/d}$$

MB ’24 (upcoming)

Asymptotic/Semi-classical analysis gives

Theorem MB '24 (upcoming)

$$\chi(\overline{\mathcal{M}\mathcal{G}}_g) \sim \begin{cases} c_e\,(-1)^{g/2}\,(C_e\,g)^{g-\frac{3}{2}} & \text{for even } g \to \infty \\ c_o\,\sin\left(\frac{\sqrt{\pi g}}{2} - \frac{\pi}{4}(g-\frac{3}{2})\right)\,(C_o')^{\sqrt{g}}(C_o\,g)^{\frac{g}{2}-1} & \text{for odd } g \to \infty \end{cases}$$

Implication for $\mathcal{M}_g$

Theorem MB '24 (upcoming)

$$\sum_k \dim \mathrm{Gr}^W_{\mathrm{top}} H^k(\mathcal{M}_g) > \begin{cases} (C\,g)^g & \text{for even } g \\ (C\,g)^{\frac{g}{2}} & \text{for odd } g \end{cases}$$

Theorem Harer-Zagier '86

$$\sum_k \dim H^k(\mathcal{M}_g) > (C\,g)^{2g}$$

$\Rightarrow$ There are super-exponential amounts of top-weight cohomology in $\mathcal{M}_g$ but still only a small fraction of the cohomology has top-weight.

Summary

- Physics-inspired methods are fantastically effective to solve state-of-the-art questions in geometry & topology
- Current research in algebraic geometry very applicable to problems in physics

⇒Fascinating connections between fundamental objects in different domains

$$\chi(\mathcal{M}_g) \sim (-1)^{g+1} g^{2g} \int \underbrace{\frac{d^D k_1 \cdots d^D k_L}{D_1 \cdots D_{|E|}}}_{\mu}$$

$$\chi(gr^W_{top}\mathcal{M}_g) \sim (-1)^{\frac{g}{2}} \left(\frac{g}{e}\right)^g \int f(k_1,\dots,k_L)\,\mu$$

$$\Rightarrow \chi(gr^W_{non\text{-}top}\mathcal{M}_g) \sim (-1)^{g+1} g^{2g}$$