Title: Arithmetic Electric-Magnetic Duality

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Abstract: The Langlands program is a grand organizing vision for a large slice of number theory and representation theory. A shockingly accurate metaphor for the Langlands program has emerged as electric-magnetic duality in four-dimensional gauge theory, but where the role of spacetime is played by objects from arithmetic. I will describe recent work with Yiannis Sakellaridis and Akshay Venkatesh, in which we apply ideas from QFT (the Gaiotto-Witten electric-magnetic duality for boundary theories) to a fundamental problem in number theory, predicting the relation between L-functions of Galois representations and integrals of automorphic forms.

Zoom link

Arithmetic Electric-Magnetic Duality

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Colloquium
Perimeter Institute
February 15 2024

Overview

Describe a perspective on number theory
(integral representation of L-functions)
inspired by physics
(boundaries in topological gauge theory).

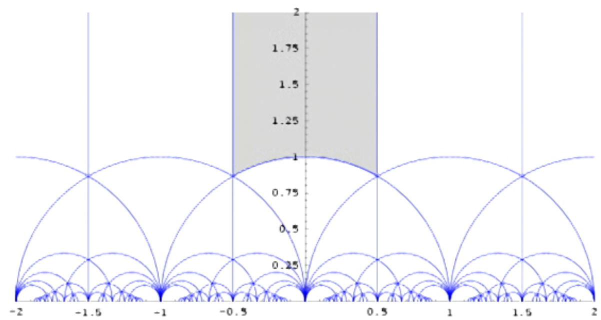
Based on joint work with
- Yiannis Sakellaridis (Johns Hopkins) and
- Akshay Venkatesh (IAS)

Arithmetic Quantum Mechanics

Automorphic forms: QM on arithmetic locally symmetric spaces

e.g., $\Gamma = SL_2(\mathbb{Z}) \circlearrowright \mathbb{H} = SL_2(\mathbb{R})/SO(2)$,

study L^2 or H^* of $\Gamma\backslash\mathbb{H}$



Arithmetic Quantum Mechanics

General story:

G reductive group (e.g., GL_n, Sp_n, E_8,..)

$\rightsquigarrow$

study (T)QM on arithmetic locally symmetric space

$$[G] = G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Structure in Arithmetic Quantum Mechanics

Very special QM problem:

- Hecke operators

Large family of commuting observables
T_p, p prime
"quantum integrable system"

General G:
Hecke operators at $p \leftrightarrow$ reps of Langlands dual group $\check{G}_{\mathbb{C}} \circlearrowright V$

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Structure in Arithmetic Quantum Mechanics

- Langlands correspondence:

Automorphic forms $\leftrightarrow$ Galois representations

"Fourier transform":

joint spectrum of observables $\sim$ representations of Galois group $\{\rho : Gal(\overline{F}/F) \to \check{G}\}$

Hecke operator $T_{p,V}$
$\leftrightarrow$
function $\rho \mapsto Tr_V(\rho(Fr_p))$

Structure in Arithmetic Quantum Mechanics

- Langlands correspondence:

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"Fourier transform":

joint spectrum of observables $\sim$ representations of Galois group
$\{\rho : Gal(\overline{F}/F) \to \check{G}\}$

$F = \mathbb{Q}$

Hecke operator $T_{p,V}$
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More Bells and Whistles

Lots of arithmetic locally symmetric spaces:

- Can vary ramification:
$\Gamma = SL_2(\mathbb{Z}) \rightsquigarrow$ subgroups defined by congruences mod N, study

$$\Gamma_N \backslash \mathbb{H}$$

- Can vary number field:
$F/\mathbb{Q}$ finite $\rightsquigarrow$

$$[SL_2]_F = SL_2(\mathcal{O}_F)\backslash SL_2(F \otimes \mathbb{R})/K$$

(e.g., $F = \mathbb{Q}(\sqrt{-d}) \rightsquigarrow$ arithmetic quotients of $\mathbb{H}^3$)

Automorphic forms as TQFT

Much richer paradigm:

theory of G-automorphic forms captured by

arithmetic 4d quantum field theory $\mathcal{A}_G$

Arithmetic extension of

- topologically twisted 4d $\mathcal{N} = 4$ super-Yang-Mills for G_c [1]
- Kapustin-Witten interpretation of Geometric Langlands

[1] GL twist at $\Psi = 0$

Arithmetic topology

- Arithmetic Topology: (refinement of Weil's "Rosetta Stone")[2]

F number field $\leftrightarrow$ 3-manifold Σ

Primes $\leftrightarrow$ Knots in Σ

local fields $\mathbb{Q}_p, \mathbb{R}, \mathbb{F}_p((t))$ $\leftrightarrow$ 2-manifolds

...

Based on Poincaré duality in étale cohomology
e.g. $\mathbb{F}_p \sim S^1$, with π_1 generated by Frobenius

[2] Mazur, Morishita, Kapranov, Reznikov. M. Kim: arithmetic Chern-Simons

Arithmetic TQFT

Idea: TQM on $[G]_F$

$\leftrightarrow$

Hilbert space of TQFT on $\Sigma \times \mathbb{R}$ – topology of space of gauge fields on Σ

$[G]_F = \mathrm{Bun}_G X$

$X = \mathrm{Spec}\, \mathcal{O}_F$

Arithmetic gauge theory: Hecke and Loop Operators

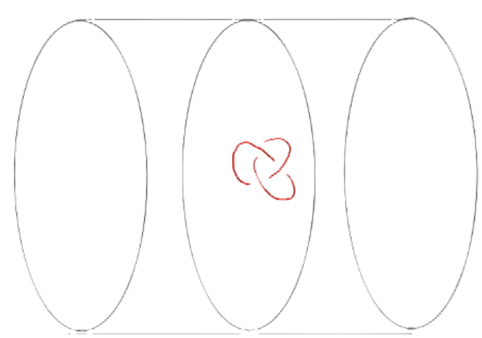
- Hecke operators at prime p

$\leftrightarrow$

1d defects: 't Hooft loop operators along a knot K:

insert magnetic monopole along $K \times \frac{1}{2} \subset \Sigma \times [0, 1]$

singularity measured by rep of dual group $\check{G}$.

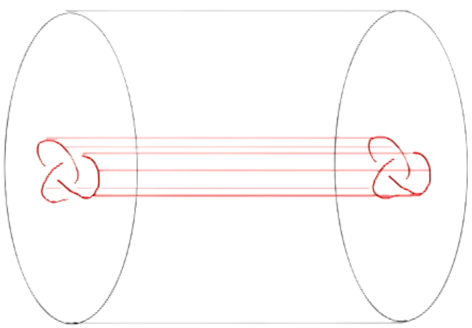


Arithmetic gauge theory: Ramification and Surface Operators

- Ramification mod N

$\leftrightarrow$

2d defects[3]: solenoids along a link $L \times \mathbb{R} \subset \Sigma \times \mathbb{R}$



[3] Gukov-Witten

Gukov-Witten

Arithmetic gauge theory

- Langlands correspondence $\leftrightarrow$ Electric-Magnetic Duality [4] for twisted $\mathcal{N} = 4$ SYM

4d A-model $\mathcal{A}_G \sim$ topology of spaces of connections
$\leftrightarrow$
4d B-model $\mathcal{B}_{\check{G}} \sim$ algebraic geometry of spaces of flat connections / monodromy representations

$$\{\pi_1(\Sigma) \to \check{G}\}$$

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[4] Montonen-Olive S-duality

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Langlands / Electric-Magnetic Duality

automorphic	spectral
magnetic	electric
$\mathcal{A}_G$	$\mathcal{B}_{\check{G}}$
topology: - of spaces of connections - of arith. loc. sym. spaces	algebraic geometry: - of flat connections - of Galois representations
1d defects (loops): Hecke / 't Hooft	Wilson / trace
2d defects (solenoids): congruence subgroups	singularities of flat connections
BZSV: 3d defects (bdry theories) periods	L-functions

Outline

Arithmetic and Quantum Field Theory

Periods and L-functions

Relative Langlands Duality

BZSV

- Our work (BZ-Sakellaridis-Venkatesh):

Apply paradigm to theory of integral representations of L-functions as periods :

understand using E-M duality for 3d defects (boundary conditions) in gauge theory[5].

[5] Gaiotto-Witten

Hot Take

What are L-functions?

Number Theory's counterparts of partition functions in Quantum Field Theory

The fundamental invariants we wish to calculate, whose direct definition is nonsensical without regularization

Example:

Q: How many positive integers are there?

A: $-\frac{1}{2}$

- $1 =$ class number of $\mathbb{Q}$ = rank of unit group of $\mathbb{Q}$
- $2 = \#$roots of unity in $\mathbb{Q}$

Riemann ζ

Science behind this:

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}$$

Formally $\zeta(0) = \sum_{n\in\mathbb{N}} 1$,
define by regularization (analytic continuation)
like path integrals...

Riemann ζ

Two dual descriptions of ζ:

- Euler product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

spectral / Galois theory
Encodes arithmetic origin

- Riemann's period integral

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty y^{s/2} \sum_{n=0}^{\infty} e^{-n^2 \pi y}\, dy$$

automorphic / rep. theory
Encodes analytic properties

L-functions

Deep arithmetic invariants of algebraic varieties.
Functions of representations $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \check{G}$ into $\check{G}$,
Recover ζ for ρ trivial

General form:

$$L(\rho, V, s) = \prod_p \frac{1}{\det(I - p^{-s}\rho(F_p))}$$

$\check{G} \circlearrowright V$ finite dimensional representation,

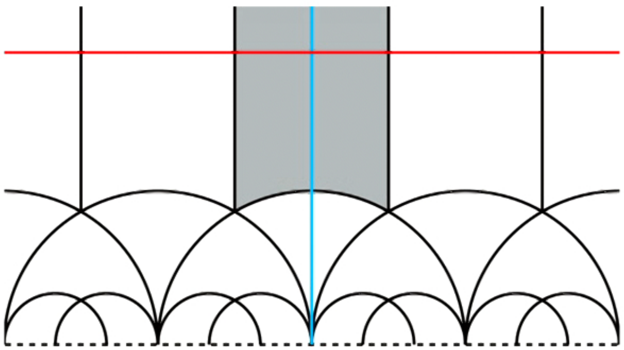
F_p =Frobenius at p

Hecke Period

Hecke period: φ modular form on $\Gamma\backslash\mathbb{H}$

$$\mathcal{P}_{G/T}(\varphi) := \int_0^\infty \varphi(iy) y^s \frac{dy}{y}$$

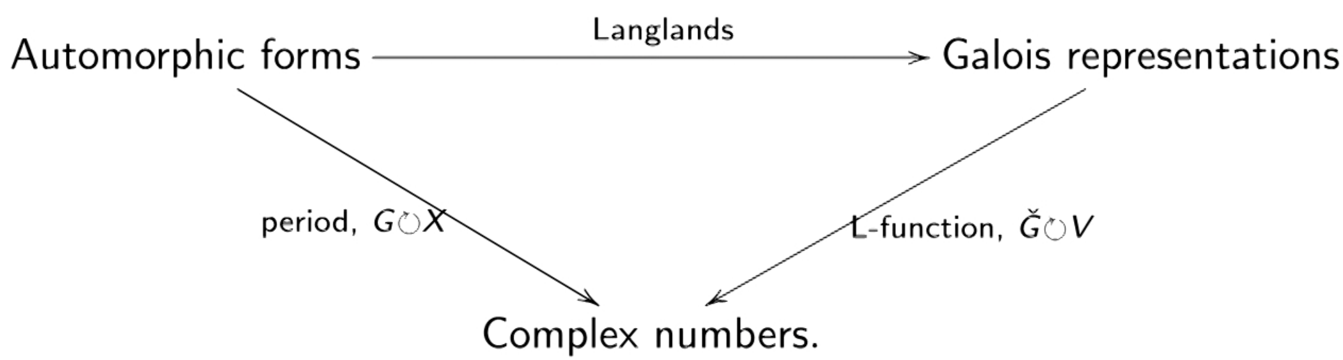
$T \subset G = SL_2\mathbb{R}$ torus[6] $\rightsquigarrow$ T-period – integral over $[T] \subset [G]$



[6] $N \subset SL_2\mathbb{R} \rightsquigarrow$ integral over $[N] \subset [G]$, constant term/Eisenstein period

The big picture

Integral representation: most important tool to access L-functions (analytic continuation, functional equation,...):



The big picture

e.g. Shimura-Taniyama:

φ modular form $\Longleftrightarrow$ E elliptic curve,
matching
Hecke G/T-period of φ $\Longleftrightarrow$ standard L-function of ρ_E:

$$\mathcal{P}_{G/T}(\varphi) = \frac{\Gamma(s)}{(2\pi)^s} L(\rho_E, V^{std}, s)$$

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More general periods

Can define period integral $\mathcal{P}_X$ for any G-space X

e.g. Riemann's integral generalized by Iwasawa, Tate to express abelian L-functions[7]:

periods for $G = GL_1 \circlearrowright X = \mathbb{A}^1$.

[7] Riemann and Dedekind ζ-, Dirichlet and Hecke L-functions

Periods vs. L-functions

Problem: Basic mismatch of data!

L-functions of $\rho : Gal \to \check{G}$ naturally labeled by representations V of $\check{G}$

Completely unrelated to data of G-varieties $G \circlearrowright X$

$$\mathcal{P}_{X=??}(\varphi) \longleftrightarrow L(s, \rho, V)?$$

Huge collection of examples of integral formulas, seek coherent theory

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Outline

Arithmetic and Quantum Field Theory

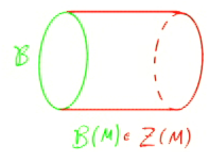
Periods and L-functions

Relative Langlands Duality

Boundary conditions

BZSV: Periods, L-functions extend to boundary theories for $\mathcal{A}_G$, $\mathcal{B}_{\check{G}}$:

For any Σ, QFT on $\Sigma \times [0,1]$ with local boundary theory produces state



Implies much richer structure
(locality, defects, arithmetic vs. geometric ...)
for periods and L-functions,
uniformly encodes the *relative Langlands program*

E-M duality of boundary conditions

Gaiotto-Witten: duality $\mathcal{A}_G \simeq \mathcal{B}_{\check{G}}$ identifies boundary theories

- Obtain SUSY boundary theories in $\mathcal{N} = 4$ SYM for G from σ-models into hyperkähler hamiltonian G-spaces.

- Can also extract hamiltonian $\check{G}$-spaces from these as Coulomb branches[8]

$\rightsquigarrow$

suggests a duality between Hamiltonian actions of dual groups!

Periods as boundary theories

BZSV:

periods for $G \circlearrowright X \leftrightarrow$

integrals over G-twisted maps to $X \leftrightarrow$

partition functions of 3d σ-model[9] into $M = T^*X$
coupled to G-gauge theory $\mathcal{A}_G$:

automorphic quantization $\Theta_M \in \mathcal{A}_G$ of $G \circlearrowright M$

Periods as boundary theories

Role of $M = T^*X$: QM on X really microlocal.
e.g. Tate: fun. eqn. for $\zeta(s) \Leftarrow$ Fourier transform on $\mathbb{A}^1$.

Replacing $X \rightsquigarrow M$: exhibit more symmetry, covers more examples

Local version (codim 2): harmonic analysis for $G(F) \circlearrowright L^2(X(F))$ $(F = \mathbb{R}, \mathbb{Q}_p, ..)$

Boundary theory encodes Plancherel measure, Θ-series operators, local-global compatibilities...

L-functions as boundary theories

Key point: understand L-functions for $\check{G} \circlearrowright V$ as partition functions for boundary theories.

$L(\rho, V, 0) \sim$ volume of $\check{G}$-twisted maps into $\check{X} := V$
(derived) Galois fixed points on V
$\leftrightarrow$

partition function of 3d σ-model (Rozansky-Witten) into
$\check{M} = T^*\check{X}$
coupled to $\check{G}$-gauge theory $\mathcal{B}_{\check{G}}$

spectral quantization $\mathcal{L}_{\check{M}} \in \mathcal{B}_{\check{G}}$ of $\check{G} \circlearrowright \check{M}$

Characteristic polynomials revisited

Why?

What is a characteristic polynomial?

$$det(I - tF)|_V = Tr_t(F, \Lambda^\cdot V)$$

graded trace of F on exterior algebra...

i.e., on spin representation for orthogonal[10] space $\check{M} = V \oplus V^*$

[10] or odd symplectic

Characteristic polynomials revisited

What is an inverse characteristic polynomial?

$$\frac{1}{det(I - tF)|_V} = Tr_t(F, Sym^{\cdot} V)$$

Think: $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$

...geometric quantization of symplectic space $\check{M} = V \oplus V^*$

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Characteristic polynomials revisited

What is an inverse characteristic polynomial?

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...geometric quantization of symplectic space $\check{M} = V \oplus V^*$

L-functions as spectral quantization

Consequences of geometric perspective:

- Nonlinear L-functions: replace V by any $\check{G} \circlearrowright \check{X}$

- Microlocal L-functions: depends not on $\check{X}$ but on $\check{M} = T^*\check{X}$,
encodes functional equation $V \leftrightarrow V^*$,
expect to extend to general hamiltonian $\check{G} \circlearrowright \check{M}$:

Exhibit more symmetry, cover more examples

Relative Langlands Duality

Relative Langlands Duality Conjecture:

- there is a duality

$$\boxed{G \circlearrowright M \quad \longleftrightarrow \quad \check{G} \circlearrowright \check{M}}$$

between hyperspherical[11] hamiltonian actions of dual groups

- automorphic and spectral quantizations exchanged by Langlands correspondence (global / local, arithmetic / geometric):

$$\boxed{\Theta_M \in \mathcal{A}_G \longleftrightarrow \mathcal{L}_{\check{M}} \in \mathcal{B}_{\check{G}}}$$

e.g. M-periods match $\check{M}$-L-functions

[11] new class, containing cotangents to spherical varieties

Sea of Conjectures

	automorphic	spectral
hyperspherical varieties:	$G \circlearrowright M$	$\check{G} \circlearrowright \check{M}$
global arithmetic: (numbers)	M-periods of automorphic forms	$\check{M}$-L-function of Galois reps
global geometric: (vector spaces)	M-period sheaf on Bun_G	$\check{M}$-L-sheaf on $Loc_{\check{G}}$
local arithmetic: (vector spaces)	spherical functions on $X(K_v)$	functions on $(\check{M})^{Frob}$
local geometric: (categories)	spherical sheaves on $X(K_v)$	quasicoherent sheaves on $\check{M}$

Duality for boundary conditions

Can build hamiltonian $\check{G}$-variety $\check{M}$ directly out of $G \circlearrowright M$

as moduli space of vacua[12] for boundary theory

i.e., via algebra of local operators on boundary.

Identify $\check{M}$ as geometrization of Plancherel formula for $L^2(X)$

$\rightsquigarrow$

Precise conjectural description of $\check{M}$ via local harmonic analysis

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[12] Coulomb branch, cf. Braverman-Finkelberg-Nakajima

Hyperspherical varieties

Strong finiteness condition: $M, \check{M}$ need to be hyperspherical, e.g. $M = T^*X$ for X spherical:

$G \circlearrowright X$ is spherical if $\mathbb{C}[X]$ multiplicity one G-rep

Spherical variety: nonabelian version of toric variety

- Tate: Toric varieties
- Hecke: PGL_2/T
- Eisenstein: Flag varieties G/P (or G/U as $G \times L$-space)
- Group: $G = H \times H \circlearrowright X = H$
- Symmetric spaces G/K
- Branching, Gan-Gross-Prasad : $GL_{n+1} \times GL_n \circlearrowright GL_{n+1}$, $SO_{2n+1} \times SO_{2n} \circlearrowright SO_{2n+1}, \cdots$

Some current directions

- Duality for domain walls $\rightsquigarrow$ Langlands functoriality
- Include surface defects $\rightsquigarrow$ ramified version
- Interfaces of boundary theories $\rightsquigarrow$ γ-factors, unfoldings
- Anomalies for unpolarized M $\rightsquigarrow$ ϵ-factors
- Nahm pole boundary conditions $\rightsquigarrow$ Arthur parameters
- Lift to 5d SYM $\rightsquigarrow$ quasi-Hamiltonian duals
- ...