Title: The r-matrix structure of Hitchin systems via loop group uniformization

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Abstract: The Hitchin systems are a remarkable family of integrable models associated to the moduli space of principal bundles on a compact Riemann surface. In this talk, I explain how the loop group uniformization of this moduli space can be used to construct an r-matrix for the Hitchin systems. This r-matrix has been previously used in the description of the Friedan-Schenker connection on the space of conformal blocks.

Zoom link

The r-matrix structure of Hitchin systems

Hitchin systems

X = cpt. Riemann surface

G = ctd. semisimple alg group

$\mathcal{M}$ = moduli space of G-bundles on X (reg. stable)

$\mathcal{M} \ni P$

Fact: $T_P\mathcal{M} = H^1(X, \mathrm{Ad}(P))$, $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$

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Fact: $T_P \mathcal{M} = H^1(X, \underbrace{\mathrm{Ad}(P)}_{= P \times_G \mathfrak{g}})$

$$\Longrightarrow \quad T_P^* M = H^1(X, Ad(P))^* \overset{\text{Serre}}{\cong} H^0(X, Ad(P) \otimes \Omega_X)$$

$$\Big\downarrow \ \text{homogeneous pol on } \mathfrak{g} \text{ of deg. } d$$

$$H^0(X, \Omega_X^{d})$$

$\underline{\text{Fact.}}$ $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \dots, P_{rk(\mathfrak{g})}]$, P_i hom. of deg d_i, $rk(\mathfrak{g}) = \ell$

$$T^* M \longrightarrow \bigoplus_{i=1}^{\ell} H^0(X, \Omega_X^{d_i})$$

$$T^*M \longrightarrow \bigoplus_{i=1} H^0(X, \Omega_X^{d_i}) \simeq B$$ Hitchin fibration

$$\Rightarrow \mathbb{C}[B] \hookrightarrow \mathbb{C}[T^*M]$$

$$\mathbb{C}[B] \cong \mathbb{C}[H_1, \dots, H_n]$$

<u>Thm (Hitchin, Faltings)</u>:

$n = \dim(B) = \dim(M)$, $\{H_i, H_j\} = 0$

$\rightsquigarrow$ complete integrable system on T^*M

$\bigoplus_{i=1} H(X, \Omega_X^{d_i}) \simeq B$ Hitchin fibration

$\to \mathbb{C}[T^*M]$

alltings):

$\dim(M)$, $\{H_i, H_j\} = 0$

mplete integrable system on T^*M (Hitchin system)

The r-matrix structure of Hitchin systems

Hitchin systems

X = cpt. Riemann surface $\supseteq S = \{p_1, \dots, p_N\}$

G = ctd. semisimple alg group, $\mathfrak{g} = \mathrm{Lie}(G)$

$\hat{M}_S$ = moduli space of G-bundles on X (reg. with formal triv. at S)

$\ni P$

<u>Fact</u>: $T_P M = H^1(X, \mathrm{Ad}(P))$, $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$

P with $\underset{\text{conn}}{\text{triv.}}$ at S

<u>Fact:</u> $T_p M = H^1(X, \mathrm{Ad}(P))$ $= P \times_G \mathfrak{g}$

2. The r-matrix method

$(M, \{,\})$ cl mech system w H Hamiltonian

$$\dot{x} = \{H, x\} \iff \dot{L} = [P, L] \quad , \; P, L : M \longrightarrow \mathfrak{g} \text{ Lie alg}$$

$p \in \mathbb{C}[\mathfrak{g}]^G$

$$\implies I_p = p(L) \quad , \quad \{I_p, H\} = 0$$

$$\{I_p, I_q\} \iff \exists r : M \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \text{ r-matrix of } (M, \{,\}, H)$$

$$\{L \otimes L\} = [1 \otimes L, r] - [L \otimes 1, r^{21}]$$

$M \longrightarrow \bigoplus_{i=1} H^0(X, \Omega_X^{d_i}) \simeq B$ Hitchin fibration

s the r-matrix of Hitchin?

$\sum b_i^* \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}, \quad K(b_i^*, b_j) = \delta_{ij}, \quad \{p_1, \dots, p_N\} \subseteq \mathbb{C}$

$\sum_{j \neq i} \frac{\gamma^{ij}}{p_j - p_i} \in \mathbb{C}[\mathfrak{g}^{*N}] \cong \mathbb{C}[\mathfrak{g}^*]^N$

$= 0, \quad L = \sum_i r(z, p_i) \quad , \quad r = r(\lambda, \mu) = \frac{\gamma}{\lambda - \mu}$

e r-matrix of Hitchin?

$\otimes b_i \in \mathfrak{g}\otimes\mathfrak{g}$, $K(b_i^*, b_j)=\delta_{ij}$, $\{p_1,\dots,p_N\}\subseteq \mathbb{C}$

$\frac{r^{ij}}{p_j-p_i}+ \in \mathbb{C}[\mathfrak{g}^{*N}] \cong \mathbb{C}[\mathfrak{g}^*]^N$

$\mathfrak{g}\otimes\mathfrak{g}[\lambda,\mu]$

$L=\sum_i r(z,p_i)$, $r=r(\lambda,\mu)=\frac{\gamma}{\lambda-\mu}+\dots$

3. Loop group uniformization

$p \in X$
$\|$
(t)

X
$\cup\!\!\!/ \quad \cup$
$X^\circ \times G \quad X^\circ = X \setminus \{p\} \quad D = \mathrm{Spec}(\mathbb{C}[[t]]) \quad D \times G$
$\cup \quad \cup\!\!\!/$
$D^\circ = \mathrm{Spec}(\mathbb{C}((t)))$

$G(X^\circ) \circlearrowleft$
$\|$
$\{X^\circ \to G\}$

Gluing together via $g \in \{D^\circ \to G\} = G((t))$

$G((t)) \longrightarrow M$
$g \longmapsto P_g$

3. Loop group uniformization

$p \in X$

$\;\|$

(t)

$$X \supset X^\circ = X \setminus \{p\} \qquad X \supset D = \mathrm{Spec}(\mathbb{C}[[t]])$$

$$X^\circ \supset D^\circ = \mathrm{Spec}(\mathbb{C}((t))) \subset D$$

$X^\circ \times G$ $\qquad$ $D \times G$

$G(X^\circ) \circlearrowleft$ $\qquad$ $\circlearrowright \{D \to G\} = G[[t]]$

$\;\|$

$\{X^\circ \to G\}$

Gluing together via $g \in \{D^\circ \to G\} = G((t))$

stacks

$$G(X^\circ) \backslash G((t)) / G[[t]] \xrightarrow{\cong} \mathrm{Bun}_G$$

$$g \longmapsto P_g$$

3. Loop group uniformization

$p \in X$
$\quad \|$
$\ (t)$

stacks

$$G(X^\circ)\backslash G((t))/G[[t]] \xrightarrow{\ \simeq\ } \mathrm{Bun}_G$$
$$g \longmapsto P_g$$

$$\begin{array}{ccc} & X & \\ \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} \\ X^\circ = X\setminus\{p\} & & D = \mathrm{Spec}(\mathbb{C}[[t]]) \\ \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} \\ & D^\circ = \mathrm{Spec}(\mathbb{C}((t))) & \end{array}$$

$X^\circ \times G$

$G(X^\circ) \circlearrowleft$
$\quad \|$
$\{X^\circ \to G\}$

$D \times G$

$\circlearrowleft \{D \to G\} = G[[t]]$

Gluing together via $g \in \{D^\circ \to G\} = G((t))$

system on T^*M (Hitchin sy

4. r-matrix of Hitchin systems

• Fact: $r(t_1,t_2) = \dfrac{\gamma}{t_1 - t_2} + \underset{\substack{\Downarrow \\ (\mathfrak{g}\otimes\mathfrak{g})[[t_1,t_2]]}}{s(t_1,t_2)} \longleftrightarrow \mathfrak{g}((t)) = \mathfrak{g}[[t]] \oplus$

$, I_q\} \Longleftrightarrow \exists\, r : M \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ r-matrix of $(M, \{,\}, H)$

$$\{L \otimes L\} = [1 \otimes L, r] - [L \otimes 1, r^{21}]$$

$\mathrm{Aut}(P) = \mathbb{C}(G)$

$S = \{p_1, \dots, p_n\} = \{p\}$

Bun_G reg. stable $\Longrightarrow H^0(X, \mathrm{Ad}(P)) = 0$

$\Downarrow$

formal triv. of at p $\Longrightarrow \Gamma(X^\circ, \mathrm{Ad}(P)) \cap \mathfrak{g}[[t]] = \{0\}$

$$\mathfrak{g}((t)) \big/ \mathfrak{g}[[t]] + \Gamma(X^\circ, \mathrm{Ad}(P)) = T_P \mathrm{Bun}_G \neq 0$$

$$\begin{tikzcd} \mathrm{Bun}_G \arrow[r, phantom, "\supseteq"] & \underset{\{\text{reg. stable bundles}\}}{\underline{\ddot{M}}} \arrow[r] & M \\ G((t)) \arrow[u] & & U \arrow[ll, "\gamma"'] \arrow[u, "p"', "\text{étale neighborhood}"] \end{tikzcd}$$

U choose coordinates $(\lambda_\alpha, \partial_\alpha)$

$$P(u) = P_{\gamma(u)}$$

$$\Longrightarrow T_u U = T_{P(u)} M$$

$$T_u U = \mathrm{Span}\{\gamma^{-1}(u) \cdot \partial_\alpha \gamma(u)\} \subset \mathfrak{g}((t))$$

$$\Gamma(X^\circ, \mathrm{Ad}(P))$$

M

$\uparrow$ étale neighbourhood

U

choose coordinates $(\lambda_\alpha, \partial_\alpha)$

$$P(u) = P_{\gamma(u)}$$

$$\Rightarrow T_u U = T_{P(u)} M$$

$$= \operatorname{Span}\{\gamma^{-1}(u) \cdot \partial_\alpha \gamma(u)\}$$

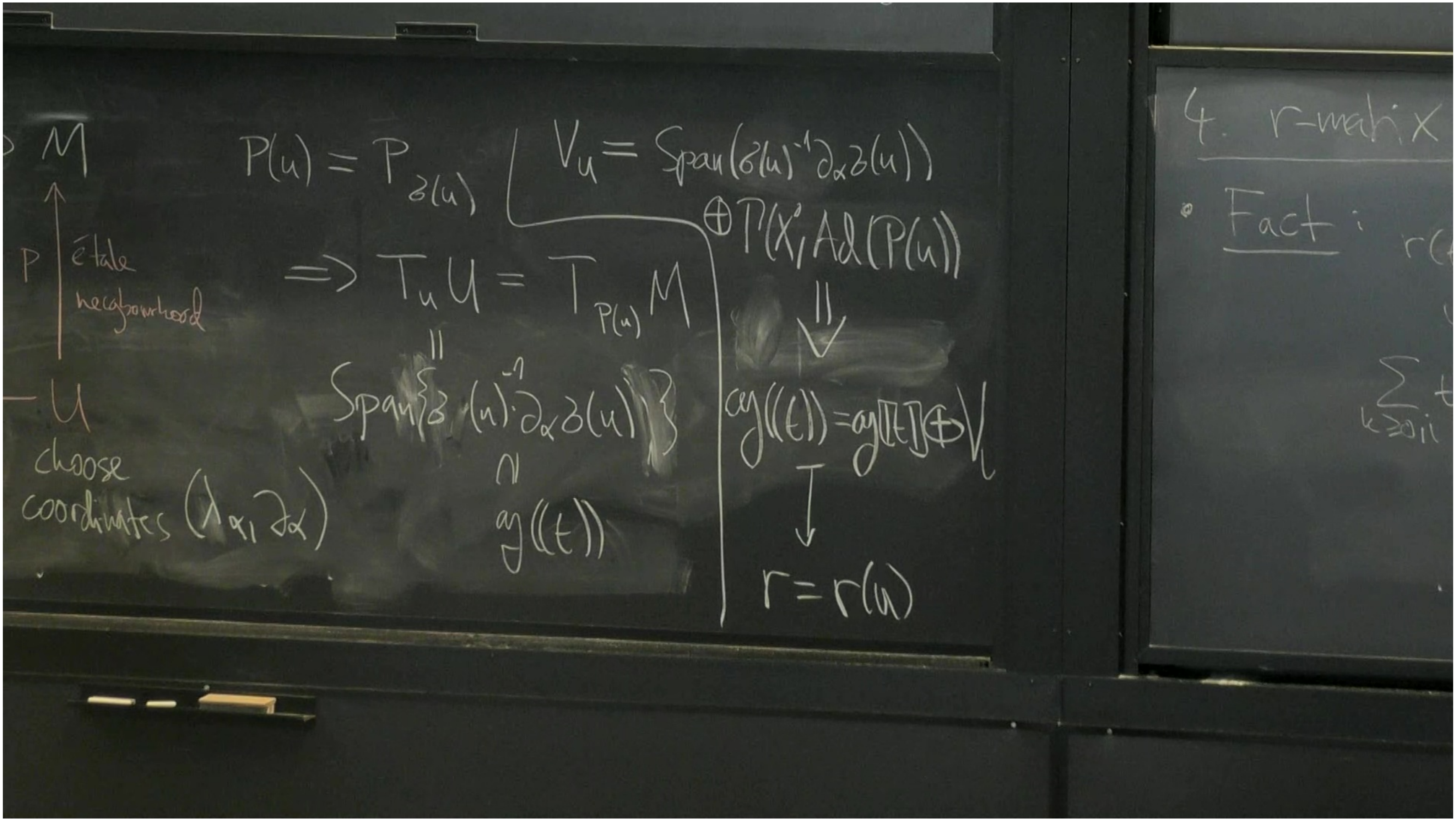
$$\subset \mathfrak{g}((t))$$

$$V_u = \operatorname{Span}(\gamma(u)^{-1} \partial_\alpha \gamma(u)) \oplus \Gamma(X, \operatorname{Ad}(P(u)))$$

4. r-matrix

- Fact: $r(t_1$

$$\sum_{k \geq 0, i} t_1^k$$


M
p
étale
neighbourhood
U
choose
coordinates $(\lambda_{\alpha}, \partial_\alpha)$
$P(u) = P_{\delta(u)}$
$\Rightarrow T_u U = T_{P(u)} M$
$\mathrm{Span}\{\delta^{-1}(u) \cdot \partial_\alpha \delta(u)\}$
$\cap$
$\mathfrak{g}((t))$
$V_u = \mathrm{Span}(\delta(u)^{-1} \partial_\alpha \delta(u))$
$\oplus P(X_i^{\circ}, \mathrm{Ad}(P(u)))$
$\mathfrak{g}((t)) = \mathfrak{g}[[t]] \oplus V_u$
$r = r(u)$
4. r-matrix
Fact:

systems

$(\mathfrak{g}\otimes\mathfrak{g})[[t_1,t_2]]$

ψ

$+\ S(t_1,t_2) \longleftrightarrow \mathfrak{g}((t)) = \mathfrak{g}[[t]] \oplus V_r$

subalgebra

ψ controls $[V_r, V_r] \notin V_r$

$\longrightarrow V_r = \mathrm{Span}_{\mathbb{C}}\{r_{k,i}\}$

$r_{k,i} = \frac{b_i}{t^{-k-1}} + s_{k,i}$

$r^{23}] + [r^{13}, r^{23}] = \psi$

$= 0$

$$n = \dim(B) = \dim(M), \quad \{H_i, H_j\} = 0$$

4. r-matrix of Hitchin systems

ψ controls $[V_r$

• Fact: $r(t_1,t_2) = \frac{\gamma}{t_1 - t_2} + s(t_1,t_2) \longleftrightarrow \mathfrak{g}((t)) = \mathfrak{g}[[t$

$(\mathfrak{g}\otimes\mathfrak{g})[[t_1,t_2]]$

$\sum_{k\geq 0, i} t_1^k b_i \otimes r_{ki} \longmapsto V_r = \mathrm{Span}_{\mathbb{C}}$

$$[r^{13}, r^{12}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = \psi$$

$$= 0$$

$\omega_\alpha = d\lambda_\alpha$

$$\sum_\alpha \Big(\underbrace{\omega^1_\alpha \partial_\alpha r^{23} - \omega^2_\alpha \partial_\alpha r^{13}}_{} \Big) = 0$$

|| in our case

<u>Thm</u>: $L \quad T^* \hat{M}_S \longrightarrow \mathfrak{g}((t))dt$ Lax matrix for Hitchin

satisfying $\{L \otimes L\} = [1 \otimes L, r] - [L \otimes 1, r^{21}]$