Abstract: A central goal of quantum information theory is to determine the capacities of a quantum channel for sending different sorts of information. I'll highlight the new and fundamentally quantum aspects that arise in quantum information theory compared to the classical theory. These include the central role of entanglement, nonadditivity, and synergies between resources. I will also discuss some challenging open questions that we will have to solve to push the theory forward.
Mathematical Challenges in QIT

\[
\begin{align*}
\mathcal{C}(N) &= \max_{m' \leq m} \mathcal{I}(X;Y) \\
\mathcal{I}(X;Y) &= H(X) + H(Y) - H(X|Y) \\
\text{recipe: choose randomly } n \times\frac{m}{5}
\end{align*}
\]
CPTP.

\[ N(\rho) = T_{\text{E}} U \rho U^+ \]

\[ Q(N) = \frac{\# \text{gabits}}{\# \text{channel}} \]

\[ C(N) = \frac{\# \text{bits}}{\# \text{channel}} \]

\[ P(N) = \frac{\# \text{private bits}}{\# \text{channel}} \]

\[ Q \leq P \leq C \]
Coding Theorems.

\[ Q^{(n)}(N) = \max \left( S(B) - S(E) \right) \]
\[ = \frac{1}{N} \max \left[ I(R;B) - I(R;E) \right] \]

\[ Q(N) = Q^{(n)}(N) \quad \text{"achievable rate"} \]
\[ Q(N) = \lim_{n \to \infty} \frac{1}{n} Q^{(n)}(N^n) \quad \text{"regularized"} \]

Similar stories for C&P, \( E \in \mathbb{N} \) where \( Q^{(n)}(N^n) > kQ^{(n)}(N) \)
Central Challenge of quantum info theory:

Additivity & non-additivity of entropy formulas

When is $Q^c(N^n) = n Q^c(N)$?

- systematic understandings

Theme:

Difficult question: given $N$, $Q(N) > 0$, or 0? $P(N) > 0$, or 0?
Additivity:

\[ N(p) = \text{Tr}_E U p U^+ \]
\[ N^c(p) = \text{Tr}_B U p U^+ \]

"degreatable"

3D, \( D_o N = N^c \) can simulate \( E \)

\[ Q(N) = Q^n(N) \quad Q^n(N \otimes \mathbb{1}) = \mathbb{n} Q^n(N) \]
Additivity,
\[ N(\rho) = \text{Tr}_E U \rho U^+ \]
\[ N^c(\rho) = \text{Tr}_B U \rho U^+ \]
"degenerate"

3D, \( D\omega N = N \), \( B \) can simulate \( E \)
\[ Q(N) = Q^{(1)}(N) \]
\[ Q^{(n)}(N \otimes n) = nQ^{(1)}(N) \]
Informally Degradable channels

$\forall \phi_{VA} \rightarrow \rho_{VBE} \quad N \text{ is ID if } \inf_{\phi_{VA}} \left[ I(V;B) - I(V;E) \right] \geq 0$

$I(V;B) \geq I(V;E)$

$Q(N) = Q''(N)$

$Q(N \otimes M) = Q(N) + Q(M)$ if all ID.
Informally Degradable Channels

$\forall \varphi_{VA} \rightarrow \rho_{VBE}$, $N$ is ID if

$$\inf_{\varphi_{VA}} \left[ I(V;B) - I(V;E) \right] \geq 0$$

$I(V;B) \geq I(V;E)$

$Q(N) = Q^{(d)}(N)$

$Q(N \times M) = Q(N) + Q(M)$ if all ID

1. $|W|$ could be infinite.
2. $N$ constrains $N^c$

Q: Are there ID channels that aren't Deg?

Classically? Yes.

Quantum case? Challenging.
Additivity.

\[ N(p) = \text{Tr}_E U p U^+ \]
\[ N^c(p) = \text{Tr}_B U p U^+ \]

"degenerate"

3D, \( D_o N = N^c \), \( B \) can simulate \( E \)

\[ Q(N) = Q^{(1)}(N) \]
\[ Q^{(1)}(N \otimes n) = n Q^{(0)}(N) \]
Inf. Deg. for flagged mixtures

\[ A(V; B) = I(V; B) - I(V; E). \]

\[ N(\rho) = (1-p)N_1(\rho) \otimes 1 \times 1 + pN_2(\rho) \otimes 1 \times 2 \]

\[ A(V; B) = (1-p)(I(V; B) - I(V; E)) + p(I(V; B_2) - I(V; E_2)) \]

\[ N_1(\rho) = A_{\chi_1}(\rho) = A_0 \rho A_0^+ + A_1 \rho A_1^+ \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]
\[ A_{\gamma_1} = A_{1-\gamma_1} \quad N_1 = A_{\delta+\epsilon} \]

\[ N_2 = A_{1-\gamma} \leq \gamma, \epsilon \leq 1 \]

\[ P \leq \frac{1}{2} \]

\[ N(p) = (1-p)A_{\gamma+\epsilon}(p) \otimes 1 \times 1 \]

\[ pA_{1-\gamma}(p) \otimes 1 \times 2 ] \]

\[ N^{c}(p) = (1-p)A_{1-(\gamma+\epsilon)}(p) \otimes 1 \times 1 \]

\[ pA_{\delta}(p) \otimes 1 \times 2 \]
\[ N_1 = A_{1-\gamma} \quad N_2 = A_{1-\gamma} \]

\[ N(p) = (1-p)A_{\text{me}}(p)\otimes |1x1| + pA_{1-	ext{me}}(p)\otimes |1x1| \]

\[ N^c(p) = (1-p)A_{1-	ext{me}}(p)\otimes |1x1| + pA_{\text{me}}(p)\otimes |1x1| \]

\[ A(V;B) = (1-p)A(V;B_{\text{me}}) + pA(V;B_{\text{me}}) \]

\[ = (1-p)A(V;B_{\text{me}}) - pA(V;B_{\text{me}}) \geq 0 \quad \text{cng} \]

\[ A(V;B_{\text{me}}) \geq (1-p)A(V;B_{\text{me}}) \quad \text{cng} \]

Classically: bound on \( IV \).
\[ Q_{ss}(N) = \frac{1}{2} \sup_{P_{RAV}} \left[ I(R;B|V) - I(R;E|V) \right] f(1A,1B,1E) \]

1) \( Q_{ss}(N) = Q''(N) \) for degradable channels.
2) \( Q_{ss}(N) = Q'''(N) \) for ID channels.
3) \( Q_{ss}(N) = 0 \) if example \( N \) is degradable.
4) \( I(R;E|V) \geq I(R;B|V) \) for i.i.d. channels.
Symmetric Assistance

\[
Q(A) = 0 
\]

\[
Q_{ss}^{(N)}(N) = Q^{(\ast)}(N \times A)
\]

\[
Q^{(\ast)}(N \times M) = Q_{ss}^{(N)}(M) + Q_{ss}^{(M)}(M)
\]

\[
Q(N) \leq Q_{ss}(N)
\]