Title: Mathematical Physics Core Lecture

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Associated vector bundle

$\pi : P \to M$, V se

$$E = P \times_\rho V = (P \times V)/G = \{[p, v] \mid p \in P, v \in V\}$$

$$\pi_E : [p, v] \in E \mapsto \pi(p) \in M \qquad \pi(pg) = \pi(p)$$

$P \to M$, V vector space $\rho : G \to GL(V)$ representation

$\} \mid p \in P, v \in V \}$ $[p, v] = \{ (p \triangleleft g, \rho(g^{-1}) v) \mid g \in G \}$

$g) = \pi(p)$

Vect. space structure on fibres

fibres: $u \in M$, $E_u = \pi_P^{-1}(\{u\}) = \left\{ [P, v] \,\middle|\, P \in \pi^{-1}\{u\}, v \in V \right\}$

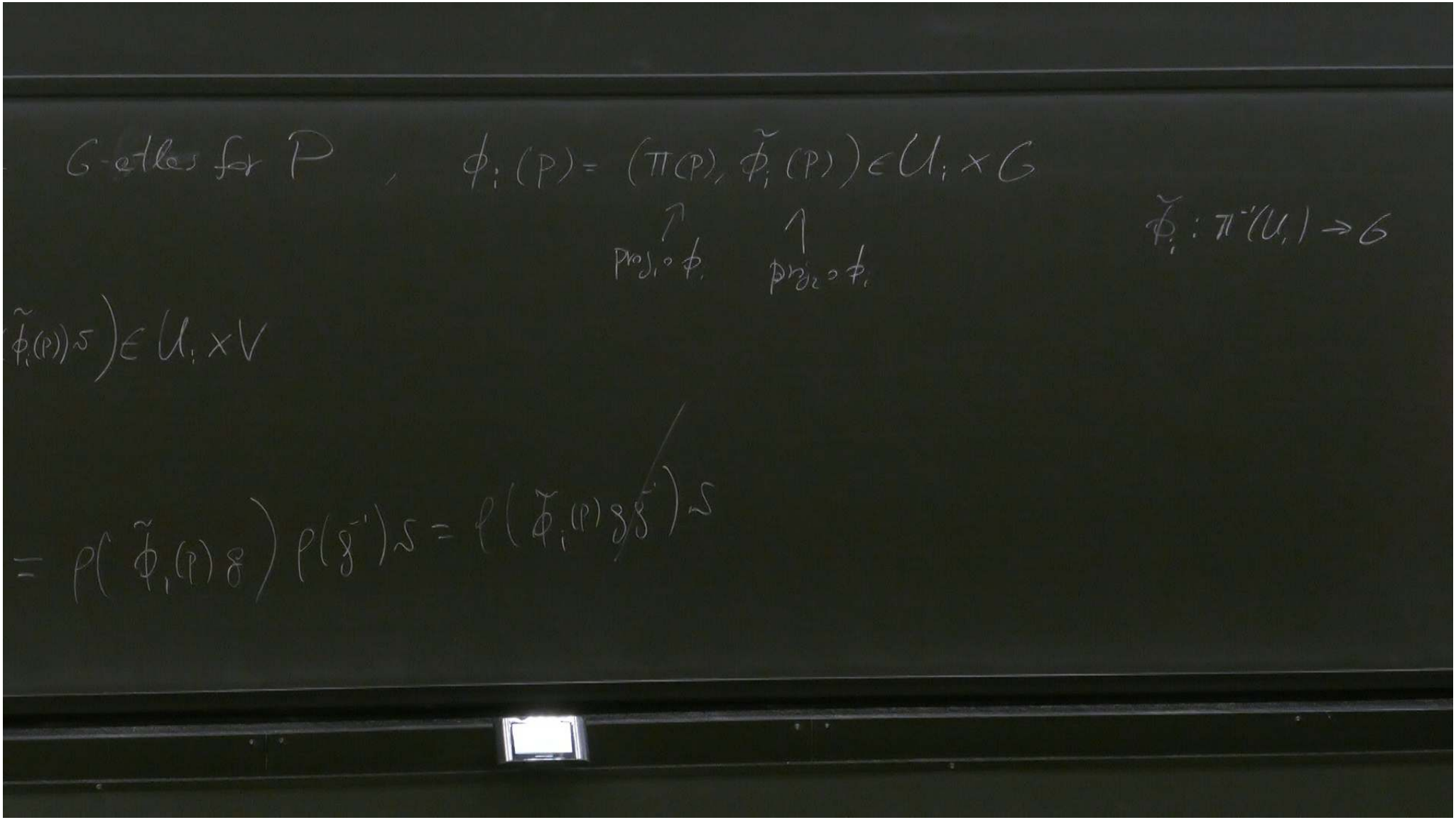
$\alpha [P, v_1] + \beta [P, v_2] = [P, \alpha v_1 + \beta v_2] = [P \triangleleft g, \underbrace{\rho(g^{-1})}_{\uparrow \text{linear}} (\alpha v_1 + \beta v_2)]$

trivialisations if $\{(U_i, \phi_i)\}_{i \in I}$ G-atlas for P, $\phi_i(p) =$

define $\Psi_i : [p, s] \in \pi_P^{-1}(U_i) \mapsto \left(\pi(p), \rho(\tilde{\phi}_i(p))s\right) \in U_i \times V$

well-defined $\rightarrow$ $\pi(pg) = \pi(p)$

$$\rho(\tilde{\phi}_i(pg))\rho(g^{-1})s = \rho(\tilde{\phi}_i(p)g)\rho(g^{-1})s$$

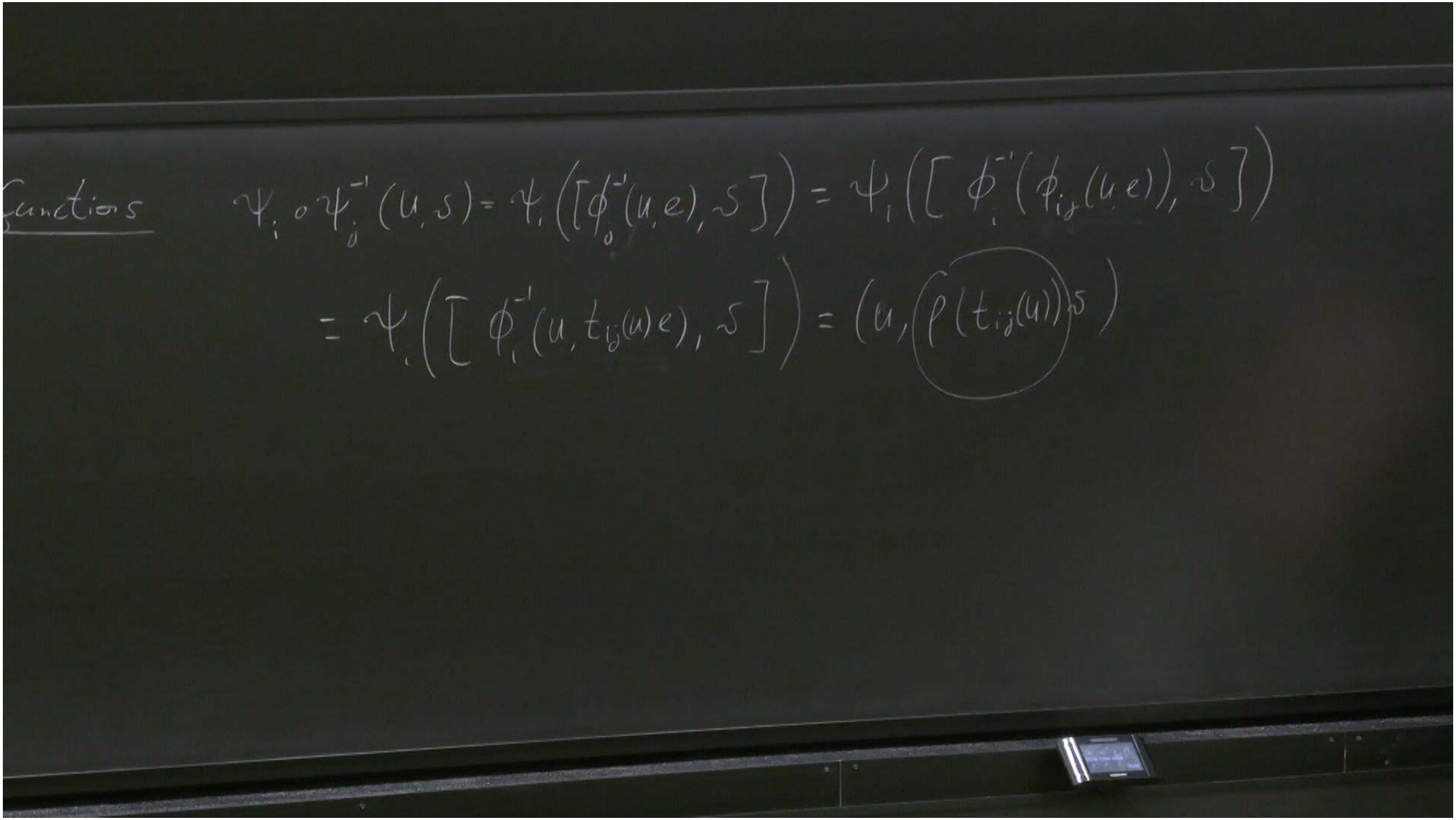

G-etles for P
$\phi_i(p) = (\pi(p), \tilde{\phi}_i(p)) \in U_i \times G$
$\text{proj}_1 \circ \phi_i$
$\text{proj}_2 \circ \phi_i$
$\tilde{\phi}_i : \pi^{-1}(U_i) \to G$
$\tilde{\phi}_i(p))s) \in U_i \times V$
$= \rho(\tilde{\phi}_i(p)g)\rho(g^{-1})s = \rho(\tilde{\phi}_i(p)gg^{-1})s$

$\phi_i(p) = (\pi(p), \tilde{\phi}_i(p)) \in U_i \times G$

$\uparrow \text{proj}_1 \circ \phi_i$ $\quad \uparrow \text{proj}_2 \circ \phi_i$ $\quad \tilde{\phi}_i : \pi^{-1}(U_i) \to G$

$\psi_i^{-1} : (u, s) \in U_i \times V \mapsto [\tilde{\phi}_i^{-1}(u, e), s] \in \pi_E^{-1}(U_i)$

$\rho(\tilde{\phi}_i(p) g) s$


Functions
$\psi_i \circ \psi_j^{-1}(u,s) = \psi_i([\phi_0^{-1}(u,e), s]) = \psi_i([\phi_i^{-1}(\phi_{ij}(u,e)), s])$
$= \psi_i([\phi_i^{-1}(u, t_{ij}(u)e), s]) = (u, \rho(t_{ij}(u))s)$

Transition functions $\psi_i \circ \psi_j^{-1}(u, s) = \psi_i\left([\phi_0^{-1}(u, e), s]\right) =$

$= \psi_i\left([\phi_i^{-1}(u, t_{ij}(u)e), s]\right) =$

Sections of E

suppose $s_i : U_i \to G$ local gauges ($s_i(u) =$

sections $U_i \to E$ are in 1–1 correspondence with smooth fcns $U_i \to V$

- if $f : U_i \to V$ smooth, $u \in U_i \mapsto [s_i(u), f(u)] \in E$ section
- if $\tau : U_i \to E$ section, $\tau(u) = [A(u), B(u)]$ $A(u) = s_i(u) \triangleleft g(u)$

$(s_i(u) = \phi_i^{-1}(u, e))$

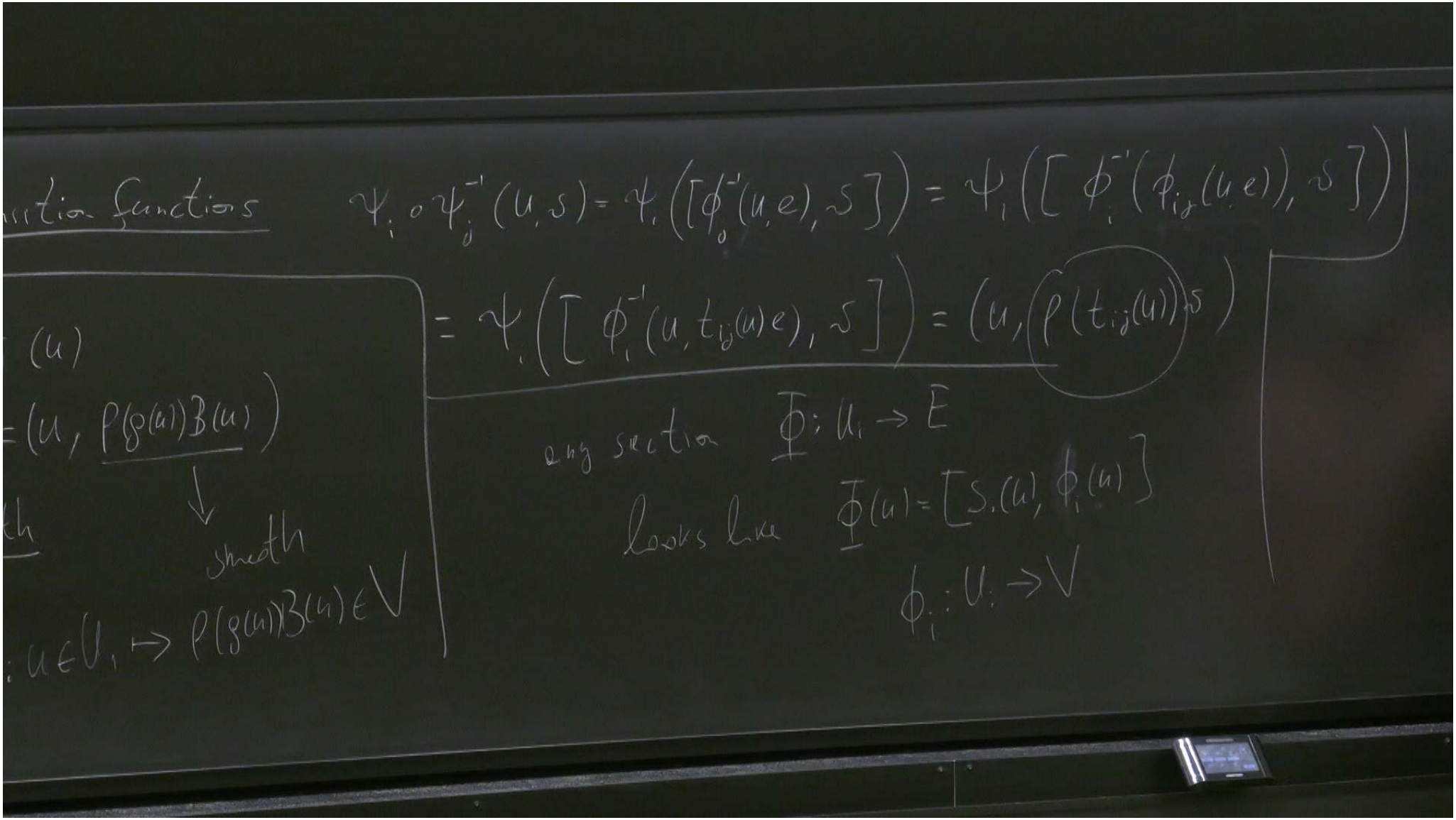
$U_i \to V$

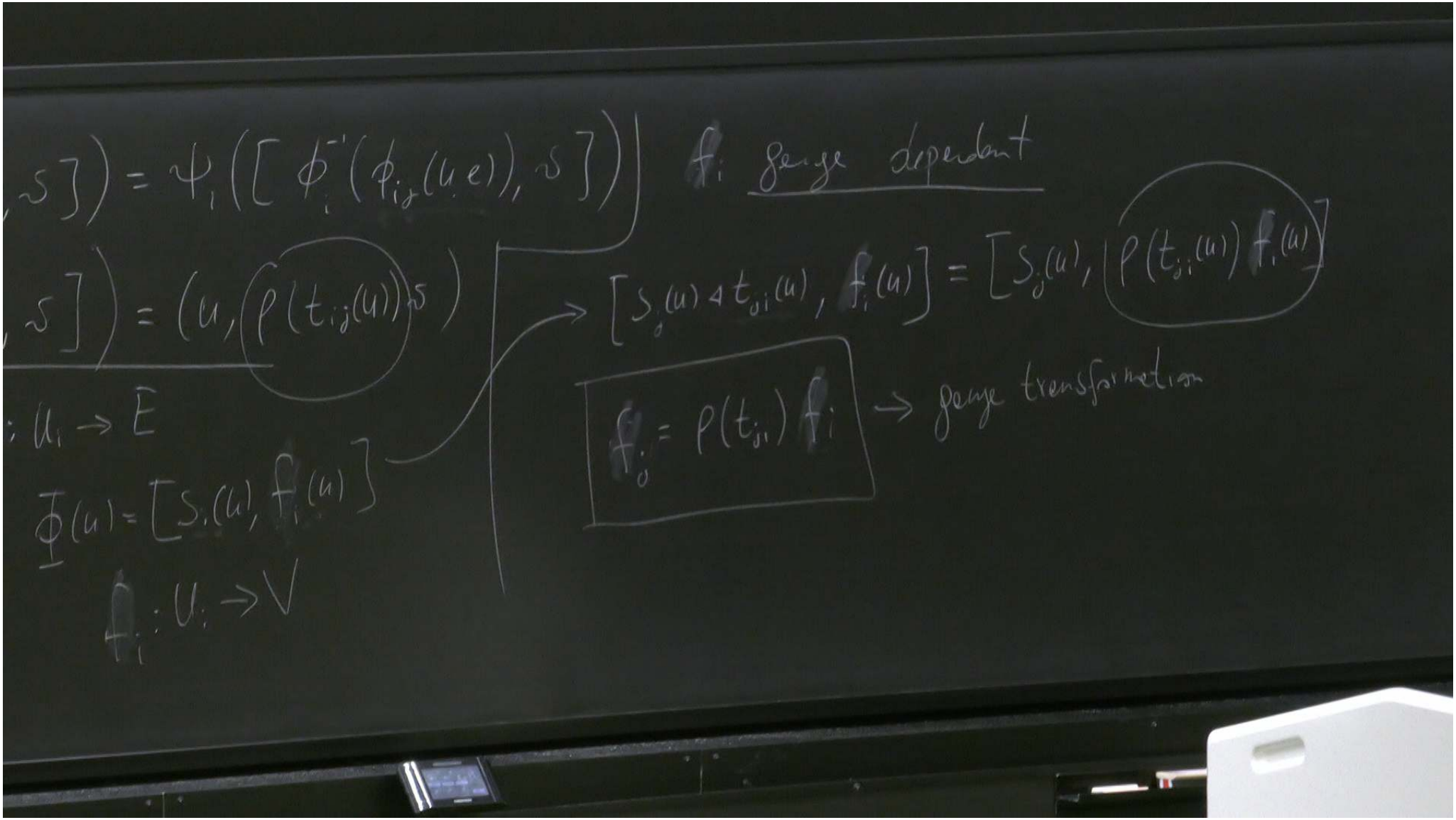
A, B, g don't have to be smooth

$\tau(u) = [s_i(u) \triangleleft g, B(u)] = [s_i(u), \rho(g(u)) \cdot B(u)]$

$\vdots$

$(u) \triangleleft g(u)$ for some $g: U_i \to G$


ition functions
$\psi_i \circ \psi_j^{-1}(u,s) = \psi_i([\phi_0^{-1}(u,e), s]) = \psi_i([\phi_i^{-1}(\phi_{ij}(u,e)), s])$
$= \psi_i([\phi_i^{-1}(u, t_{ij}(u)e), s]) = (u, \rho(t_{ij}(u))s)$
(u)
$= (u, \rho(g(u))\beta(u))$
smooth
$u \in U_i \mapsto \rho(g(u))\beta(u) \in V$
any section $\Phi: U_i \to E$
looks like $\Phi(u) = [s_i(u), \phi_i(u)]$
$\phi_i: U_i \to V$


f_i gauge dependant
$[s_j(u) \cdot t_{ji}(u), f_i(u)] = [s_j(u), \rho(t_{ji}(u)) f_i(u)]$
$f'_i = \rho(t_{ji}) f_i$ → gauge transformation
$\Phi(u) = [s_i(u), f_i(u)]$
$f_i : U_i \to V$

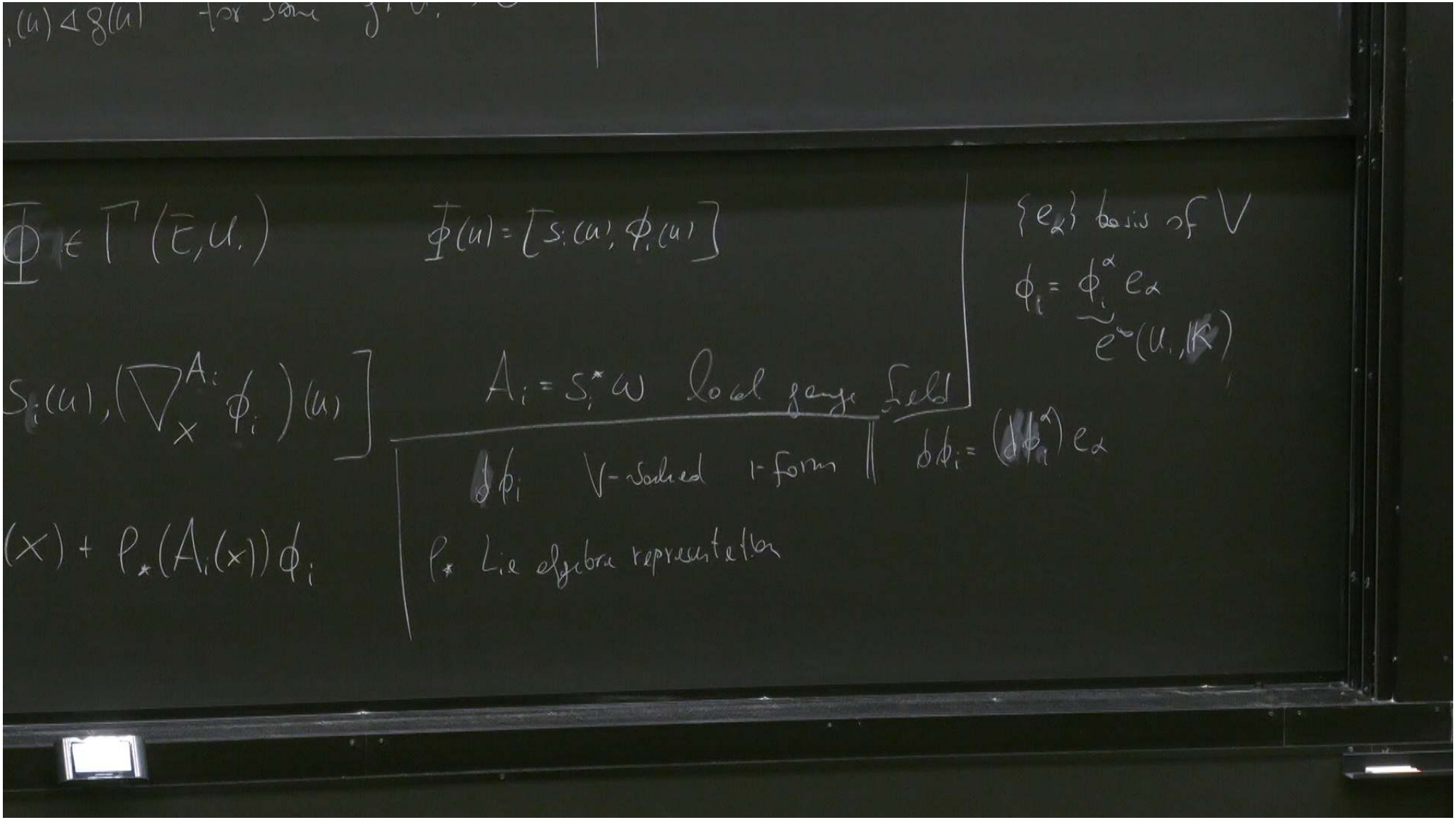
Covariant derivatives

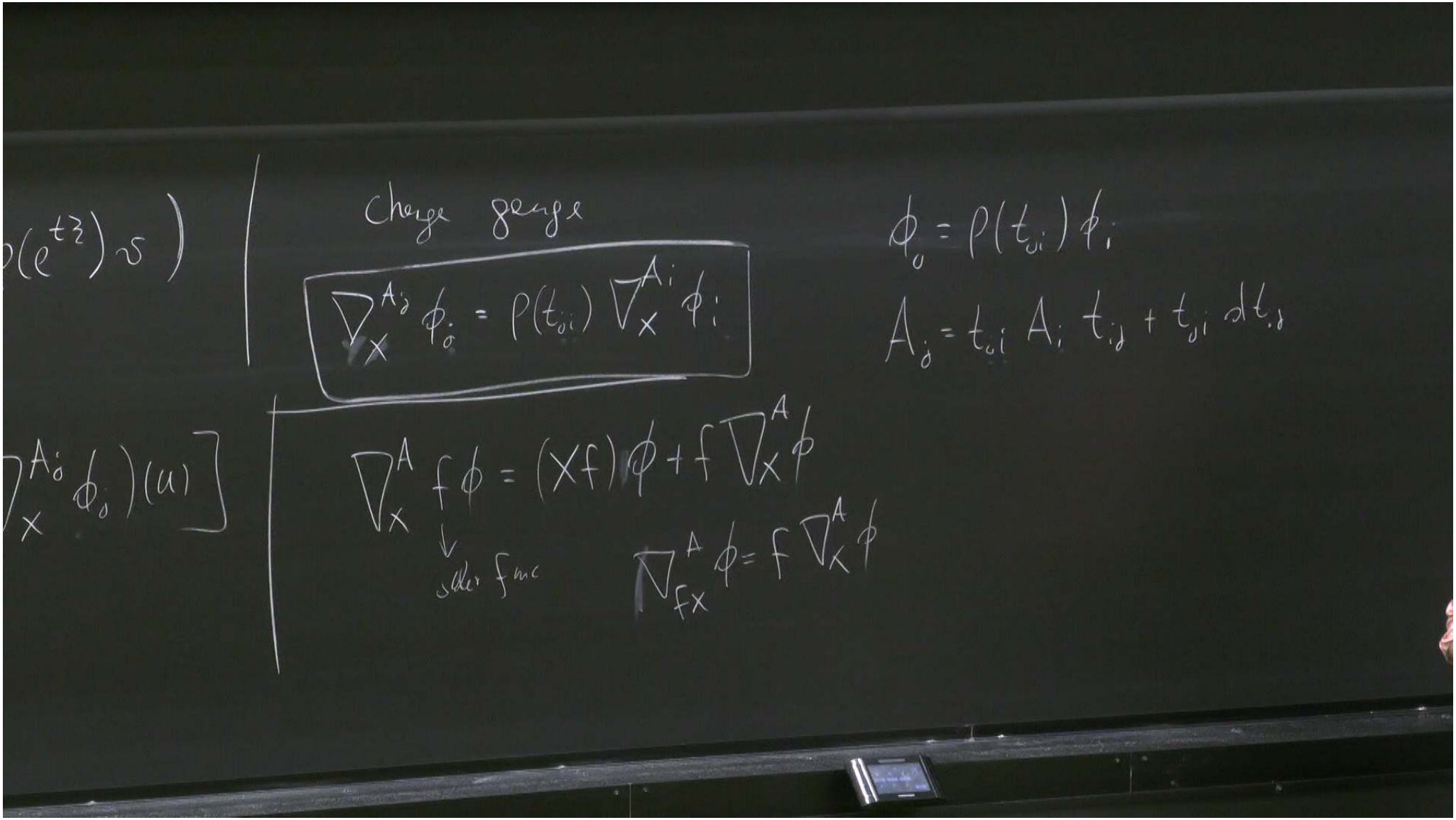
$\partial_\mu \phi \longrightarrow \partial_\mu \phi + A_\mu \phi$

given $X : M \to TM$

$\nabla_X^W : \Gamma(E, U_i) \to \Gamma(E, U_i)$

$\{\Phi : U \to E \text{ section}\}$


$\Phi \in \Gamma(E, U_i)$
$\Phi(u) = [s_i(u), \phi_i(u)]$
$\{e_\alpha\}$ basis of V
$\phi_i = \phi_i^\alpha e_\alpha$
$\phi_i^\alpha \in C^\infty(U_i, \mathbb{K})$
$s_i(u), (\nabla_X^{A_i} \phi_i)(u)]$
$A_i = s_i^* \omega$ local gauge field
$\delta\phi_i$ V-valued 1-form
$\delta\phi_i = (\delta\phi_i^\alpha) e_\alpha$
$(x) + \rho_*(A_i(x))\phi_i$
ρ_* Lie algebra representation


Charge gauge
$\nabla_x^{A_0} \phi_0 = \rho(t_{0i}) \nabla_x^{A_i} \phi_i$
$\phi_0 = \rho(t_{0i}) \phi_i$
$A_j = t_{0i} A_i t_{ij} + t_{0i} \partial t_{ij}$
$\nabla_x^A f\phi = (Xf)\phi + f \nabla_x^A \phi$
scalar func
$\nabla_{fx}^A \phi = f \nabla_x^A \phi$

$[A(u), B(u)]$ $A(u) = s_i(u) \triangleleft g(u)$ for some g

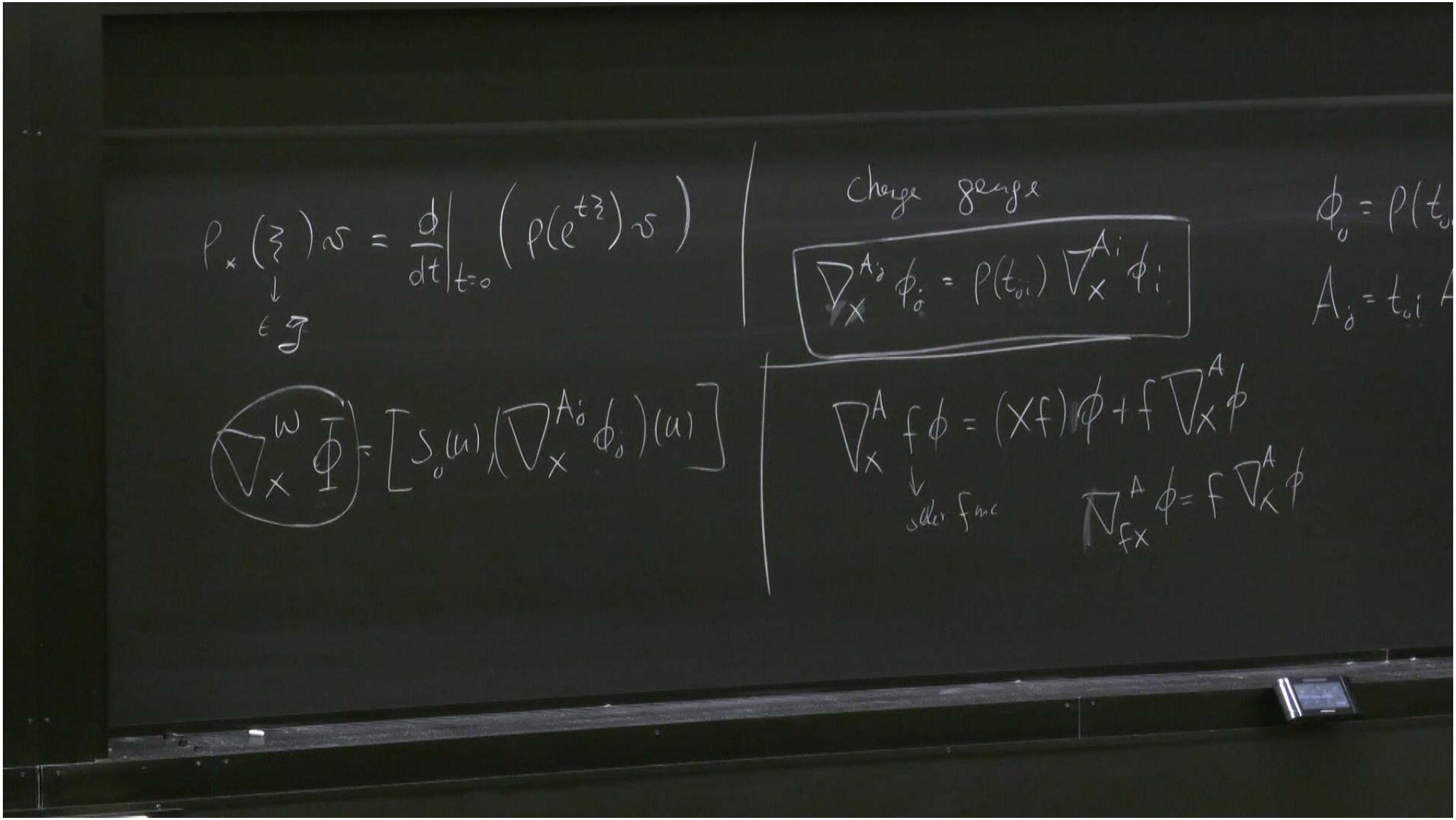
$\to \partial_m \phi + A_m \phi$ $\Phi \in \Gamma(E, U_i)$ $\Phi(u) = [s_i(u), \phi_i(u)]$

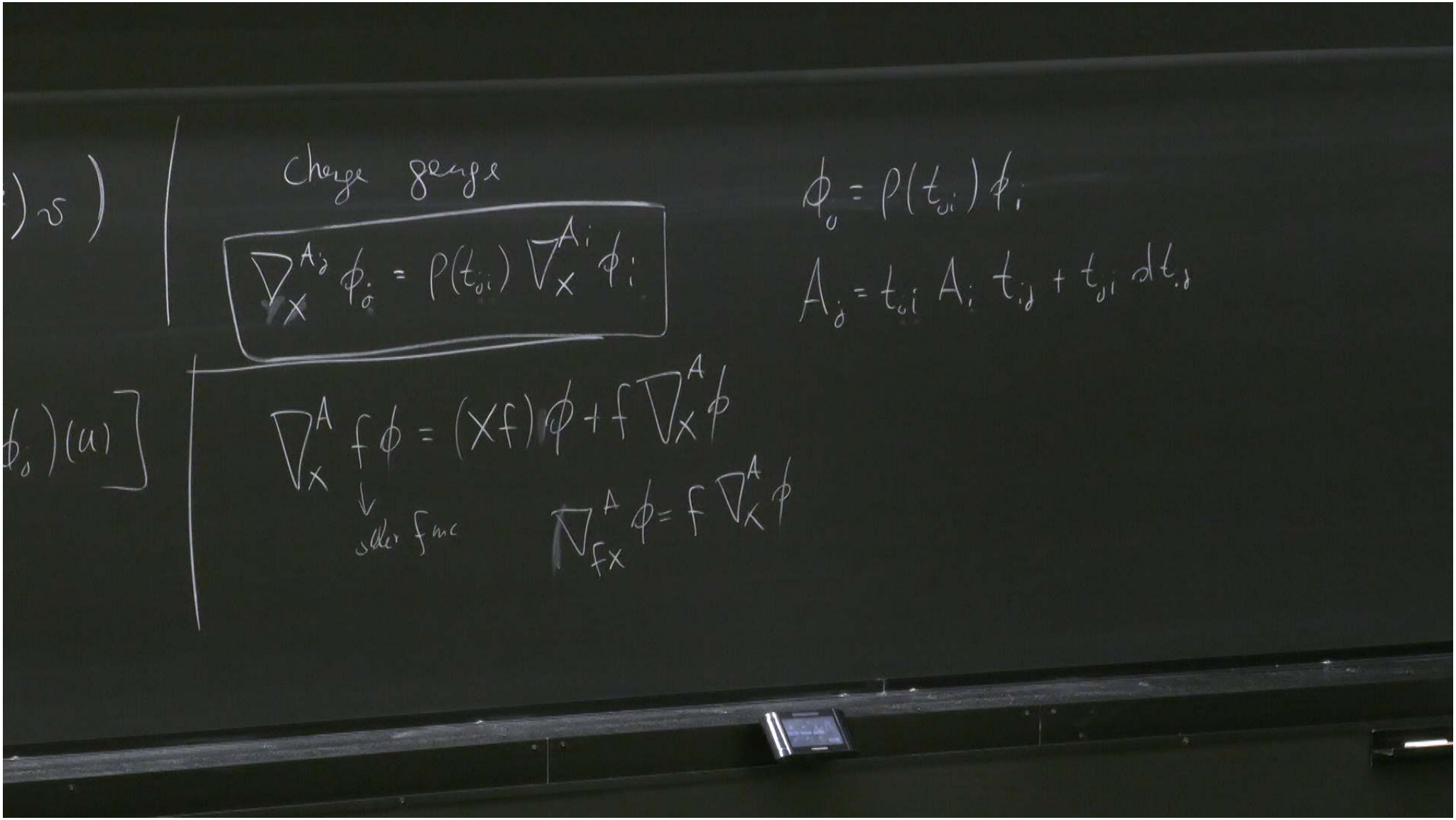
$(\nabla_x^{\omega} \Phi)(u) = [s_i(u), (\nabla_x^{A_i} \phi_i)(u)]$ $A_i = s_i^* \omega$ local gauge fie

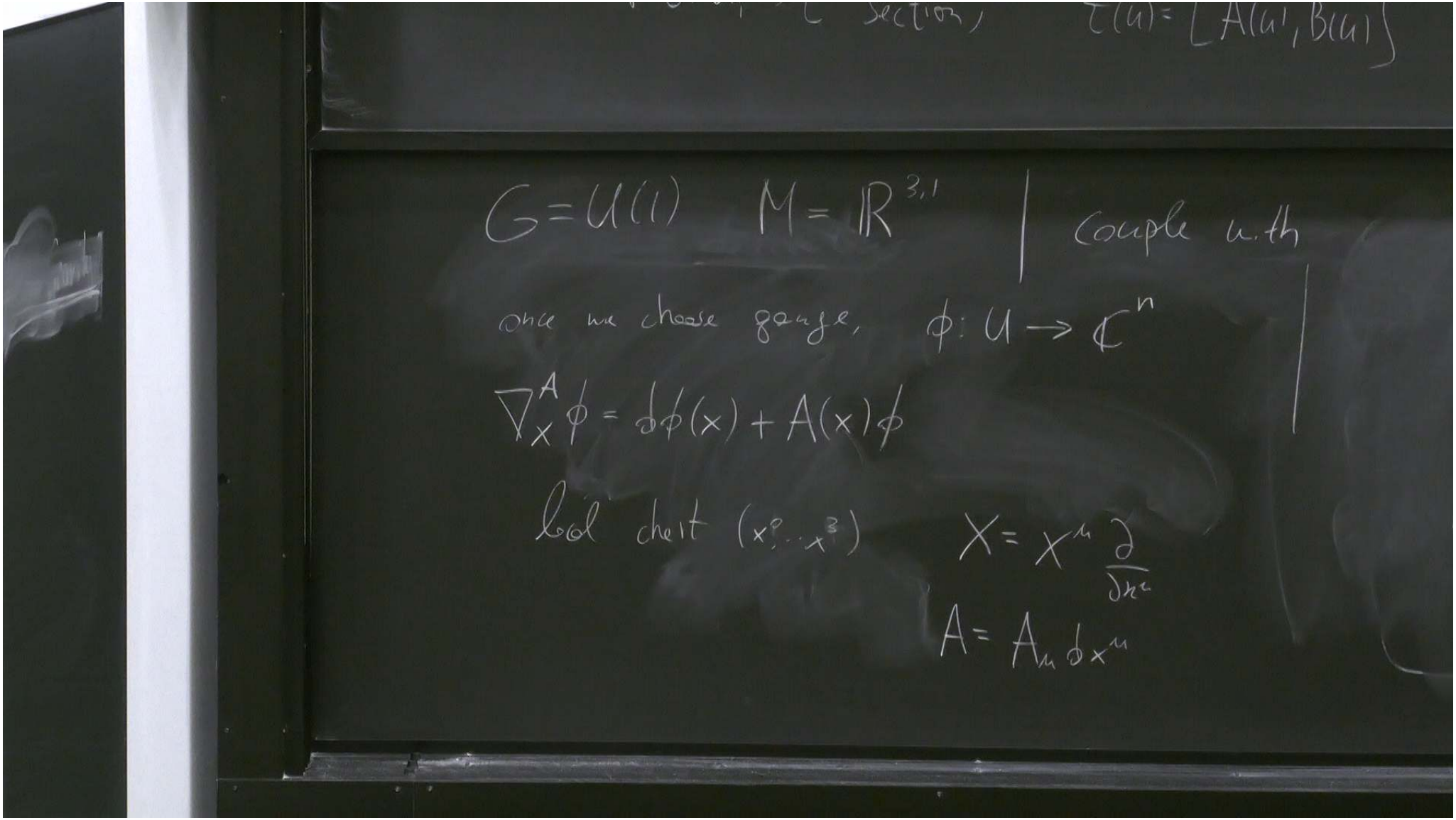
$\nabla_x^{A_i} \phi_i = d\phi_i(x) + \rho_*(A_i(x))\phi_i$

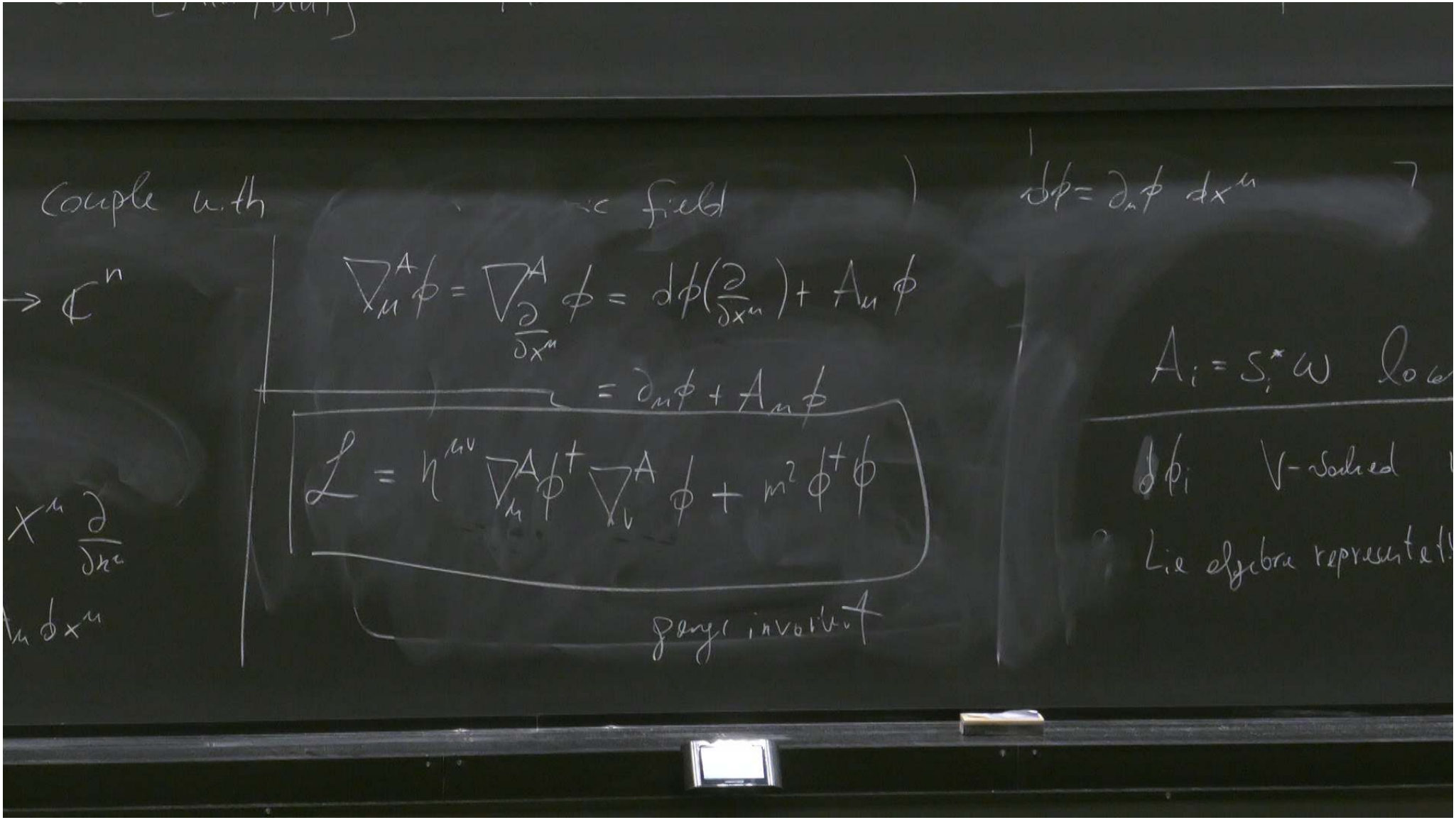
ϕ_i V-valued 1-form

ρ_* Lie algebra representation


Charge gauge
$\nabla_x^{A_j}\phi_0 = \rho(t_{0i})\,\nabla_x^{A_i}\phi_i$
$\phi_0 = \rho(t_{0}$
$A_j = t_{0i}$ A
$\rho_*(\xi)\omega = \frac{d}{dt}\Big|_{t=0}\left(\rho(e^{t\xi})\omega\right)$
$\epsilon \mathfrak{g}$
$\nabla_x^{\omega}\bar{\phi} = [S_0(u), (\nabla_x^{A_0}\phi_0)(u)]$
$\nabla_x^A f\phi = (Xf)\phi + f\nabla_x^A\phi$
scalar fnc
$\nabla_{fx}^A\phi = f\nabla_x^A\phi$


Charge gauge
$\nabla_X^{A_j} \phi_0 = \rho(t_{0i}) \nabla_X^{A_i} \phi_i$
$\phi_0 = \rho(t_{0i}) \phi_i$
$A_j = t_{0i} A_i t_{ij} + t_{0i} \partial t_{ij}$
$\nabla_X^A f\phi = (Xf)\phi + f \nabla_X^A \phi$
$\nabla_{fX}^A \phi = f \nabla_X^A \phi$


section
$\xi(u) = [A(u), B(u)]$
$G = U(1)$ $M = \mathbb{R}^{3,1}$
couple w. th
once we choose gauge, $\phi: U \to \mathbb{C}^n$
$\nabla_x^A \phi = \partial\phi(x) + A(x)\phi$
lab chart $(x^0 \ldots x^3)$
$X = X^\mu \frac{\partial}{\partial x^\mu}$
$A = A_\mu dx^\mu$


Couple w. th
ic field
$\mathbb{C}^n$
$\nabla_\mu^A \phi = \nabla_{\frac{\partial}{\partial x^\mu}}^A \phi = d\phi\left(\frac{\partial}{\partial x^\mu}\right) + A_\mu \phi$
$= \partial_\mu \phi + A_\mu \phi$
$\mathcal{L} = \eta^{\mu\nu} \nabla_\mu^A \phi^\dagger \nabla_\nu^A \phi + m^2 \phi^\dagger \phi$
gauge invariant
$d\phi = \partial_\mu \phi \, dx^\mu$
$A_i = S_i^* \omega$
ϕ_i V-valued
Lie algebra represent
$x^\mu \frac{\partial}{\partial x^\mu}$
$A_\mu dx^\mu$