Title: Mathematical Physics Core Lecture

Speakers: Giuseppe Sellaroli

Collection: Mathematical Physics - Core 2023/24

Date: February 05, 2024 - 11:30 AM

URL: https://pirsa.org/24020020

Associated vector bundle

$\pi : P \to M$, V secto

$$E = P \times_\rho V = (P \times V)/G = \{ [p, v] \mid p \in P, v \in V \}$$

$\pi_\rho : [p, v] \in E \mapsto \pi(p) \in M$

$\pi(p \triangleleft g) = \pi(p)$

$P \to M$, V vector space $\qquad \rho : G \to GL(V)$ representation

$\ldots] \mid p \in P, v \in V \} \qquad [p, v] = \{ (p \triangleleft g, \rho(g^{-1}) v) \mid g \in G \}$

$\ldots g) = \pi(p)$

Vect. space structure on fibres

fibres: $u \in M$, $E_u = \pi_\rho^{-1}(\{u\}) = \left\{ [p,v] \;\middle|\; p \in \pi^{-1}\{u\},\ v \in V \right\}$

$$\alpha[p,v_1] + \beta[p,v_2] = [p, \alpha v_1 + \beta v_2] = [p \triangleleft g, \underbrace{\rho(g^{-1})}_{\uparrow\ \text{linear}} (\alpha v_1 + \beta v_2)]$$

trivialisations if $\{(U_i, \phi_i)\}_{i\in I}$ G-atlas for P , $\phi_i(p) = ($

pr

defn $\psi_i : [p, s] \in \pi_\rho^{-1}(U_i) \mapsto \left(\pi(p), \rho(\tilde{\phi}_i(p)) s\right) \in U_i \times V$

well-defined $\rightarrow$ $\pi(p \triangleleft g) = \pi(p)$

$$\rho(\tilde{\phi}_i(p \triangleleft g))\, \rho(g^{-1}) s = \rho\left(\tilde{\phi}_i(p)\, g\right) \rho(g^{-1}) s$$

G-atlas for P , $\phi_i(p) = (\pi(p), \tilde{\phi}_i(p)) \in U_i \times G$

$\uparrow$ $\mathrm{proj}_1 \circ \phi_i$ $\uparrow$ $\mathrm{proj}_2 \circ \phi_i$

$\tilde{\phi}_i : \pi^{-1}(U_i) \to G$

$(\tilde{\phi}_i(p))s) \in U_i \times V$

$= \rho(\tilde{\phi}_i(p) g) \rho(g^{-1}) s = \rho(\tilde{\phi}_i(p) \cancel{g g^{-1}}) s$

$\phi_i(p) = (\pi(p), \tilde{\phi}_i(p)) \in U_i \times G$

$\uparrow$ $pr_1 \circ \phi_i$ $\quad$ $\uparrow$ $pr_2 \circ \phi_i$

$\tilde{\phi}_i : \pi^{-1}(U_i) \to G$

$\psi_i^{-1} : (u,s) \in U_i \times V \mapsto [\phi_i^{-1}(u,e), s] \in \pi_\rho^{-1}(U_i)$

$\rho(\tilde{\phi}_i(p)\, g \cancel{g^{-1}})\, s$

functions

$$\psi_i \circ \psi_j^{-1}(u,s) = \psi_i\left(\left[\phi_j^{-1}(u,e), s\right]\right) = \psi_i\left(\left[\phi_i^{-1}\left(\phi_{ij}(u,e)\right), s\right]\right)$$

$$= \psi_i\left(\left[\phi_i^{-1}(u, t_{ij}(u)e), s\right]\right) = \left(u, \rho(t_{ij}(u))s\right)$$

transition functions

$$\psi_i \circ \psi_j^{-1}(u, s) = \psi_i\left(\left[\phi_j^{-1}(u, e), s\right]\right) =$$

$$= \psi_i\left(\left[\phi_i^{-1}(u, t_{ij}(u)e), s\right]\right) =$$

sections of E

suppose $s_i : U_i \to G$ local gauges $\left(s_i(u) = \right.$

sections $U_i \to E$ are in 1-1 correspondence with smooth funcs $U_i \to V$

- if $f : U_i \to V$ smooth, $u \in U_i \mapsto [s_i(u), f(u)] \in E$ section
- if $\tau : U_i \to E$ section, $\tau(u) = [A(u), B(u)]$ $A(u) = s_i(u) \triangleleft g(u)$

$\left(s_i(u) = \phi_i^{-1}(u, e) \right)$

$U_i \to V$

A, B, g don't have to be smooth

$\tau(u) = [s_i(u) \triangleleft g, B(u)] = [s_i(u), \rho(g(u)) \cdot B(u)]$

ion

$(u) \triangleleft g(u)$ for some $g : U_i \to G$

...sition functions

$$\psi_i \circ \psi_j^{-1}(u,s) = \psi_i\left([\phi_j^{-1}(u,e), s]\right) = \psi_i\left([\phi_i^{-1}(\phi_{ij}(u,e)), s]\right)$$

$$= \psi_i\left([\phi_i^{-1}(u, t_{ij}(u)e), s]\right) = (u, \rho(t_{ij}(u))s)$$

(u)

$= (u, \underline{\rho(g(u))B(u)})$

$\downarrow$

smooth

...th

$: u \in U_i \mapsto \rho(g(u))B(u) \in V$

any section $\Phi : U_i \to E$

looks like $\Phi(u) = [s_i(u), \phi_i(u)]$

$\phi_i : U_i \to V$

$$\ldots, s]) = \psi_i([\phi_i^{-1}(\phi_{ij}(u,e)), s]))$$

f_i: gauge dependent

$$\ldots, s]) = (u, \rho(t_{ij}(u))\, s)$$

$: U_i \to E$

$$\Phi(u) = [s_i(u), f_i(u)]$$

$$f_i : U_i \to V$$

$$[s_j(u) \triangleleft t_{ji}(u), f_i(u)] = [s_j(u), \rho(t_{ji}(u))\, f_i(u)]$$

$$\boxed{f_j = \rho(t_{ji})\, f_i} \to \text{gauge transformation}$$

Covariant derivatives

$$\partial_\mu \phi \longrightarrow \partial_\mu \phi + A_\mu \phi$$

given $X : M \to TM$

$$\nabla^{\omega}_{X} : \underline{\Gamma(E, U_i)} \longrightarrow \Gamma(E, U_i)$$

$$= \{ \Phi : U_i \to E \text{ section} \}$$

$,(u) \Delta g(u)$ for some $g: U_i$

$\Phi \in \Gamma(E, U_i)$

$\Phi(u) = [s_i(u), \phi_i(u)]$

$\{e_\alpha\}$ basis of V

$\phi_i = \phi_i^\alpha e_\alpha$

$\phi_i^\alpha \in C^\infty(U_i, \mathbb{K})$

$s_i(u), (\nabla_X^{A_i} \phi_i)(u)]$

$A_i = s_i^* \omega$ local gauge field

$d\phi_i$ V-valued 1-form

$d\phi_i = (d\phi_i^\alpha) e_\alpha$

$(X) + \rho_*(A_i(x))\phi_i$

ρ_* Lie algebra representation

$\rho(e^{t\xi})v\,)$

$\nabla^{A_j}_X \phi_j)(u)\,]$

Change gauge

$$\boxed{\nabla^{A_j}_X \phi_j = \rho(t_{ji})\,\nabla^{A_i}_X \phi_i}$$

$$\phi_j = \rho(t_{ji})\,\phi_i$$

$$A_j = t_{ji}\,A_i\,t_{ij} + t_{ji}\,dt_{ij}$$

$$\nabla^A_X f\phi = (Xf)\phi + f\,\nabla^A_X\phi$$

$f \downarrow$ smooth func

$$\nabla^A_{fX}\phi = f\,\nabla^A_X\phi$$

$[A(u), B(u)]$ $\quad A(u) = s_i(u) \triangleleft g(u)$ for some g ...

$\to \partial_\mu \phi + A_\mu \phi$

$\Phi \in \Gamma(E, U_i)$ $\quad \Phi(u) = [s_i(u), \phi_i(u)]$

$$(\nabla^{\omega}_{X} \Phi)(u) = [s_i(u), (\nabla^{A_i}_{X} \phi_i)(u)]$$

$A_i = s_i^* \omega$ local gauge fiel...

$$\boxed{\nabla^{A_i}_{X} \phi_i = d\phi_i(X) + \rho_*(A_i(x))\phi_i}$$

$d\phi_i$ V-valued 1-form $\|$ d...

ρ_* Lie algebra representation

$$\rho_*(\xi)v = \frac{d}{dt}\Big|_{t=0}\left(\rho(e^{t\xi})v\right)$$

$$\in \mathfrak{g}$$

$$\nabla^{\omega}_X \Phi = \left[s_0(u), (\nabla^{A_j}_X \phi_j)(u) \right]$$

Change gauge

$$\boxed{\nabla^{A_j}_X \phi_j = \rho(t_{ji}) \nabla^{A_i}_X \phi_i}$$

$$\phi_j = \rho(t_{j\dots}$$

$$A_j = t_{ji} \dots$$

$$\nabla^A_X f\phi = (Xf)\phi + f\nabla^A_X \phi$$

smooth func

$$\nabla^A_{fX}\phi = f\nabla^A_X \phi$$

$) v)$

$\phi_j)(u)]$

Change gauge

$$\boxed{\nabla_X^{A_j}\phi_j = \rho(t_{ji})\,\nabla_X^{A_i}\phi_i}$$

$$\phi_j = \rho(t_{ji})\,\phi_i$$

$$A_j = t_{ji}\,A_i\,t_{ij} + t_{ji}\,dt_{ij}$$

$$\nabla_X^A f\phi = (Xf)\phi + f\nabla_X^A\phi$$

$f \downarrow$ scalar fnc

$$\nabla_{fX}^A\phi = f\nabla_X^A\phi$$

section) $\quad \tau(u) = [A(u), B(u)]$

$G = U(1) \qquad M = \mathbb{R}^{3,1}$ | couple with

once we choose gauge, $\phi : U \to \mathbb{C}^n$

$$\nabla^A_X \phi = d\phi(X) + A(X)\phi$$

local chart $(x^0, \dots, x^3)$ $\qquad X = X^\mu \frac{\partial}{\partial x^\mu}$

$$A = A_\mu dx^\mu$$

Couple with ... field

$\to \mathbb{C}^n$

$x^\mu \frac{\partial}{\partial x^\mu}$

$A_\mu dx^\mu$

$$\nabla^A_\mu \phi = \nabla^A_{\frac{\partial}{\partial x^\mu}} \phi = d\phi\left(\frac{\partial}{\partial x^\mu}\right) + A_\mu \phi$$

$$= \partial_\mu \phi + A_\mu \phi$$

$$\boxed{\mathcal{L} = \eta^{\mu\nu} \nabla^A_\mu \phi^\dagger \nabla^A_\nu \phi + m^2 \phi^\dagger \phi}$$

gauge invariant

$d\phi = \partial_\mu \phi \, dx^\mu$

$A_i = s_i^* \omega$ lo...

$d\phi_i$ V-valued

Lie algebra represent...