Title: Mathematical Physics Core Lecture

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Curvature of Connection $\quad \pi : P \to M$ princ $G$-bundle, $\omega \in \Omega^1(P, \mathfrak{g})$ connection

2-form $\quad \Omega = d\omega + \frac{1}{2}[\omega, \omega] = d\omega + \omega \wedge \omega \quad \to \quad \omega = \underbrace{\omega^a}_{\in \Omega^1(P, \mathbb{R})} \overbrace{T_a}^{\in \mathfrak{g}} \qquad \omega \wedge \omega = \omega^a T_a \wedge \omega^b T_b = \omega^a \wedge \omega^b \, T_a T_b$

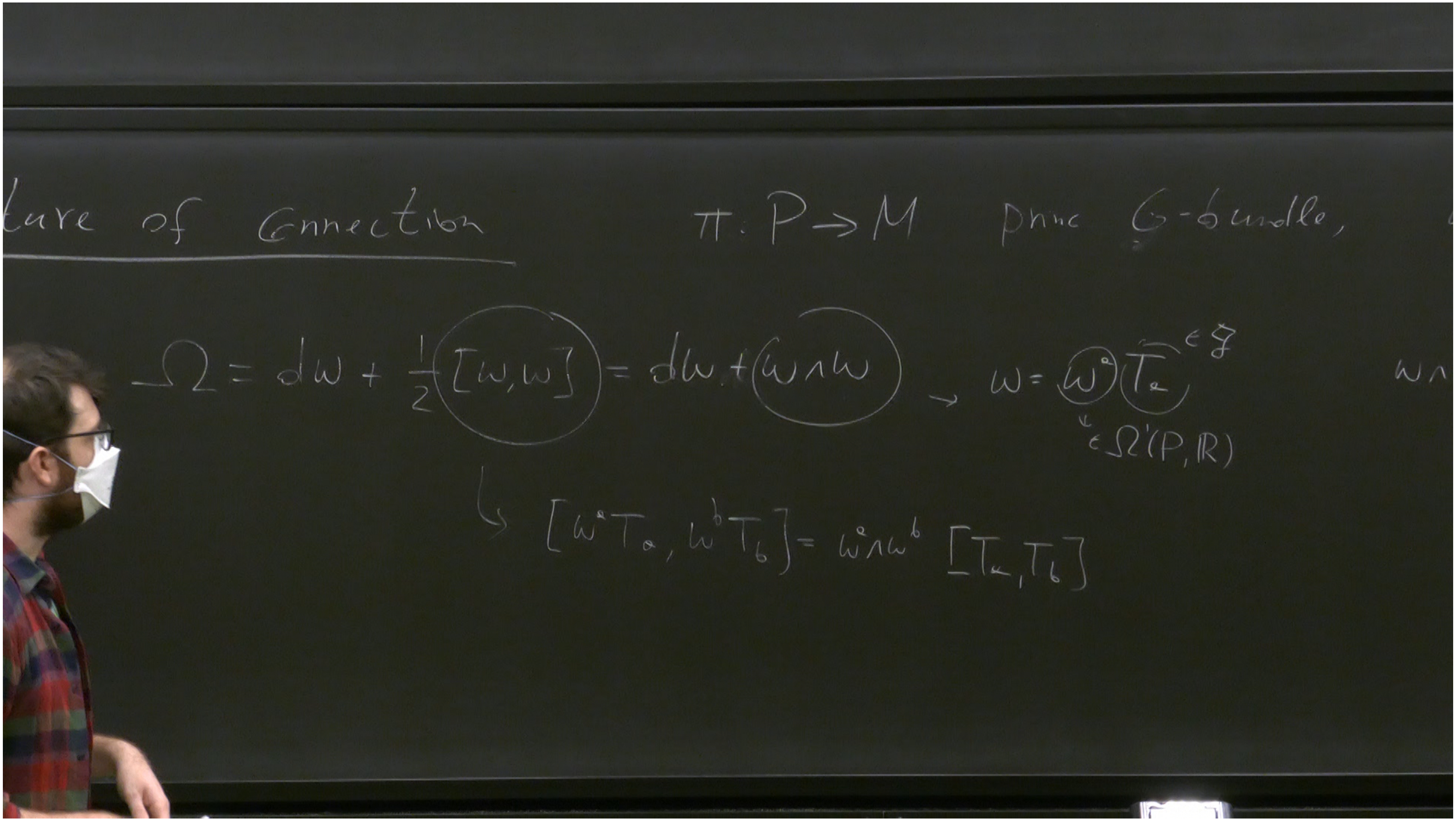
bundle, $\quad \omega \in \Omega^1(P, \mathfrak{g})$ connection

$\in \mathfrak{g}$

$$\omega \wedge \omega = \omega^a T_a \wedge \omega^b T_b = \omega^a \wedge \omega^b \; T_a T_b = \frac{1}{2}\left(\omega^a \wedge \omega^b - \omega^b \wedge \omega^a\right) T_a T_b$$

$$= \frac{1}{2} \omega^a \wedge \omega^b \left(T_a T_b - T_b T_a\right)$$

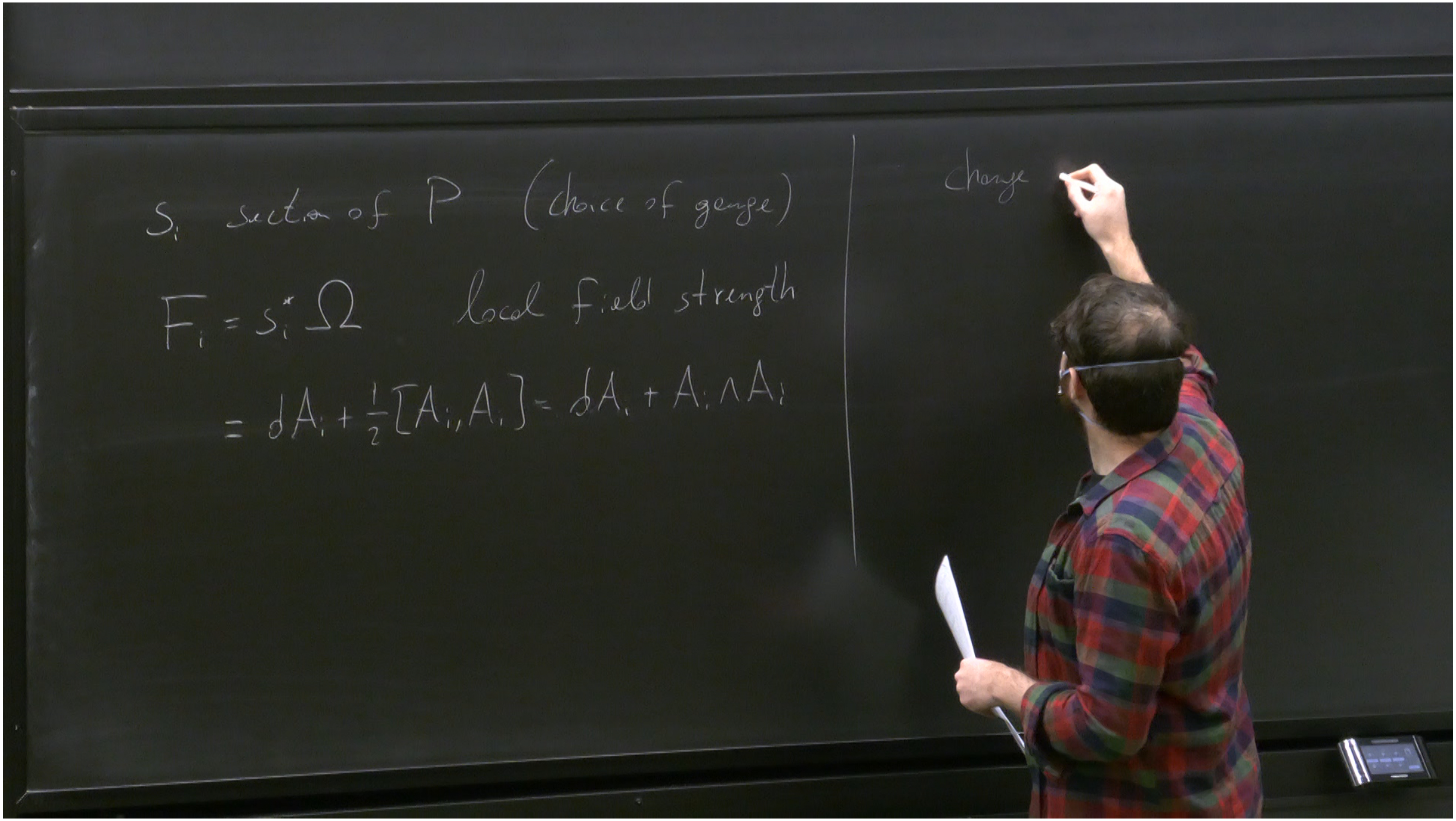
$\mathbb{R})$

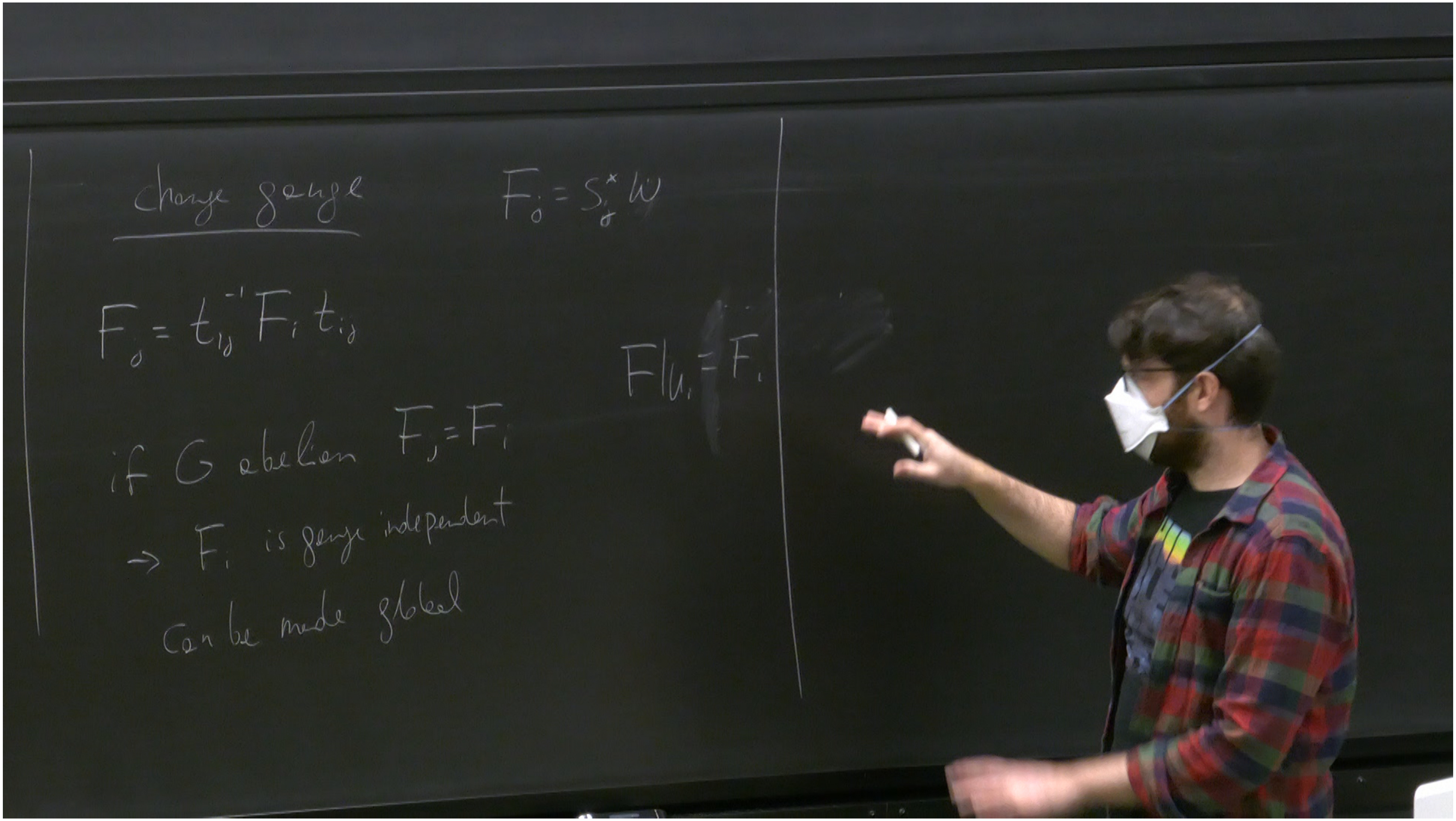

ture of Connection
π: P → M princ G-bundle,
Ω = dω + 1/2 [ω,ω] = dω + ω∧ω → ω = ω^a T_a ∈ g
∈ Ω¹(P, ℝ)
ω∧
[ω^a T_a, ω^b T_b] = ω^a∧ω^b [T_a, T_b]

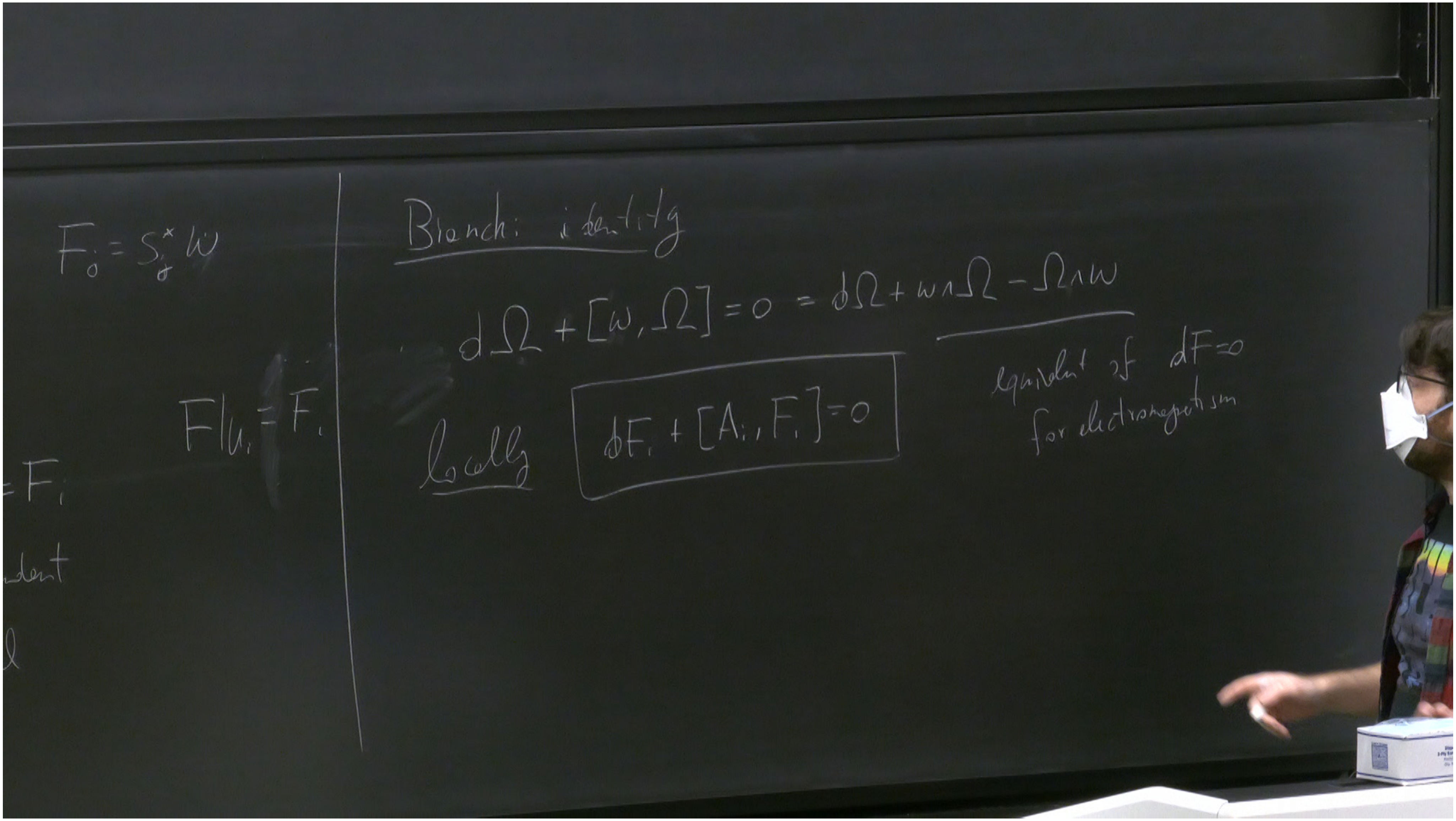
ture of Connection $\quad \pi : P \to M$ princ $G$-bundle,

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega] = d\omega + \omega\wedge\omega \quad \to \quad \omega = \underset{\in \Omega^1(P,\mathbb{R})}{\omega^a}\,\overset{\in \mathfrak{g}}{T_a} \qquad \omega\wedge$$

$$\hookrightarrow \; [\omega^a T_a, \omega^b T_b] = \omega^a \wedge \omega^b \, [T_a, T_b]$$

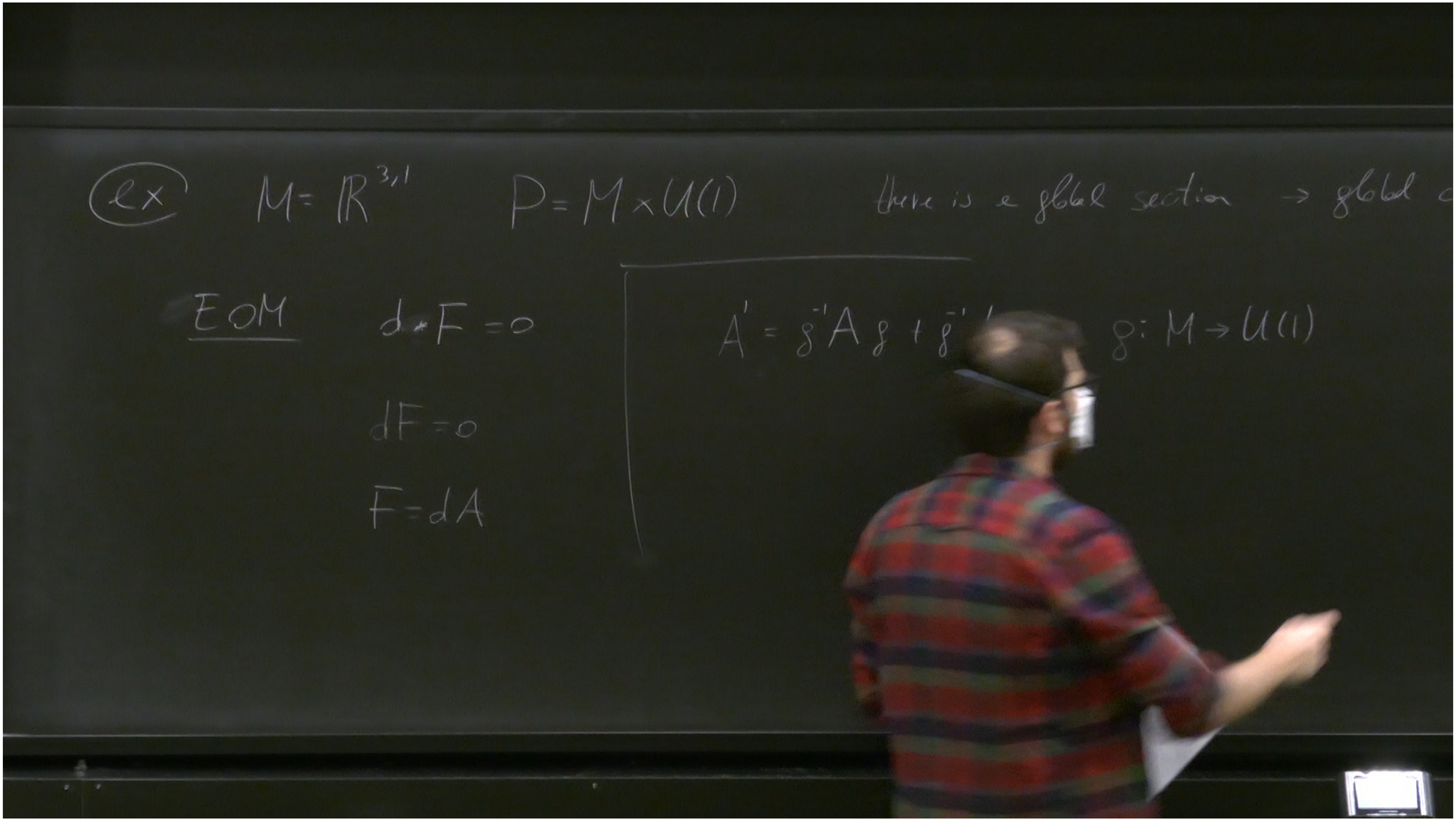

$s_i$ section of $P$ (choice of gauge)
$F_i = s_i^* \Omega$ local field strength
$= dA_i + \frac{1}{2}[A_i, A_i] = dA_i + A_i \wedge A_i$
change


change gauge
$F_j = s_j^* \omega$
$F_j = t_{ij}^{-1} F_i t_{ij}$
$F|_{U_i} = F_i$
if G abelian $F_j = F_i$
→ $F_i$ is gauge independent
can be made global


$F_i = s_i^* \omega$
$F|_{U_i} = F_i$
$= F_i$
Bianchi identity
$d\Omega + [\omega, \Omega] = 0 = d\Omega + \omega \wedge \Omega - \Omega \wedge \omega$
locally $\boxed{dF_i + [A_i, F_i] = 0}$
equivalent of $dF = 0$
for electromagnetism

ex $M = \mathbb{R}^{3,1}$ $P = M \times U(1)$ there is a global section $\rightarrow$ global choice of gauge $\rightarrow$ $A \in \Omega^1(M, \underline{u(1)})$

global choice of gauge $\rightarrow$ $A \in \Omega^1(M, \underline{u}(1))$ | fully specifies $\omega \in \Omega^1(P, \underline{u}(1))$


ex
M = ℝ^{3,1}
P = M × U(1)
there is a global section → global
EOM
d*F = 0
dF = 0
F = dA
A' = g^{-1}A g + g^{-1}
g: M → U(1)

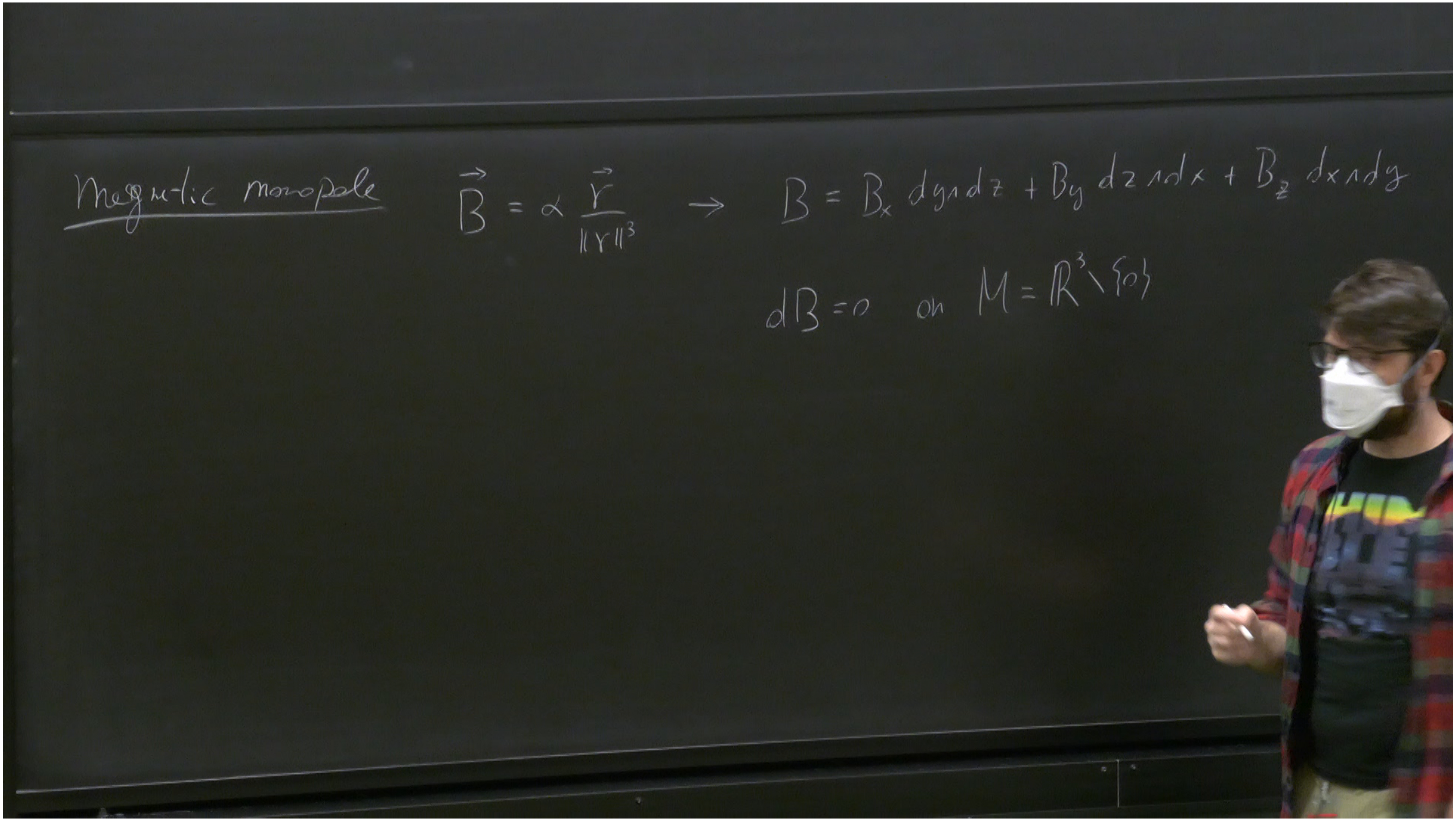
$U(1)$ there is a global section $\to$ global choice of gauge $\to$ $A \in \Omega^1(M, \underline{u(1)})$ | full

$u(1) = i\mathbb{R}$

$A' = g^{-1} A g + g^{-1} dg \qquad g : M \to U(1) \qquad g(u) = e^{i\alpha(u)} \qquad \alpha$ is smooth

$= A + g^{-1} dg$

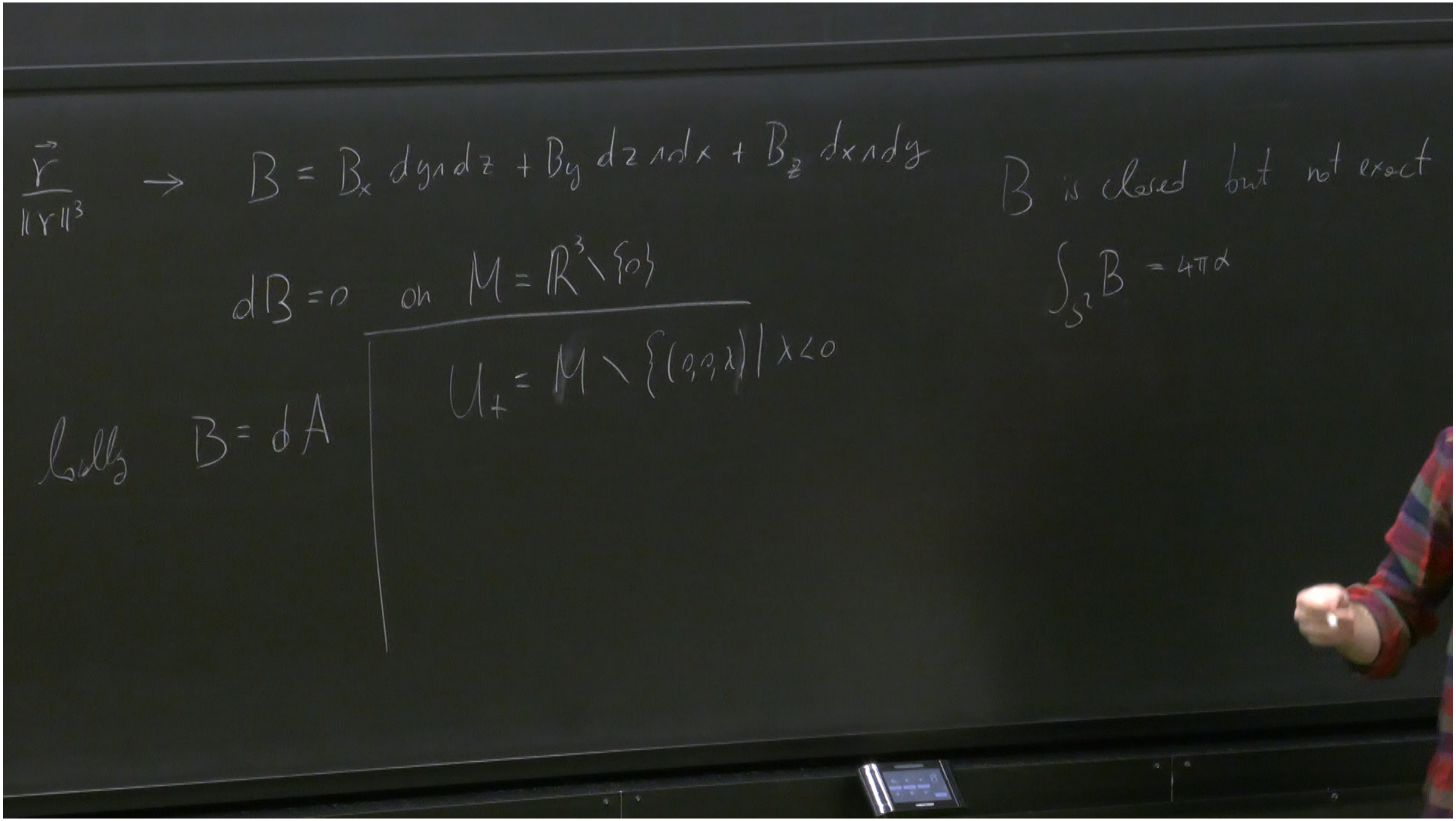
$= A + e^{-i\alpha} d(e^{i\alpha}) = A + i\, d\alpha \left( \cancel{e^{-i\alpha} e^{i\alpha}} \right)$


Magnetic monopole
$\vec{B} = \alpha \frac{\vec{r}}{\|r\|^3} \rightarrow B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy$
$dB = 0$ on $M = \mathbb{R}^3 \setminus \{0\}$

Magnetic monopole $\quad \vec{B} = \alpha \dfrac{\vec{r}}{\|r\|^3} \quad \rightarrow \quad B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy$

$dB = 0$ on $M = \mathbb{R}^3 \setminus \{0\}$

$G = U(1)$


$\frac{\vec{r}}{\|r\|^3} \to B = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy$
B is closed but not exact
$dB = 0$ on $M = \mathbb{R}^3 \setminus \{0\}$
$\int_{S^2} B = 4\pi\alpha$
$U_+ = M \setminus \{(0,0,\lambda) \mid \lambda < 0$
locally $B = dA$

$$\frac{\vec{r}}{\|r\|^3} \rightarrow \quad B = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy$$

$B$ is closed but not exact

$$dB = 0 \quad \text{on} \quad M = \mathbb{R}^3 \setminus \{0\}$$

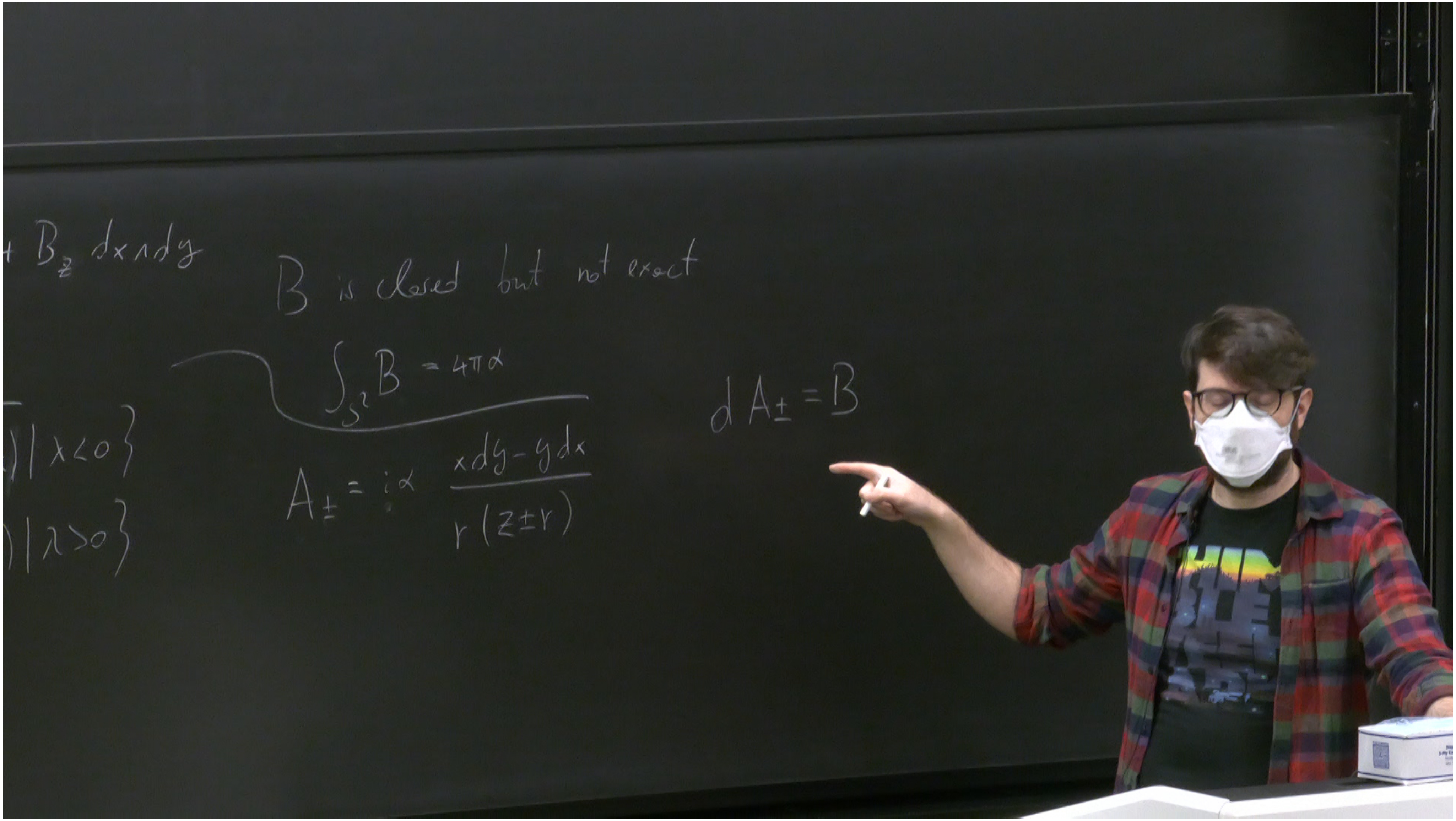
$$\int_{S^2} B = 4\pi\alpha$$

locally $B = dA$

$$U_+ = M \setminus \{(0,0,\lambda) \mid \lambda < 0\}$$

$$U_- = M \setminus \{(0,0,\lambda) \mid \lambda > 0\}$$

$$A_\pm = i\alpha \, \frac{x\,dy - y\,dx}{r\,(z \pm r)}$$


+ $B_z \, dx \wedge dy$
B is closed but not exact
$\int_{S^2} B = 4\pi\alpha$
$d A_\pm = B$
$| x < 0 \}$
$A_\pm = i\alpha \, \frac{x\,dy - y\,dx}{r(z \pm r)}$
$| \lambda > 0 \}$

global choice of gauge $\rightarrow$ $A \in \Omega^1(M, \mathfrak{u}(1))$ | fully specifies $\omega \in \Omega^1(P, \mathfrak{u}(1))$

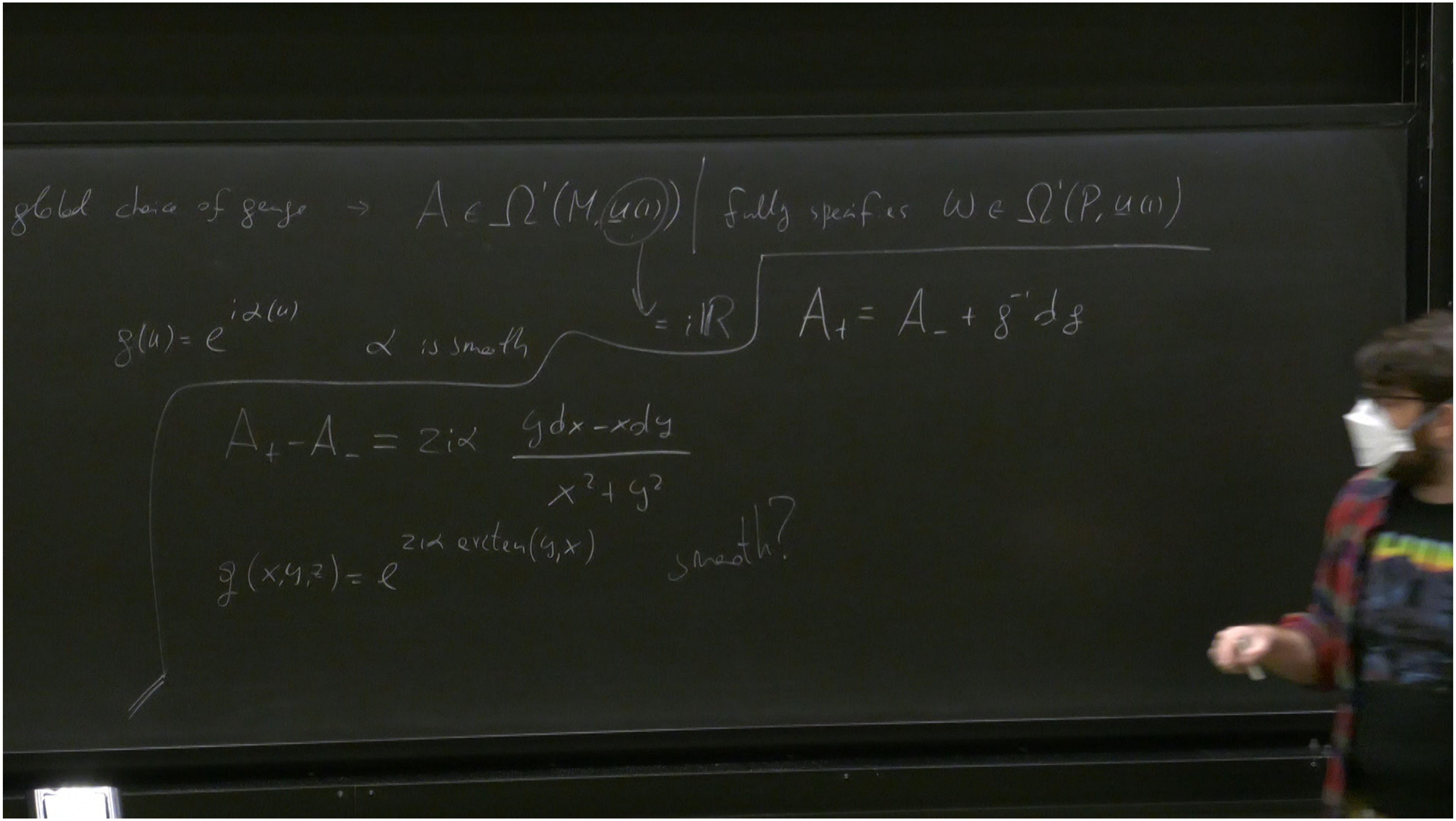
$\mathfrak{u}(1) = i\mathbb{R}$

$A_+ = A_- + g^{-1} dg$

$g(u) = e^{i\alpha(u)}$ $\quad \alpha$ is smooth

$$A_+ - A_- = 2i\alpha \, \frac{y\,dx - x\,dy}{x^2 + y^2}$$

$$g(x,y,z) = e^{2i\alpha \arctan(y,x)}$$


global choice of gauge → $A \in \Omega^1(M, \underline{u}(1))$ | fully specifies $\omega \in \Omega^1(P, \underline{u}(1))$
$= i\mathbb{R}$
$A_+ = A_- + g^{-1} dg$
$g(u) = e^{i\alpha(u)}$
$\alpha$ is smooth
$A_+ - A_- = 2i\alpha \dfrac{y\,dx - x\,dy}{x^2 + y^2}$
$g(x,y,z) = e^{2i\alpha \, \mathrm{arctan}(y,x)}$
smooth?

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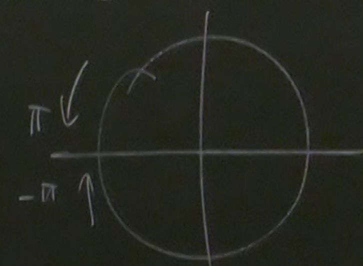


$$A_+ - A_- = 2i\alpha \, \frac{y\,dx - x\,dy}{x^2 + y^2}$$

$$g(x,y,z) = e^{2i\alpha \, \mathrm{arctan}(y,x)}$$

smooth?

$\rightarrow$ $2\alpha$ must be $\in \mathbb{Z}$

Magnetic monopole $\vec{B} = \alpha \frac{\vec{r}}{\|r\|^3} \to B = B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx \wedge dy$

$dB = 0$ on $M = \mathbb{R}^3 \setminus \{0\}$

$G = U(1)$

$U_+ = M \setminus \{(0,0,\lambda) \mid \lambda < 0\}$

locally $B = dA$

$U_- = M \setminus \{(0,0,\lambda) \mid \lambda > 0\}$

$d(A + d\lambda) = dA$

Magnetic monopole

$$\vec{B} = \alpha \frac{\vec{r}}{\|r\|^3} \rightarrow B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy$$

$$\nabla \cdot \vec{B} \propto \delta(x)$$

$$dB = 0 \quad \text{on} \quad M = \mathbb{R}^3 \setminus \{0\}$$

$$G = U(1)$$

$$U_+ = M \setminus \{(0,0,\lambda) \mid \lambda < 0\}$$

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Magnetic monopole

$$\vec{B} = \alpha \frac{\vec{r}}{\|r\|^3} \rightarrow B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy$$

$$\nabla \cdot \vec{B} \propto \delta(x)$$

$dB = 0$ on $M = \mathbb{R}^3 \setminus \{0\}$

$$\frac{\partial B}{\partial t} = 0$$

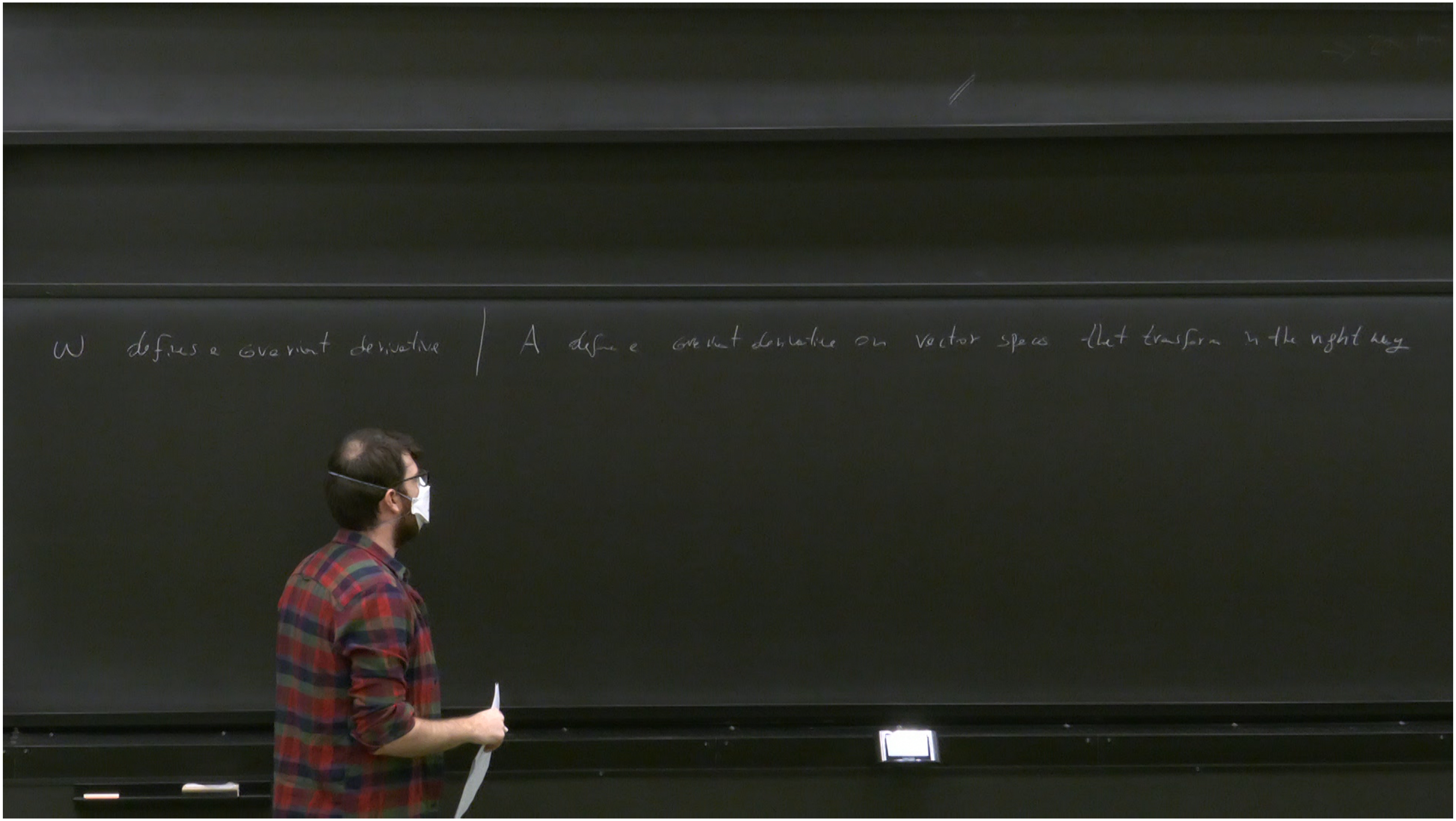
$$G = U(1)$$

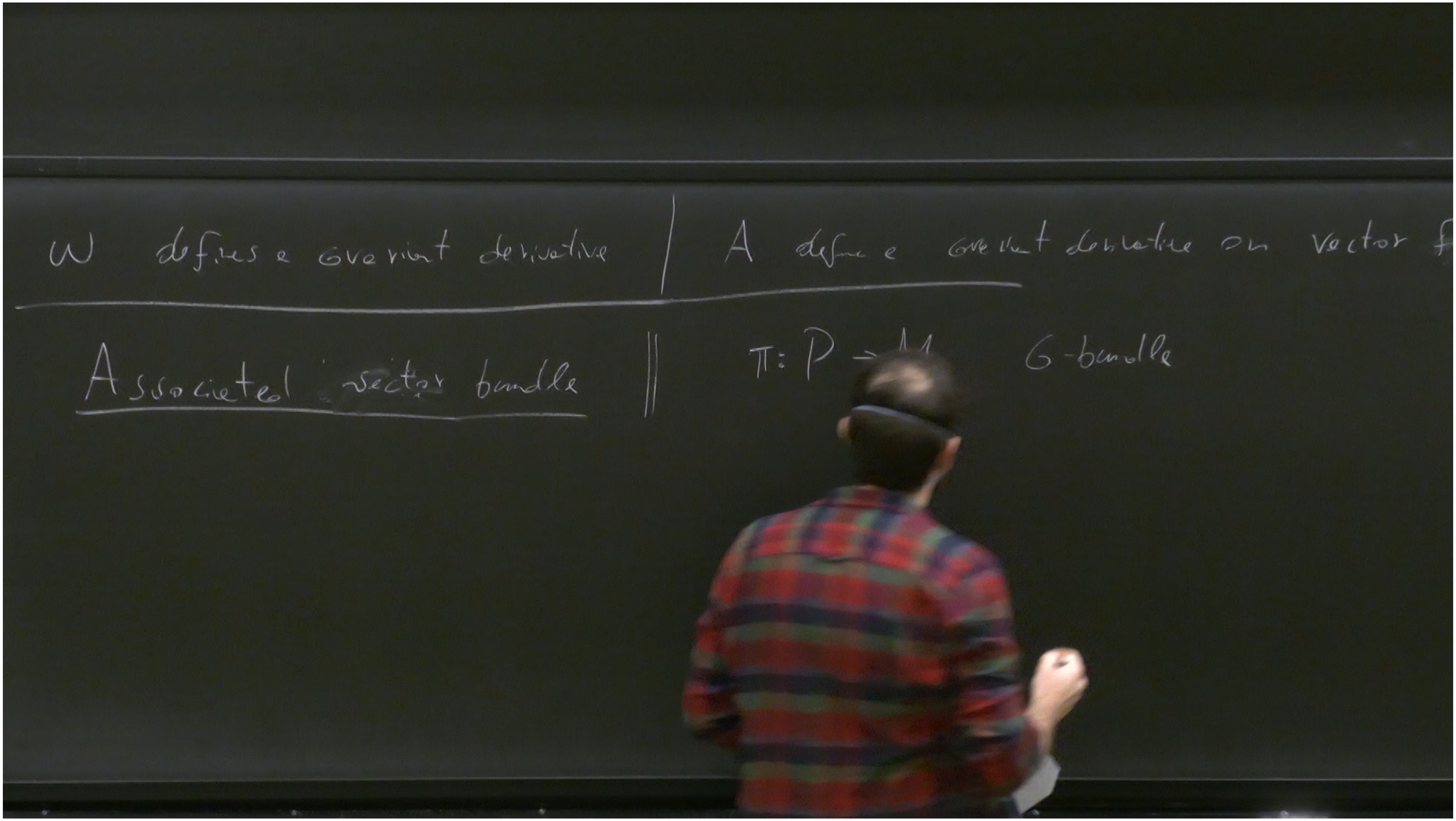
locally $B = dA$

$$d(A + d\lambda) = dA$$

$$U_+ = M \setminus \{(0,0,\lambda) \mid \lambda < 0\}$$

$$U_- = M \setminus \{(0,0,\lambda) \mid \lambda > 0\}$$


ω defines a covariant derivative
A defines a covariant derivative on vector spaces that transform in the right way


ω defines a covariant derivative
A defines a covariant derivative on vector f
Associated vector bundle
π: P →
G-bundle

on vector field that transform in the right way

$V$ real or complex v. space (fibre)

rep. of $G$

$\rho : G \to GL(V) = Aut(V)$

$\{ \Phi : V \to V \mid$ V space isomorphism $\}$

$\rho(e) v = v$

$\rho(gh) v = \rho(g)(\rho(h) v)$

A define covariant derivative on vector field that transform in the right way

$\pi: P \to M$ G-bundle

V real or complex v. space (fibre)

rep. of

$\rho: G \to$

$\rho(e)v = v$

$\rho(gh)v = \rho$

defines a covariant derivative | A defines a covariant derivative on vector fields that transfor

ociated vector bundle || $\pi : P \to M$ G-bundle | V real or complex v. space (

right action of G on $P \times V$ : $\boxed{(p,v) \triangleleft g = (p \triangleleft g,\ \rho(g^{-1})v)}$

$\rho(h^{-1})\rho(g^{-1})v = \rho(h^{-1}g^{-1})v = \rho((gh)^{-1})v = v \triangleleft gh$

$$(P\times V)/G = \{[p,v] \mid p\in P,\ v\in V\}$$

$$[p,v] = \{(p,v)\triangleleft g \mid g\in G\}$$

<u>Vector bundle</u>: $(P\times V)/G$

Vector bundle: $(P \times V)/G = P \times_G V$