Title: Quantum Field Theory for Cosmology - Lecture 20240229

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: February 29, 2024 - 4:00 PM

URL: https://pirsa.org/24020016

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

QFT for Cosmology, Achim Kempf, Lecture 14

Quantum field theory on FRW spacetimes.

Observations:

On scales > 1 GLy:

- The universe is spatially very flat
- The cosmic expansion is very isotropic.

Friedmann Robertson Walker (FRW) spacetimes:

- Simplifying approximation:

  Spacetime is modeled as having

  - no spatial curvature at all.
  - entirely isotropic expansion

(Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

There are even solutions that only temporarily get very close to flatness. The Einstein eqns are nonlinear!)

With these assumptions, we choose convenient coordinates:

* Time coordinate t:

Definition: The motion of galaxies due to the cosmic expansion is called the Hubble flow.

Definition: The peculiar velocity is the "small" extra random velocity that galaxies can possess

* Space coordinates:

It is convenient to use "comoving coordinates", x_1, x_2, x_3:

- At one time, t_0, (say today) we set up an ordinary rectangular coordinate system.
- Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

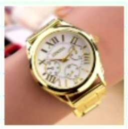
1/24

1

expansion is called the Hubble flow.

Definition: The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

Definition: As the time coordinate, t, let us use the proper time, t, of a freely streaming observer who has no peculiar velocity.



(to a good approximation, you can use your wrist watch on earth)

rectangular coordinate system.

- Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

Advantages:
- In the comoving coordinate system, galaxies have constant coordinates, except for possible peculiar motion.
- Waves keep their wave lengths numerically constant even while they get physically stretched.

* The metric:

Recall that $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$ is the invariant 4-distance.

In our coordinates, $g_{\mu\nu}(x)$ must read:

because we use wrist watch "proper" time

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$$

because our coordinate system's unit of length means over time a larger and larger proper length.

* The "scale factor":

- The scale factor function $a(t)$ is needed to take into account the expansion when calculating distances.
  - Example: The proper distance d between two galaxies with comoving distance $(\Delta x_1, \Delta x_2, \Delta x_3)$ at proper time t is:

$$d = \sqrt{|g_{\mu\nu}(t_0)\, \Delta x^\mu \Delta x^\nu|}$$
$$= a(t)\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

Note: $\Delta x_0 = t_0 - t_0 = 0$ since we are l[illegible] at the distance between the galaxies at equal times.

3/24

3

because we use wrist watch "proper" time

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$$

because our coordinate system's unit of length means over time a larger and larger proper length.

o Example: The proper distance d between two galaxies with comoving distance $(\Delta x_1, \Delta x_2, \Delta x_3)$ at proper time t is:

$$d = \sqrt{|g_{\mu\nu}(t_0)\,\Delta x^\mu \Delta x^\nu|}$$

$$= a(t)\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

Note: $\Delta x_0 = t_0 - t_0 = 0$ since we are looking at the distance between the galaxies at equal time.

* Dynamics of $a(t)$:

The function $a(t)$ is determined by **all** equations of motion:

1. Calculate the energy momentum tensor $T_{\mu\nu}(t,\vec{x})$ contributions of at least the most important fields, say $\phi_i(t,\vec{x})$.

2. Solve, simultaneously:

   * The equations of motion for the fields ϕ_i

   * The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(t,x) = \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix}$:

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

* Semi-classical approximation

We can solve these classically, but not quantum mechanically:

Can quantize only ϕ_i, not $g_{\mu\nu}$.

$\Rightarrow$ need to "make quantum $T_{\mu\nu}(t,\vec{x})$ classical" for Einstein eqn!

$\rightsquigarrow$ One uses: $\overline{T}_{\mu\nu}(x) := \langle \Omega | T_{\mu\nu}(t,\vec{x}) | \Omega \rangle$

Problem: Energy & Momentum are naturally nonlocal because of uncertainty principle.

Remark: $\dot{a}(t)$ is related to curvature between space & time.

> For now, we will assume that the expansion's scale factor function $a(t)$ is given.

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

2. Solve, simultaneously:

* The equations of motion for the fields ϕ_i

* The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(t,x) = \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix}$:

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

~> One uses: $\overline{T}_{\mu\nu}(x) := \langle \Omega | T_{\mu\nu}(t,\vec{x}) | \Omega \rangle$

Problem: Energy & Momentum are naturally nonlocal because of uncertainty principle.

Remark: $\dot{a}(t)$ is related to curvature between space & time.

$\boxed{\text{For now, we will assume that the expansion's scale factor function } a(t) \text{ is given.}}$

Convenient Definition: The conformal time coordinate, η.

- Recall that:

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & 0 \\ & & -a^2(t) & \\ & 0 & & -a^2(t) \end{pmatrix}$$

- It would be convenient if $g_{\mu\nu}$ were proportional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

- This can be achieved by choosing a new time coordinate η, so that time also has a prefactor a^2, i.e., so that:

$$(\Delta t)^2 = a^2(t_0)(\Delta \eta)^2$$

- To this end, we need: $a\, d\eta = dt$

i.e.: $\dfrac{d\eta}{dt} = \dfrac{1}{a}$

and therefore $\eta(t) = \displaystyle\int_{t_0}^{t} \frac{1}{a(t')}\, dt'$ ← yields arbitrary integration constant.

- The variable η is called the "**conformal time**".

(...because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale-factor-only transformations)

- Using conformal time and comoving spatial coordinates the metric reads:

$$g_{\mu\nu}(\eta,\vec{x}) = a^2(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^2(\eta)\, \eta_{\mu\nu}$$

do not mix up

7

7 / 24

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

- It would be convenient if $g_{\mu\nu}$ were propotional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

- This can be achieved by choosing a new time coordinate η, so that time also has a prefactor a^2, i.e., so that:

$$(\Delta t)^2 = a^2(t_0)(\Delta\eta)^2$$

- To this end, we need: $a\, d\eta = dt$

- This also implies:

$$g^{\mu\nu}(\eta,\vec{x}) = a^{-2}(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^{-2}(\eta)\,\eta^{\mu\nu}$$

Recall: $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu{}_\sigma$, i.e., $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverse to another.

- We easily obtain the integral measure needed for the action:

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu}(\eta,\vec{x}))|} = a^4(\eta)$$

(...because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale-factor-only transformations)

- Using conformal time and comoving spatial coordinates the metric reads:

$$g_{\mu\nu}(\eta,\vec{x}) = a^2(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^2(\eta)\,\eta_{\mu\nu}$$

do not mix up

The Klein Gordon field in FRW spacetimes

- Neglecting a potential $V(\phi)$ for now, we obtain the action of the "free K.G. field on the FRW background":

$$S_{KG} = \int \left(\frac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)\sqrt{|g|}\, d^4x$$

$$\overset{\text{here}}{=} \int \left(\frac{1}{2} a^{-2}(\eta)\,\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right) a^4\, d\eta\, d^3x$$

- Thus, from the general Euler Lagrange equation

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

9 / 24

9

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

Recall: $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu{}_\sigma$, i.e., $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverse to another.

- We easily obtain the integral measure needed for the action:

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu}(\eta,\vec{x}))|} = a^4(\eta)$$

the action of the free K.G. field on the FRW background:

$$S_{KG} = \int \left(\frac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)\sqrt{|g|}\, d^4x$$

$$\overset{\text{here}}{=} \int \left(\frac{1}{2} a^{-2}(\eta)\,\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right) a^4\, d\eta\, d^3x$$

- Thus, from the general Euler Lagrange equation

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

$$\left(\frac{1}{a^4(\eta)}\frac{\partial}{\partial x^\mu}\eta^{\mu\nu}a^2\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

$$\left(\frac{1}{a^4(\eta)}\eta^{\mu\nu}a^2(\eta)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \frac{1}{a^4(\eta)}2a'a\frac{\partial}{\partial x^0} + m^2\right)\phi(x) = 0 \qquad a' := \frac{da}{d\eta}$$

$$\boxed{\phi''(\eta,\vec{x}) + 2\frac{a'(\eta)}{a(\eta)}\phi'(\eta,\vec{x}) - \Delta\phi(\eta,\vec{x}) + a^2(\eta)m^2\phi(\eta,\vec{x}) = 0}$$

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi'' + \divideontimes\,\phi' + \divideontimes\,\phi = 0$$

a time-dependent friction-like term that is entirely new.

a term that also occurs in the usual harmonic oscillator. Notice though that it is now time-dependent.

Strategy: Use a new, re-scaled, field variable χ:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

This simple ansatz succeeds:

$$\chi(\eta,\vec{x}) := a(\eta)\,\phi(\eta,\vec{x})$$

Namely:

we have: $\phi' = \frac{\partial}{\partial\eta}\frac{1}{a}\chi = -\frac{a'}{a^2}\chi + \frac{1}{a}\chi'$

and: $\phi_{,i} = \frac{\partial}{\partial x^i}\frac{1}{a(\eta)}\chi(\eta,\vec{x}) = \frac{1}{a}\chi_{,i}$ for $i=1,2,3$

File Edit View Insert Actions Tools Help

Sharing Desktop ▲ ■ Stop Sharing

Recall: $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu{}_\sigma$, i.e., $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverse to another.

- We easily obtain the integral measure needed for the action:

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu}(\eta,\vec{x}))|} = a^4(\eta)$$

$$\left(\frac{1}{a^4(\eta)}\frac{\partial}{\partial x^\mu}\eta^{\mu\nu}a^2\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

$$\left(\frac{1}{a^4(\eta)}\eta^{\mu\nu}a^2(\eta)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \frac{1}{a^4(\eta)}2a'a\frac{\partial}{\partial x^0} + m^2\right)\phi(x) = 0 \qquad a' = \frac{da}{d\eta}$$

$$\boxed{\phi''(\eta,\vec{x}) + 2\frac{a'(\eta)}{a(\eta)}\phi'(\eta,\vec{x}) - \Delta\phi(\eta,\vec{x}) + a^2(\eta)m^2\phi(\eta,\vec{x}) = 0}$$

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi'' + \ast\phi' + \ast\phi = 0$$

a time-dependent friction-like term that is entirely new.

a term that also occurs in the usual harmonic oscillator. Notice though that it is now time-dependent.

$$S_{KG} = \int\left(\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)\sqrt{|g|}\,d^4x$$

$$\overset{\text{here}}{=} \int\left(\frac{1}{2}a^{-2}(\eta)\,\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right)a^4\,d\eta\,d^3x$$

- Thus, from the general Euler Lagrange equation

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

Strategy: Use a new, re-scaled, field variable χ:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

This simple ansatz succeeds:

$$\chi(\eta,\vec{x}) := a(\eta)\,\phi(\eta,\vec{x})$$

Namely:

we have: $\phi' = \frac{\partial}{\partial\eta}\frac{1}{a}\chi = -\frac{a'}{a^2}\chi + \frac{1}{a}\chi'$

and: $\phi_{,i} = \frac{\partial}{\partial x^i}\frac{1}{a(\eta)}\chi(\eta,\vec{x}) = \frac{1}{a}\chi_{,i}$ for $i=1,2,3$

11 / 24

11

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi'' + \ast\,\phi' + \ast\,\phi = 0$$

a time-dependent friction-like term that is entirely new.

a term that also occurs in the usual harmonic oscillator. Notice though that it is now time-dependent.

This simple ansatz succeeds:

$$\chi(\eta,\vec{x}) := a(\eta)\,\phi(\eta,\vec{x})$$

Namely:

we have: $\phi' = \frac{\partial}{\partial\eta}\frac{1}{a}\chi = -\frac{a'}{a^2}\chi + \frac{1}{a}\chi'$

and: $\phi_{,i} = \frac{\partial}{\partial x^i}\frac{1}{a(\eta)}\chi(\eta,\vec{x}) = \frac{1}{a}\chi_{,i}$ for $i=1,2,3$

Using these, the action in terms of χ becomes:

$$S_{\chi} = \int \frac{1}{2}\left(\chi'^2 - \sum_{i=1}^{3}\chi_{,i}^2 - \underbrace{\left(m^2a^2 - \frac{a''}{a}\right)}\chi^2\right) d\eta\, d^3x$$

Note that this term is like a time-dependent mass term $m^2_{eff}(\eta)$

Exercise: verify

Equation of motion:

* Do

$$\frac{\delta S}{\delta\phi(\eta,\vec{x})} = 0 \quad \text{and} \quad \frac{\delta S}{\delta\chi(\eta,\vec{x})} = 0$$

yield equivalent equations of motion?

* Yes, because:

$$0 = \frac{\delta S}{\delta\phi} = \frac{\delta S}{\delta\chi}\frac{\delta\chi}{\delta\phi}$$

if $\delta S/\delta\chi$ vanishes then also $\delta S/\delta\phi$ vanishes.

* Thus, we may calculate the equation of motion directly in terms of χ from $S[\chi]$, to obtain:

Exercise: verify!

$$\chi'' - \Delta\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0 \qquad \textbf{(EoM 1)}$$

Remark:

We could have obtained this equation of motion directly from that of ϕ by change of variable. But finding the action for χ was still worthwhile, namely to get the conjugate to χ'

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

Exercise: verify

Equation of motion:

* Do
$$\frac{\delta S'}{\delta\phi(\eta,\vec{x})}=0 \quad \text{and} \quad \frac{\delta S'}{\delta\chi(\eta,\vec{x})}=0$$
yield equivalent equations of motion?

Preparation for quantization:

* We need the canonically conjugate field
$$\pi^{(\chi)}(\eta,\vec{x})$$
to the field $\chi(\eta,\vec{x})$, i.e., the Legendre transform of χ'.

* To this end, we consider the Lagrangian:
$$L=\int \frac{1}{2}\left(\chi'^2-\sum_{i=1}^{3}\chi_{,i}^{\;2}-\left(m^2a^2-\frac{a''}{a}\right)\chi\right)d^3x$$

* Thus, the Legendre transformed variable reads:
$$\pi^{(\chi)}(\eta,\vec{x}):=\frac{\delta L}{\delta\chi'(\eta,\vec{x})}=\chi'(\eta,\vec{x}) \qquad \text{(EoM 2)}$$

directly in terms of χ from $S[\chi]$, to obtain:

Exercise: verify!
$$\chi''-\Delta\chi+\left(m^2a^2-\frac{a''}{a}\right)\chi=0 \qquad \text{(EoM 1)}$$

Remark:
We could have obtained this equation of motion directly from that of ϕ by change of variable. But finding the action for χ was still worthwhile, namely to get the conjugate to χ!

* Which is the field that is conjugate to ϕ?
$$S_{K.G.}=\int\left(\frac{1}{2}a^{-2}(\eta)\,\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}m^2\phi^2\right)a^4\,d\eta\,d^3x$$

$\Rightarrow$ The field $\pi^{(\phi)}$ which is conjugate to ϕ reads:
$$\pi^{(\phi)}:=\frac{\delta L}{\delta\phi'}=a^2\phi'$$

* Compare:
$$\begin{aligned}\pi^{(\chi)}&=\chi'\\&=(a\phi)'\\&=a\phi'+a'\phi\\&=\frac{1}{a}\pi^{(\phi)}+a'\phi\end{aligned}$$
, i.e., $\pi^{(\phi)}$, $\pi^{(\chi)}$ are different!

15 / 24

15

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

* To this end, we consider the Lagrangian:

$$L = \int \frac{1}{2}\left(x'^2 - \sum_{i=1}^{3} x_{,i}^{\;2} - \left(m^2 a^2 - \frac{a''}{a}\right) x\right) d^3x$$

* Thus, the Legendre transformed variable reads:

$$\pi^{(x)}(\eta,\vec{x}) := \frac{\delta L}{\delta x'(\eta,\vec{x})} = x'(\eta,\vec{x}) \qquad \text{(EoM 2)}$$

$$\pi^{(\phi)} := \frac{\delta L}{\delta \phi'} = a^2 \phi'$$

* Compare:

$$\begin{aligned}\pi^{(x)} &= x' \\ &= (a\phi)' \\ &= a\phi' + a'\phi \\ &= \frac{1}{a}\pi^{(\phi)} + a'\phi\end{aligned}$$

, i.e., $\pi^{(\phi)}$, $\pi^{(x)}$ are different!

□ Quantization:

$$[\hat{\phi}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')] = i\,\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{\phi}(\eta,\vec{x}), \hat{\phi}(\eta,\vec{x}')] = 0$$

$$[\hat{\pi}^{(\phi)}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')] = 0$$

□ Proposition:

In terms of the fields $\hat{x} := a\hat{\phi}$, $\hat{\pi}^{(x)} := \hat{x}'$, these commutation relations become:

$$[\hat{x}(\eta,\vec{x}), \hat{\pi}^{(x)}(\eta,\vec{x}')] = i\,\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{x}(\eta,\vec{x}), \hat{x}(\eta,\vec{x}')] = 0$$

$$[\hat{\pi}^{(x)}(\eta,\vec{x}), \hat{\pi}^{(x)}(\eta,\vec{x}')] = 0$$

□ Proof: Only the first CCR is nontrivial to check:

$$\begin{aligned}[\hat{x}(\eta,\vec{x}), \hat{\pi}^{(x)}(\eta,\vec{x}')] &= [a(\eta)\hat{\phi}(\eta,\vec{x}), \tfrac{1}{a(\eta)}\hat{\pi}^{(\phi)}(\eta,\vec{x}') + a'(\eta)\hat{\phi}(\eta,\vec{x}')] \\ &= [\hat{\phi}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')] \\ &= i\,\delta^3(\vec{x}-\vec{x}')\end{aligned}$$

□ Thus, the change from ϕ to x is fairly trivial.

Notice, however:

$$L \xrightarrow{\text{L.T. } \phi' \text{ replaced by } \pi^{\phi}} H^{(\phi)} := \left\{\int \phi' \pi^{(\phi)} d^3x - L\right\}$$

$$L \xrightarrow{\text{L.T. } x' \text{ replaced by } \pi^{x}} H^{(x)} := \left\{\int x' \pi^{(x)} d^3x - L\right\}$$

they have no reason to be the same!

Proposition:

In terms of the fields $\hat{x} := a\,\hat{\phi}$, $\hat{\pi}^{(x)} := \hat{x}'$, these commutation relations become:

$$[\hat{x}(\eta,\vec{x}), \hat{\pi}^{(x)}(\eta,\vec{x}')] = i\,\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{x}(\eta,\vec{x}), \hat{x}(\eta,\vec{x}')] = 0$$

$$[\hat{\pi}^{(x)}(\eta,\vec{x}), \hat{\pi}^{(x)}(\eta,\vec{x}')] = 0$$

Thus, the change from ϕ to x is fairly trivial.

Notice, however:

$$L \xrightarrow{\text{L.T. } \phi' \text{ replaced by } \pi^{\phi}} H^{(\phi)} := \int \phi' \pi^{(\phi)} d^3x - L$$

$$L \xrightarrow{\text{L.T. } x' \text{ replaced by } \pi^{x}} H^{(\pi)} := \int x' \pi^{(x)} d^3x - L$$

they have no reason to be the same!

Question:

How can both be valid generators of time evolution, i.e., how can we have:

$$i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}] \quad \text{and} \quad i\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

and yet $\hat{H}^{(\phi)} \neq \hat{H}^{(x)}$?

Should there not be one Hamiltonian for all variables?

Answer: Yes, and it is, of course $\hat{H}^{(\phi)}$.

Recall that in QM: $i\hat{Q}' = [\hat{Q}, \hat{H}] + \underbrace{i\frac{\partial}{\partial t}\hat{Q}}_{\substack{\text{This extra term is there if the variable } \hat{Q} \text{ has also explicit time-} \\ \text{dependence, e.g., } \hat{Q} = \cos(at+bt^2)\hat{q} + c\hat{p}, \text{ or here: } \hat{x} = \frac{1}{a}\hat{\phi}.}}$

Explicitly:

* From $\hat{x} = a\hat{\phi}$ and $i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}]$ we obtain:

$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\frac{1}{a}\hat{x}' - i\frac{a'}{a^2}\hat{x} = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\hat{x}' = [\hat{x}, \hat{H}^{(\phi)}] + i\frac{a'}{a}\hat{x}$$

* But we also have:

$$i\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

$\Rightarrow$ We must have: $\hat{H}^{(x)} \neq \hat{H}^{(\phi)}$

$i\hat\phi' = [\hat\phi, \hat H^{(\phi)}]$ and $i\hat x' = [\hat x, \hat H$

and get $\hat H^{(\phi)} \neq \hat H^{(x)}$?

- Should there not be one Hamiltonian for all variables?
- **Answer**: Yes, and it is, of course $\hat H^{(\phi)}$.

This extra term is there if the variable $\hat Q$ has also explicit time-dependence, e.g., $\hat Q = \cos(at+bt^2)\hat q + c\hat p$, or here: $\hat x = \frac{1}{a}\hat\phi$.

Recall that in QM: $i\hat Q' = [\hat Q, \hat H] + \underbrace{i\frac{\partial}{\partial t}\hat Q}$

$\Rightarrow \quad i\frac{1}{a}\hat x' - i\frac{a'}{a^2}\hat x = \frac{1}{a}[\hat x, \hat H^{(\phi)}]$

$\Rightarrow \quad i\hat x' = [\hat x, \hat H^{(\phi)}] + i\frac{a'}{a}\hat x$

* But we also have:

$$i\hat x' = [\hat x, \hat H^{(x)}]$$

$\Rightarrow$ We must have: $\quad \hat H^{(x)} \neq \hat H^{(\phi)}$

Since there are multiple Hamiltonians, which, if anyone, is the energy?

- One usually defines the energy as the generator of time evolution. We saw that in the presence of gravity this is ambiguous: one can define many different Hamiltonians for the same theory (same action).
- Therefore, with Einstein, we define the energy (density) not as the generator of time evolution but as a generator of curvature:
- Recall: The Einstein equation

$$\underbrace{R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)R(x)}_{\text{curvature}} + \Lambda g_{\mu\nu}(x) = \underbrace{8\pi G\, T_{\mu\nu}(x)}_{\text{"energy momentum"}}$$

- Recall: The K.G. field's energy-momentum tensor

$$T_{\mu\nu}^{(\phi)}(\eta,\vec x) = \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\tfrac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}m^2\phi^2\right]$$

- Consider $T_{00}(\eta,\vec x)$, which is called the "energy density":

Note: In differential geometry, there is also another use of the term "density": For every tensor, say $Q_{\mu\nu}$, there is a so-called "tensor density" $\tilde Q_{\mu\nu}$, defined as $\tilde Q_{\mu\nu} := Q_{\mu\nu}\sqrt{g}$, which absorbs the obligatory measure factor in integrations.

$$T_{00}(\eta,\vec x) = a^{-4}\frac{1}{2}\pi^{(\phi)^2} + \frac{1}{2}\sum_{i=1}^{3}\phi_{,i}^{\,2} + \frac{a^2}{2}m^2\phi^2 \qquad (T)$$

- Exercises:
  a) Verify (T).
  b) Calculate $H^{(\phi)}$. Notice that $H^{(\phi)}$ is not a scalar.
  c) Show that $H^{(\phi)}(\eta) = \int_{\mathbb{R}^3} T^0_{\ 0}(\eta,\vec x)\sqrt{g}\, d^3x$.
  d) Calculate $H^{(x)}(\eta)$.