Title: Quantum Field Theory for Cosmology - Lecture 20240215

Speakers: Achim Kempf

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QFT for Cosmology, Achim Kempf, Lecture 12

We are now ready to 2nd quantize:

SR, 1st Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, 1st Q. Lagrangian formalism

step 2: allow curvature

GR, 1st Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1st Q Lagrangian formalism

step 4

GR, 2nd Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, 2nd Q Lagrangian formalism (Path integral of QFT)

Solving the quantized theory is to solve:

1.) Commutation relations:

$$[\hat\phi(x,t), \hat\pi(x',t)] = i\hbar\, \delta^3(x-x')$$
$$[\hat\phi(x,t), \hat\phi(x',t)] = 0$$
$$[\hat\pi(x,t), \hat\pi(x',t)] = 0$$

(CCRs)

2.) Hermiticity:

$$\hat\phi^+(x,t) = \hat\phi(x,t), \qquad \hat\pi^+(x,t) = \hat\pi(x,t) \qquad \text{(HC)}$$

which is needed so that the expectation values are real.

3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat f(\phi,\pi) = \frac{1}{i\hbar}[\hat f, \hat H] \qquad \text{for } \hat f = \hat\phi,\ \hat f = \hat\pi,\ \text{etc}$$

Namely:

How to solve the CCR, HC and EoM equations?

Recall: the solution we obtained on Minkowski space:

The a_k, a_k^+ take care of the CCRs

$$\hat\phi(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \Big(\underbrace{e^{-i\omega_k t + ikx}}_{} a_k + \underbrace{e^{i\omega_k t - ikx}}_{} a_k^+ \Big) d^3k$$

Number-valued solutions to the K.G. equation

and $\hat\pi(x,t) = \dot{\hat\phi}(x,t)$

Strategy: * ensure hermiticity, HC, by construction

* separate the CCR and EoM problems:

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Hamiltonian formalism $\xrightarrow{\text{(equivalence)}}$ Lagrangian formalism

step 2 ↓ allow curvature

GR, 1st Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1st Q Lagrangian formalism

step 4 ↓

GR, 2nd Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, 2nd Q Lagrangian formalism (Path integral of QFT)

$$\left.\begin{array}{l}[\hat\phi(x,t),\hat\phi(x',t)]=0\\ [\hat\pi(x,t),\hat\pi(x',t)]=0\end{array}\right\}\quad \text{(CCRs)}$$

2.) Hermiticity:

$$\hat\phi^\dagger(x,t)=\hat\phi(x,t),\qquad \hat\pi^\dagger(x,t)=\hat\pi(x,t)\qquad \text{(HC)}$$

which is needed so that the expectation values are real.

3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat f(\phi,\pi)=\frac{1}{i\hbar}[\hat f,\hat H]\qquad \text{for } \hat f=\hat\phi,\ \hat f=\hat\pi,\ \text{etc}$$

Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu}+m^2\right)\hat\phi(x,t)=0\qquad \text{(EoM1)}$$

and:

$$\hat\pi(x,t)=\sqrt{|g|}\,g^{0\nu}\frac{\partial}{\partial x^\nu}\hat\phi(x,t)\qquad \text{(EoM2)}$$

How to solve the CCR, HC and EoM equations?

Recall: the solution we obtained on Minkowski space:

$$\hat\phi(x,t)=\int_{\mathbb{R}^3}\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\Big(\underbrace{e^{-i\omega_k t+ikx}}_{\text{Number-valued solutions to the K.G. equation}}\hat a_k+\underbrace{e^{i\omega_k t-ikx}}\hat a_k^\dagger\Big)d^3k$$

The $a_k, a_k^\dagger$ take care of the CCRs

and $\hat\pi(x,t)=\dot{\hat\phi}(x,t)$

Strategy:
* ensure hermiticity, HC, by construction
* separate the CCR and EoM problems:

Ansatz:

$$\boxed{\begin{array}{l}\hat\phi(x,t):=\sum_k u_k(x,t)\,a_k+u_k^*(x,t)\,a_k^\dagger\\ \hat\pi(x,t):=\sqrt{|g|}\,g^{0\nu}\frac{\partial}{\partial x^\nu}\hat\phi(x,t)\end{array}}$$

(k need not be a "momentum"!)

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Hamiltonian formalism $\xrightarrow{\text{(equivalence)}}$ Lagrangian formalism

step 2 | allow curvature

GR, 1st Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1st Q Lagrangian formalism

step 4

GR, 2nd Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, 2nd Q Lagrangian formalism (Path integral of QFT)

$$\left.\begin{aligned}[\hat\phi(x,t),\hat\phi(x',t)]&=0\\ [\hat\pi(x,t),\hat\pi(x',t)]&=0\end{aligned}\right\}\quad \text{(CCRs)}$$

2.) Hermiticity:

$$\hat\phi^+(x,t)=\hat\phi(x,t),\qquad \hat\pi^+(x,t)=\hat\pi(x,t)\qquad \text{(HC)}$$

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Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu}+m^2\right)\hat\phi(x,t)=0\qquad \text{(EoM1)}$$

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How to solve the CCR, HC and EoM equations?

Recall: the solution we obtained on Minkowski space:

The a_k, a_k^+ take care of the CCRs

$$\hat\phi(x,t)=\int_{\mathbb{R}^3}\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\Big(\underbrace{e^{-i\omega_k t+ikx}}\,\hat a_k+\underbrace{e^{i\omega_k t-ikx}}\,\hat a_k^+\Big)\,d^3k$$

Number-valued solutions to the K.G. equation

and $\hat\pi(x,t)=\dot{\hat\phi}(x,t)$

Strategy:
* ensure hermiticity, HC, by construction
* separate the CCR and EoM problems:

Ansatz:

$$\boxed{\begin{aligned}\hat\phi(x,t)&:=\sum_k u_k(x,t)\,a_k+u_k^*(x,t)\,a_k^+\\ \hat\pi(x,t)&:=\sqrt{|g|}\,g^{0\nu}\frac{\partial}{\partial x^\nu}\hat\phi(x,t)\end{aligned}}$$

(k need not be a "momentum"!)

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$$\frac{d}{dt}\hat{f}(\phi,\pi) = \frac{1}{i\hbar}[\hat{f}, \hat{H}] \quad \text{for } \hat{f}=\hat{\phi},\ \hat{f}=\hat{\pi}, \text{ etc}$$

Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{(EoM1)}$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \quad \text{(EoM2)}$$

Number-valued solutions to the K.G. equation

and $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$

Strategy:
* ensure hermiticity, HC, by construction
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Ansatz:

$$\boxed{\begin{aligned}\hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t)\end{aligned}}$$

(k need not be a "momentum"!)

- Here, we use the easy-to-construct operators that obey

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

- Try out the ansatz:
  * Hermiticity: ✓
    (HC) holds by construction.

* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{(EoM1)}$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

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Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad (\text{EoM1})$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad (\text{EoM2})$$

- Here, we use the easy-to-construct operators that obey

$$[a_k, a^+_{k'}] = \delta_{k,k'}$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, **EoM1**, i.e. to the Klein Gordon equation, called the ***Mode Functions***.

- Try out the ansatz:
  - * **Hermiticity:** ✓
    (HC) holds by construction.

Strategy:
* ensure hermiticity, **HC**, by construction
* separate the **CCR** and **EoM** problems:

Ansatz:

$$\boxed{\begin{aligned}\hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t)\end{aligned}}$$

(k need not be a "momentum"!)

* **The 1st equation of motion:** ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad (\text{EoM1})$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve **(EoM1)**.

* **The 2nd equation of motion:** ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

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Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{(EoM1)}$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \quad \text{(EoM2)}$$

Strategy: * ensure hermiticity, HC, by construction

* separate the CCR and EoM problems:

Ansatz:

$$\boxed{\begin{aligned}\hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t)\end{aligned}}$$

(k need not be a "momentum"!)

- Here, we use the easy-to-construct operators that obey

$$[a_k, a^\dagger_{k'}] = \delta_{k,k'}$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the Mode Functions.

- Try out the ansatz:

* Hermiticity: ✓

(HC) holds by construction.

* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{(EoM1)}$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

- Checking the CCRs:

Now check CCR:

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□ And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, **EoM1**, i.e. to the Klein Gordon equation, called the **Mode Functions**.

□ Try out the ansatz:

* Hermiticity: ✓

(HC) holds by construction.

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

□ **Checking the CCRs:**

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] \overset{!}{=} i\hbar\, \delta^3(x-x')$$

□ Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

□ Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

Now check CCR:

$$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$$

$$= \Big[\sum_k u_k(x',t)\, a_k + u_k^*(x',t)\, a_k^+ ,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \Big(\big(\tfrac{\partial}{\partial x^\nu} u_{k'}(x,t)\big) a_{k'} + \big(\tfrac{\partial}{\partial x^\nu} u_{k'}^*(x,t)\big) a_{k'}^+ \Big) \Big]$$

$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \Big(u_k(x',t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \Big) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum \Big(u_k(x',t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_k(x,t) \Big) \overset{!}{=} i\, \delta^3(\vec{x} - \vec{x}')$$

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$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \qquad (EoM2)$$

- Here, we use the easy-to-construct operators that obey

$$[a_k, a^+_{k'}] = \delta_{k,k'}$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the Mode Functions.

- Try out the ansatz:

  * Hermiticity: ✓

    (HC) holds by construction.

- Checking the CCRs:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] \overset{!}{=} i\hbar\, \delta^3(x-x')$$

  - Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum u_k(x,t)\, a_k + u^*_k(x,t)\, a^+_k$$

$$\boxed{\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)}$$

* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu} + m^2\right) \hat{\phi}(x,t) = 0 \qquad (EoM1)$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u^*_k(x,t)\, a^+_k$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

Now check CCR:

$$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$$

$$= \Big[\sum_k u_k(x',t)\, a_k + u^*_k(x',t)\, a^+_k,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \Big(\big(\tfrac{\partial}{\partial x^\nu} u_{k'}(x,t)\big) a_{k'} + \big(\tfrac{\partial}{\partial x^\nu} u^*_{k'}(x,t)\big) a^+_{k'} \Big) \Big]$$

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which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

□ Try out the ansatz:

* Hermiticity: ✓

(HC) holds by construction.

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

□ Checking the CCRs:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] \overset{!}{=} i\hbar\, \delta^3(x-x')$$

□ Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

□ Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

$$\Rightarrow \hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu} \sum_k \left[\left(\frac{\partial}{\partial x^\nu} u_k(x,t)\right) a_k + \left(\frac{\partial}{\partial x^\nu} u_k^*(x,t)\right) a_k^+ \right]$$

Now check CCR:

$$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$$

$$= \left[\sum_k u_k(x',t)\, a_k + u_k^*(x',t)\, a_k^+ ,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left(\left(\frac{\partial}{\partial x^\nu} u_{k'}(x,t)\right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t)\right) a_{k'}^+ \right) \right]$$

$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left(u_k(x',t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x',t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_k(x,t) \right) \overset{!}{=} i\, \delta^3(\vec{x} - \vec{x}')$$

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$$\hat\phi(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

- Express $\hat\pi$ in terms of the ansatz:

$$\hat\pi(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat\phi(x,t)$$

$$\Rightarrow \hat\pi(x,t) = \sqrt{|g|}\, g^{0\nu} \sum_k \left[\left(\frac{\partial}{\partial x^\nu} u_k(x,t) \right) a_k + \left(\frac{\partial}{\partial x^\nu} u_k^*(x,t) \right) a_k^+ \right]$$

$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left(u_k(x',t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x',t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_k(x,t) \right) \overset{!}{=} i\, \delta^3(\vec{x} - \vec{x}')$$

Conclusion so far:

Our ansatz

$$\hat\phi(x,t) := \sum u_m(x,t)\, a_m + u_m^*(x,t)\, a_m^+$$

solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{ u_m(x,t) \}$$

of the Klein Gordon equation that obeys:

$$\text{(W)} \quad \boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_k(x',t) \right) = i\, \delta^3(x - x')}$$

When do such $\{u_k(x,t)\}$ exist? I.e., when does the ansatz succeed?

Proposition:

- Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form: $\mathbb{R} \times \mathcal{M}$ ← any 3-dim differentiable manifold
- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
- Then, such a set of functions $\{u_k\}$ can be shown to exist.
- In fact there are many such sets $\{\tilde u_k\}$ obeying (W)! (And we will have to address which set to choose to solve the theory.)

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Conclusion so far:

Our ansatz

$$\hat{\phi}(x,t) := \sum_n u_n(x,t)\, a_n + u_n^*(x,t)\, a_n^+$$

solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

(W) $$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

Proof:

- Consider the vector space, V, of all real-valued solutions of the Klein Gordon equation.
- We define a bi-linear form $(\ ,\ )$ on V. For all $f, h \in V$:

$$(f,h) := \int d\Sigma_\nu \sqrt{g}\, g^{\mu\nu} (f\, \partial_\mu h - h\, \partial_\mu f)$$

When do such $\{u_n(x,t)\}$ exist? I.e., when does the ansatz succeed?

Proposition:

- Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form:

  $\mathbb{R} \times \mathcal{M}$

  ↑ any 3-dim differentiable manifold

- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
- Then, such a set of functions $\{u_n\}$ can be shown to exist.
- In fact there are many such sets $\{\tilde{u}_n\}$ obeying (W)!

  (And we will have to address which set to choose to solve the theory.)

- What can we do with $(\ ,\ )$? No diagonalization?

Theorem (Darboux):

For any nondegenerate symplectic form $(,)$, there exists a basis $\{v_n\}$ such that, in this basis, $(,)$ takes the matrix form:

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ & & \ddots \end{pmatrix}$$

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solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

(W) $$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_n \left(u_n(x,t) \frac{\partial}{\partial x'^\nu} u_n^*(x',t) - u_n^*(x',t) \frac{\partial}{\partial x^\nu} u_n(x',t) \right) = i\, \delta^3(x-x')}$$

$\mathbb{R} \times \mathcal{M}$

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- Consider the vector space, V, of all real-valued solutions of the Klein Gordon equation.
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- What can we do with $(\ ,\ )$? No diagonalization?

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For any nondegenerate symplectic form $(\ ,\ )$, there exists a basis $\{v_m\}$ such that, in this basis, $(\ ,\ )$ takes the matrix form:

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i.e., such that $(v_{2m}, v_{2m+1}) = 1$, $(v_{2m+1}, v_{2m}) = -1$
and all other pairings vanish.

- Thus, if we expand $a, b \in V$ as: $\underset{V}{a} = \underset{\mathbb{R}}{a_m}\, \underset{V}{v_m}$, $b = \underset{\mathbb{R}}{b_m}\, v_m$

Then: $(a,b) = \sum_{m=0}^{\infty} a_{2m} b_{2m+1} - a_{2m+1} b_{2m}$

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solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

$$\text{(W)} \qquad \boxed{\sqrt{|g|}\, g^{0\nu} \sum_n \left(u_n(x,t) \frac{\partial}{\partial x'^\nu} u_n^*(x',t) - u_n^*(x,t) \frac{\partial}{\partial x'^\nu} u_n(x',t) \right) = i\, \delta^3(x-x')}$$

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$\mathbb{R} \times \mathcal{M}$

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- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
- Then, such a set of functions $\{u_n\}$ can be shown to exist.
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Our ansatz

$$\hat\phi(x,t) := \sum_m u_m(x,t)\, a_m + u_m^*(x,t)\, a_m^+$$

solves the QFT, i.e., HC, EoM and CCR

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(W) $$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

Proposition: Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form:

$\mathbb{R} \times \mathcal{M}$ ← any 3-dim differentiable manifold

- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
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- Now assume we picked such a basis $\{v_m\}$ in V.
- Recall: V = space of real-valued solutions of K.G. eqn.
- Definition:

  $\bar{V}$:= space of complex-valued solutions of K.G. eqn.

- We easily find a basis of $\bar{V}$, namely $\{u_m\} \cup \{u_m^*\}$ where:

$$u_m := \frac{1}{\sqrt{2}}\left(v_{2m} + i v_{2m+1}\right), \quad u_m^* = \frac{1}{\sqrt{2}}\left(v_{2m} - i v_{2m+1}\right)$$

- What is a natural product $\langle \, , \, \rangle$ on $\bar{V}$?

- On $\bar{V}$ we define:

$$\langle f,h \rangle = i \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu}\left(f^*\partial_\nu h - (\partial_\nu f^*) h\right)$$

- Then, $(\, , \,)$ yields:

$\langle u_n, u_m \rangle = -\delta_{n,m}$, $\langle u_n^*, u_m^* \rangle = +\delta_{n,m}$, $\langle u_n, u_m^* \rangle = 0$ (I)

Exercise: verify this.

Thus, $\langle \, , \, \rangle$ is an indefinite inner product on $\bar{V}$: $\langle \, , \, \rangle = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$

Proposition: A resolution of the identity on $\bar{V}$ is given by:

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Remark: One can also turn $\overline{V}$ into a Hilbert space, namely the Krein space: Let P^+ and P^- be the projectors on the spaces