Title: Quantum Field Theory for Cosmology - Lecture 20240215

Speakers: Achim Kempf

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QFT for Cosmology, Achim Kempf, Lecture 12

We are now ready to 2nd quantize:

SR, 1st Q Hamiltonian formalism
—step 1: Legendre transform (equivalence)→ SR, 1st Q. Lagrangian formalism
step 2: allow curvature ↓
GR, 1st Q Hamiltonian formalism
step 4: ↓
GR, 2nd Q Hamiltonian formalism
←step 3: Legendre transform (equivalence)— GR, 1st Q Lagrangian formalism
↓
GR, 2nd Q Lagrangian formalism (Path integral of QFT)
Dyson Schwinger eqns are same (equivalence)

Solving the quantized theory is to solve:

1.) Commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$$
$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$
$$[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$$

(CCRs)

2.) Hermiticity:

$$\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^\dagger(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

which is needed so that the expectation values are real.

3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat{f}(x,\pi) = \frac{1}{i\hbar}[\hat{f}, \hat{H}] \qquad \text{for } \hat{f} = \hat{\phi},\ \hat{f} = \hat{\pi},\ \text{etc}$$

Namely:

How to solve the CCR, HC and EoM equations?

Recall: the solution we obtained on Minkowski space:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(e^{-i\omega_k t + ikx}\, \hat{a}_k + e^{i\omega_k t - ikx}\, \hat{a}_k^\dagger \right) d^3k$$

The $\hat{a}_k, \hat{a}_k^\dagger$ take care of the CCRs

Number-valued solutions to the K.G. equation

and $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$

Strategy: * ensure hermiticity, HC, by construction

* separate the CCR and EoM problems:

1 / 21

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Hamiltonian formalism ⟷ (equivalence) ⟷ Lagrangian formalism

step 2: allow curvature

GR, 1st Q Hamiltonian formalism ⟷ step 3: Legendre transform (equivalence) ⟷ GR, 1st Q Lagrangian formalism

step 4

GR, 2nd Q Hamiltonian formalism ⟷ Dyson Schwinger eqns are same (equivalence) ⟷ GR, 2nd Q Lagrangian formalism (Path integral of QFT)

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$

$$[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$$

} (CCRs)

2.) Hermiticity:

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Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\nu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\mu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad \text{(EoM2)}$$

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and $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$

Strategy:

* ensure hermiticity, HC, by construction
* separate the CCR and EoM problems:

Ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\,\hat{a}_k + u_k^*(x,t)\,\hat{a}_k^\dagger$$

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t)$$

(k need not be a "momentum"!)

1 / 21

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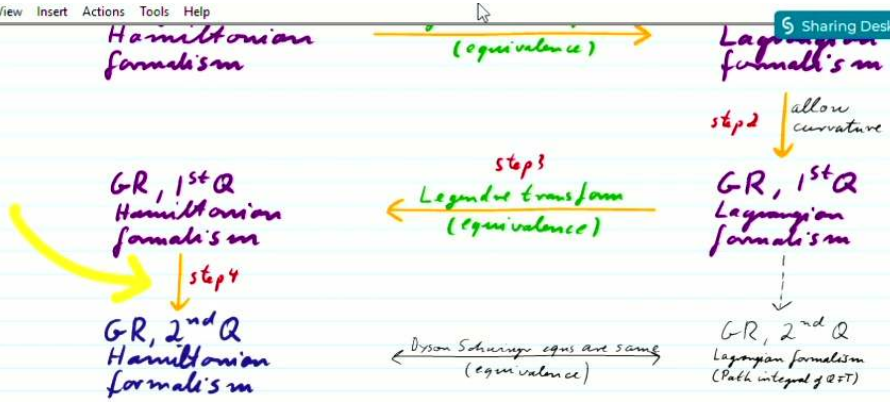

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(CCRs)

2.) **Hermiticity:**

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Namely:

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- Here, we use the easy-to-construct operators that obey

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}.$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

- Try out the ansatz:

  * Hermiticity: ✓

    (HC) holds by construction.

Number-valued solutions to the K.G. equation

and $\dot{\hat{\pi}}(x,t) = \ddot{\hat{\phi}}(x,t)$

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* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{(EoM1)}$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

3 / 21

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Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$

and:

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3 / 21

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* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

- Checking the CCRs:

Now check CCR:

3

3 / 21

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This equation holds by the way we define $\hat{\pi}(x,t)$.

- **Checking the CCRs:**

$$[\hat{\phi}(x,t),\, \hat{\pi}(x',t)] \overset{!}{=} i\,\delta^3(x-x')$$

- Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

- Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

Now check CCR:

$$[\hat{\phi}(x,t),\, \hat{\pi}(x',t)]$$

$$= \left[\sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger ,\; \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left(\left(\frac{\partial}{\partial x^\nu} u_{k'}(x,t)\right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t)\right) a_{k'}^\dagger \right) \right]$$

$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left(u_k(x,t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_k(x,t) \right) \overset{!}{=} i\,\delta^3(\vec{x}-\vec{x}')$$

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$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \qquad \text{(EoM2)}$$

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

- Here, we use the easy-to-construct operators that obey
$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}.$$
- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

- Try out the ansatz:
  - * Hermiticity: ✓
    (HC) holds by construction.

  - * The 1st equation of motion: ✓
$$\left(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} g^{\nu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu} + m^2\right) \hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$
    This eqn holds because in our ansatz,
$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$
    the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).
  - * The 2nd equation of motion: ✓
    This equation holds by the way we define $\hat{\pi}(x,t)$.

- **Checking the CCRs:**
$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] \overset{!}{=} i\hbar\, \delta^3(x-x')$$
  - Express $\hat{\phi}$ in terms of the ansatz:
$$\hat{\phi}(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

Now check CCR:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)]$$
$$= \left[\sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left(\frac{\partial}{\partial x^\nu} u_{k'}(x',t)\right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x',t)\right) a_{k'}^\dagger\right]$$

3 / 21

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which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

- Try out the ansatz:
  - * Hermiticity: ✓
    (HC) holds by construction.

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

- * The 2nd equation of motion: ✓
  This equation holds by the way we define $\hat{\pi}(x,t)$.

- **Checking the CCRs:**

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] \overset{!}{=} i\,\delta^3(x-x')$$

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$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

$$\Rightarrow \quad \hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu} \sum_k \left[\left(\frac{\partial}{\partial x^\nu} u_k(x,t)\right) a_k + \left(\frac{\partial}{\partial x^\nu} u_k^*(x,t)\right) a_k^\dagger \right]$$

Now check CCR:

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Conclusion so far:

Our ansatz

$$\hat{\phi}(x,t) := \sum_m u_m(x,t)\, a_m + u_m^*(x,t)\, a_m^\dagger$$

solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{ u_m(x,t) \}$$

of the Klein Gordon equation that obeys:

(W)
$$\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_k(x',t) \right) = i\, \delta^3(x - x')$$

When do such $\{ u_m(x,t) \}$ exist? I.e., when does the ansatz succeed?

Proposition:

- Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form:
  $\mathbb{R} \times \mathcal{M}$ ← any 3-dim differentiable manifold
- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
- Then, such a set of functions $\{u_k\}$ can be shown to exist.
- In fact there are many such sets $\{\tilde{u}_k\}$ obeying (W)!
  (And we will have to address which set to choose to solve the theory.)

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Conclusion so far:

Our ansatz

$$\hat{\phi}(x,t) := \sum_m u_m(x,t)\, a_m + u_m^*(x,t)\, a_m^\dagger$$

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When do such $\{u_m(x,t)\}$ exist? I.e., when does the ansatz succeed?

Proposition:

- Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form:

  $\mathbb{R} \times M$, M any 3-dim differentiable manifold

- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solutions everywhere.
- Then, such a set of functions $\{u_m\}$ can be shown to exist.
- In fact there are many such sets $\{\tilde{u}_m\}$ obeying (W)!

  (And we will have to address which set to choose to solve the theory.)

Proof:

- Consider the vector space, V, of all real-valued solutions of the Klein Gordon equation.
- We define a bi-linear form $(\ ,\ )$ on V. For all $f, h \in V$:

$$(f,h) := \int d\Sigma_x \sqrt{g}\, n^\mu (f \partial_\mu h - h \partial_\mu f)$$

- What can we do with $(\ ,\ )$? No diagonalization?

Theorem (Darboux):

For any non-degenerate symplectic form $(\ ,\ )$, there exists a basis $\{v_m\}$ such that, in this basis, $(\ ,\ )$ takes the matrix form:

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

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(W)
$$\sqrt{|g|}\, g^{0\nu} \sum_n \left(u_n(x,t) \frac{\partial}{\partial x^\nu} u_n^*(x',t) - u_n^*(x,t) \frac{\partial}{\partial x^\nu} u_n(x',t) \right) = i\,\delta^3(x-x')$$

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$$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{|g|}\, g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f)$$

↳ any spacelike hypersurface i.e. set of points of equal time.

- Proposition: (f,h) is independent of choice of Σ.
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9

$\mathbb{R} \times \mathcal{M}$

↳ any 3-dim differentiable manifold

- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solutions everywhere.
- Then, such a set of functions $\{u_n\}$ can be shown to exist.
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For any nondegenerate symplectic form $(\ ,\ )$, there exists a basis $\{v_m\}$ such that, in this basis, $(\ ,\ )$ takes the matrix form:

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9/21

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solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

(W)
$$\sqrt{|g|}\, g^{0\nu} \sum_n \left(u_n(x,t) \frac{\partial}{\partial x^\nu} u_n^*(x',t) - u_n^*(x,t) \frac{\partial}{\partial x^\nu} u_n(x',t) \right) = i\,\delta^3(x-x')$$

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9 / 21

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Our ansatz

$$\hat{\phi}(x,t) := \sum u_n(x,t)\, a_n + u_n^*(x,t)\, a_n^\dagger$$

solves the QFT, i.e., HC, EoM and CCR

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(W) $$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

Proposition:

- Assume spacetime is "globally hyperbolic", i.e., that it possesses a foliation by Cauchy surfaces, i.e., that it is topologically of the form: $\mathbb{R} \times \mathcal{M}$ ← any 3-dim differentiable manifold
- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.
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V $\mathbb{R}$ V $\mathbb{R}$

9 / 21

9

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$\overline{V}$:= space of complex-valued solutions of K.G. eqn.

- We easily find a basis of $\overline{V}$, namely $\{u_n\} \cup \{u_n^*\}$ where:

$$u_n := \frac{1}{\sqrt{2}} (v_{2n} + i v_{2n+1}), \quad u_n^* = \frac{1}{\sqrt{2}} (v_{2n} - i v_{2n+1})$$

- What is a natural product $\langle\ ,\ \rangle$ on $\overline{V}$?

- On $\overline{V}$ we define:

$$\langle f, h \rangle = i \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} (f^* \partial_\nu h - (\partial_\nu f^*) h)$$

- Then, $(\ ,\ )$ yields:

$$\langle u_n, u_m \rangle = -\delta_{n,m}, \quad \langle u_n^*, u_m^* \rangle = +\delta_{n,m}, \quad \langle u_n, u_m^* \rangle = 0 \qquad \text{(I)}$$

Exercise: verify this.

Thus, $\langle\ ,\ \rangle$ is an indefinite inner product on $\overline{V}$: $\langle\ ,\ \rangle = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & +1 & \\ & & & \ddots \end{pmatrix}$

Proposition: A resolution of the identity on $\overline{V}$ is given by:

$$\mathbb{1} = \sum -|u_n\rangle\langle u_n| + |u_n^*\rangle\langle u_n^*|$$

11 / 21

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Remark. One can also turn $\overline{V}$ into a Hilbert space, namely the Krein space: Let P^+ and P^- be the projectors on the spaces