Title: Quantum Field Theory for Cosmology - Lecture 20240208

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

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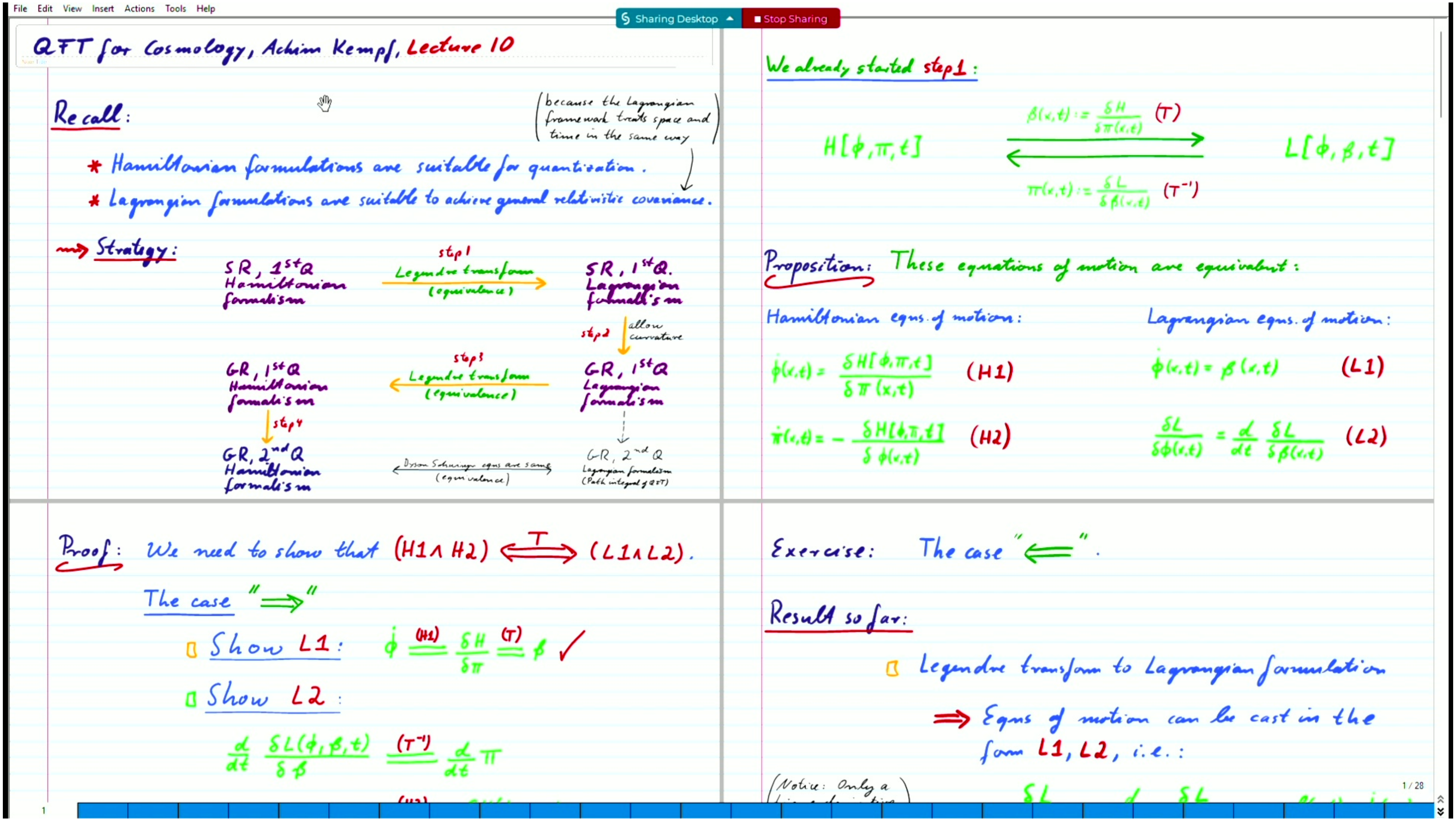


QFT for Cosmology, Achim Kempf, Lecture 10

Recall:

(because the Lagrangian framework treats space and time in the same way)

- * Hamiltonian formulations are suitable for quantization.
- * Lagrangian formulations are suitable to achieve general relativistic covariance.

→ Strategy:

$$\begin{tikzcd}
\begin{array}{c}\text{SR, }1^{st}\text{Q}\\ \text{Hamiltonian}\\ \text{formalism}\end{array} \arrow[r, "\text{step 1}\ \text{Legendre transform}", "\text{(equivalence)}"'] & \begin{array}{c}\text{SR, }1^{st}\text{Q.}\\ \text{Lagrangian}\\ \text{formalism}\end{array} \arrow[d, "\text{allow curvature}", "\text{step 2}"'] \\
\begin{array}{c}\text{GR, }1^{st}\text{Q}\\ \text{Hamiltonian}\\ \text{formalism}\end{array} \arrow[d, "\text{step 4}"] & \begin{array}{c}\text{GR, }1^{st}\text{Q}\\ \text{Lagrangian}\\ \text{formalism}\end{array} \arrow[l, "\text{step 3}\ \text{Legendre transform}"', "\text{(equivalence)}"] \arrow[d, dashed] \\
\begin{array}{c}\text{GR, }2^{nd}\text{Q}\\ \text{Hamiltonian}\\ \text{formalism}\end{array} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \begin{array}{c}\text{GR, }2^{nd}\text{Q}\\ \text{Lagrangian formalism}\\ \text{(Path integral of QFT)}\end{array}
\end{tikzcd}$$

Proof: We need to show that $(H1 \wedge H2) \xLeftrightarrow{T} (L1 \wedge L2)$.

The case "$\Longrightarrow$"

- ☐ Show L1: $\dot\phi \overset{(H1)}{=} \frac{\delta H}{\delta \pi} \overset{(T)}{=} \beta$ ✓
- ☐ Show L2:

$$\frac{d}{dt} \frac{\delta L(\phi,\beta,t)}{\delta \beta} \overset{(T^{-1})}{=} \frac{d}{dt}\pi$$

We already started step 1:

$$\begin{tikzcd}
H[\phi,\pi,t] \arrow[r, shift left, "\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)}\ (T)"] & L[\phi,\beta,t] \arrow[l, shift left, "\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)}\ (T^{-1})"]
\end{tikzcd}$$

Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:	Lagrangian eqns. of motion:
$\dot\phi(x,t) = \frac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)}$ (H1)	$\dot\phi(x,t) = \beta(x,t)$ (L1)
$\dot\pi(x,t) = -\frac{\delta H[\phi,\pi,t]}{\delta \phi(x,t)}$ (H2)	$\frac{\delta L}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta\beta(x,t)}$ (L2)

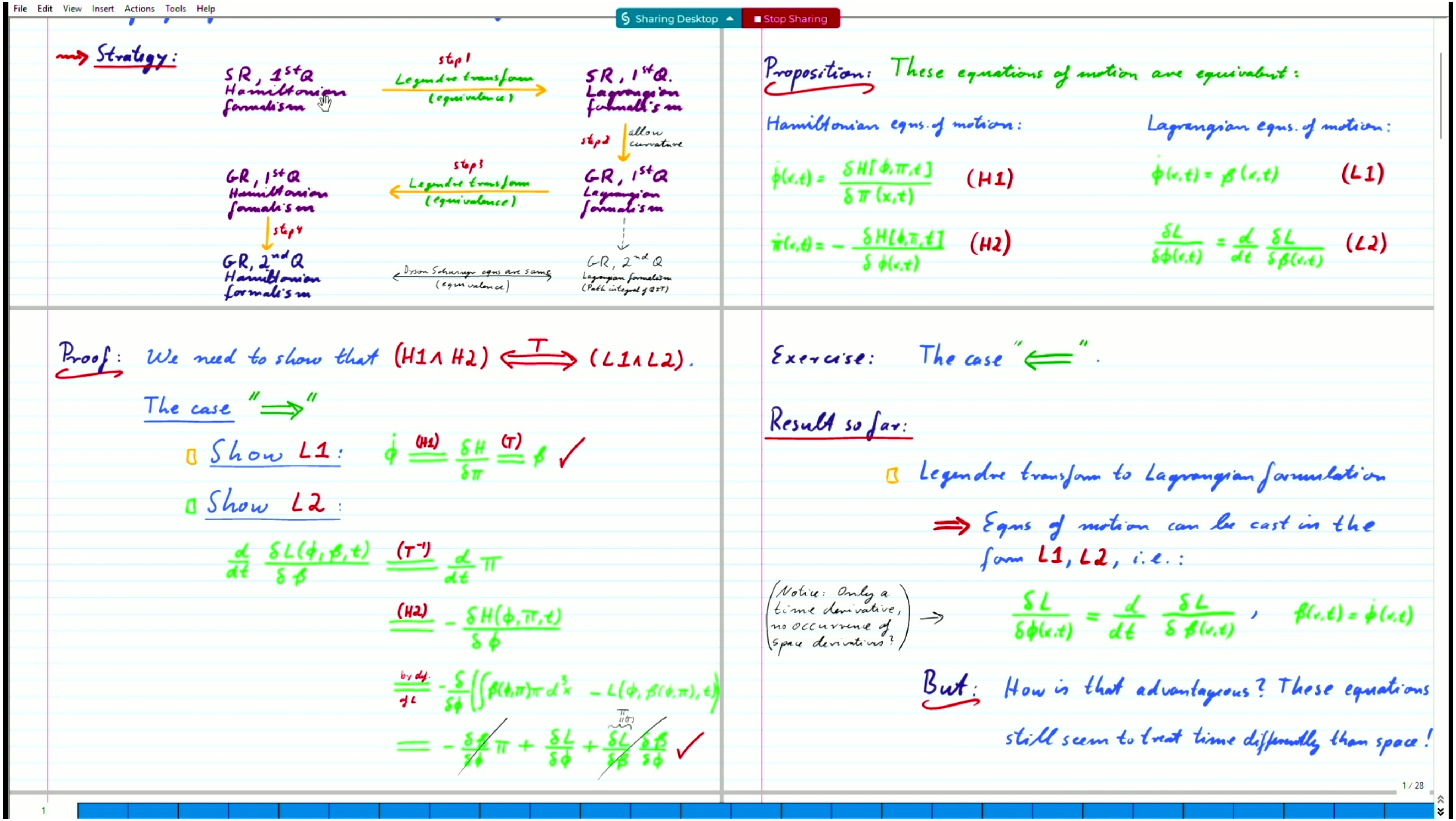
Exercise: The case "$\Longleftarrow$".

Result so far:

☐ Legendre transform to Lagrangian formulation

$\Longrightarrow$ Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a



Strategy:

$$\begin{tikzcd}
\text{SR, 1}^{\text{st}}\text{Q Hamiltonian formalism} \arrow[r, "\text{step 1 Legendre transform}", "\text{(equivalence)}"'] & \text{SR, 1}^{\text{st}}\text{Q. Lagrangian formalism} \arrow[d, "\text{allow curvature}", "\text{step 2}"'] \\
\text{GR, 1}^{\text{st}}\text{Q Hamiltonian formalism} \arrow[d, "\text{step 4}"] & \text{GR, 1}^{\text{st}}\text{Q Lagrangian formalism} \arrow[l, "\text{step 3 Legendre transform}"', "\text{(equivalence)}"] \arrow[d, dashed] \\
\text{GR, 2}^{\text{nd}}\text{Q Hamiltonian formalism} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \text{GR, 2}^{\text{nd}}\text{Q Lagrangian formalism (Path integral of QFT)}
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Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:

$$\dot{\phi}(x,t) = \frac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)} \quad (H1)$$

$$\dot{\pi}(x,t) = -\frac{\delta H[\phi,\pi,t]}{\delta \phi(x,t)} \quad (H2)$$

Lagrangian eqns. of motion:

$$\dot{\phi}(x,t) = \beta(x,t) \quad (L1)$$

$$\frac{\delta L}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta \beta(x,t)} \quad (L2)$$

Proof: We need to show that $(H1 \wedge H2) \xLeftrightarrow{T} (L1 \wedge L2)$.

The case "$\Longrightarrow$"

- ☐ Show L1: $\dot{\phi} \overset{(H1)}{=\!=} \frac{\delta H}{\delta \pi} \overset{(T)}{=\!=} \beta$ ✓
- ☐ Show L2:

$$\frac{d}{dt}\frac{\delta L(\phi,\beta,t)}{\delta\beta} \overset{(T^{-1})}{=\!=} \frac{d}{dt}\pi$$

$$\overset{(H2)}{=\!=} -\frac{\delta H(\phi,\pi,t)}{\delta\phi}$$

$$\overset{\text{by def.}}{\underset{\text{of } L}{=\!=}} -\frac{\delta}{\delta\phi}\left(\int \beta(\phi,\pi)\pi\, d^3x - L(\phi,\beta(\phi,\pi),t)\right)$$

$$= -\cancel{\frac{\delta\beta}{\delta\phi}}\pi + \frac{\delta L}{\delta\phi} + \cancel{\overbrace{\frac{\delta L}{\delta\beta}}^{\pi \overset{(T)}{=}}\frac{\delta\beta}{\delta\phi}} \checkmark$$

Exercise: The case "$\Longleftarrow$".

Result so far:

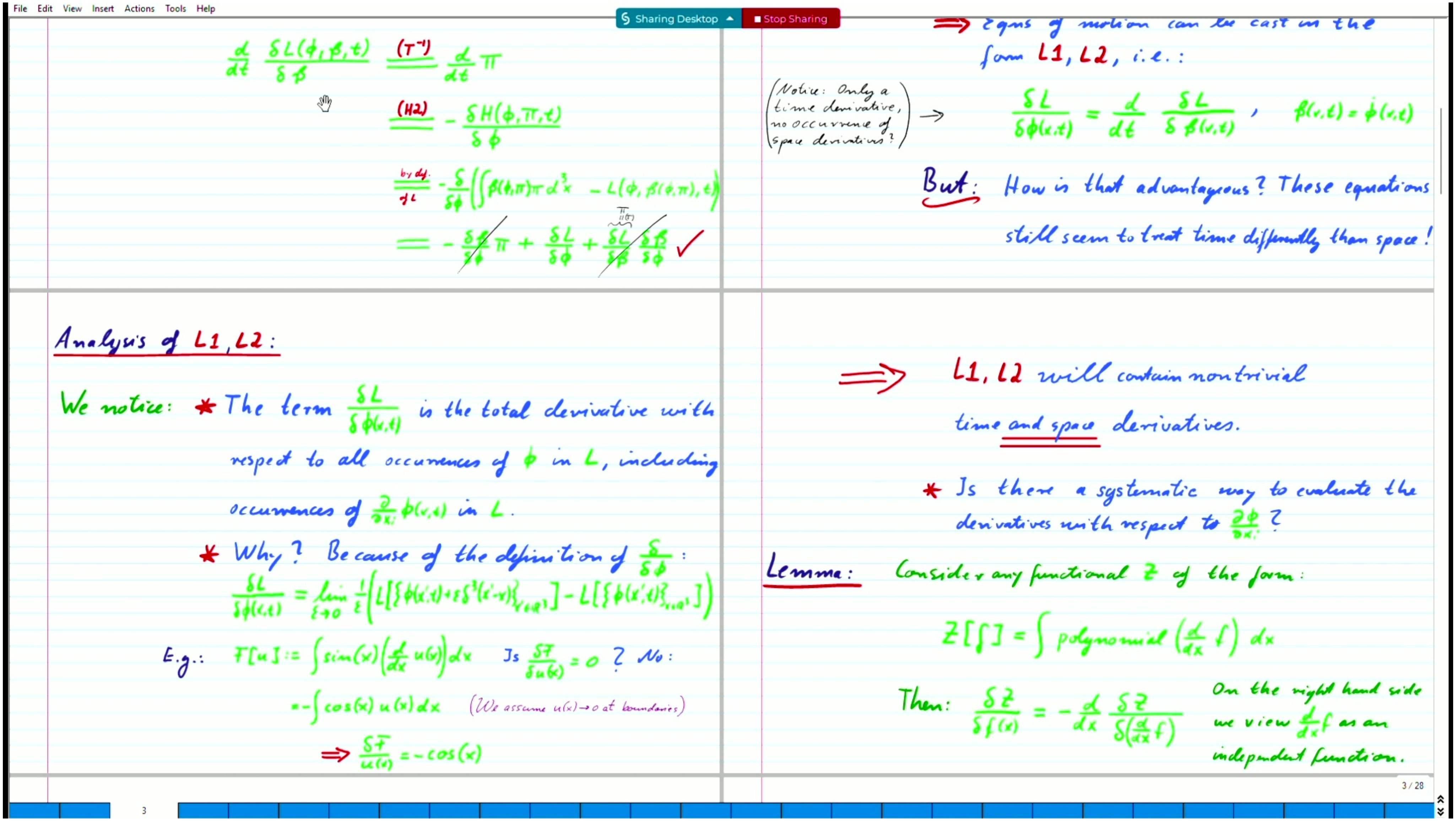
- ☐ Legendre transform to Lagrangian formulation

$\Longrightarrow$ Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a time derivative, no occurrence of space derivatives?) →

$$\frac{\delta L}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta\beta(x,t)}, \quad \beta(x,t) = \dot{\phi}(x,t)$$

But: How is that advantageous? These equations still seem to treat time differently than space!



$$\frac{d}{dt}\frac{\delta L(\phi,\beta,t)}{\delta\beta} \overset{(T^{-1})}{=\!=} \frac{d}{dt}\pi$$

$$\overset{(H2)}{=\!=} -\frac{\delta H(\phi,\pi,t)}{\delta\phi}$$

$$\underset{\text{of } L}{\overset{\text{by def.}}{=\!=}} -\frac{\delta}{\delta\phi}\left(\int\beta(\phi,\pi)\pi\, d^3x \;-\; L(\phi,\beta(\phi,\pi),t)\right)$$

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But: How is that advantageous? These equations still seem to treat time differently than space!

Analysis of L1, L2:

We notice: ✱ The term $\frac{\delta L}{\delta\phi(x,t)}$ is the total derivative with respect to all occurences of ϕ in L, including occurences of $\frac{\partial}{\partial x_i}\phi(x,t)$ in L.

✱ Why? Because of the definition of $\frac{\delta}{\delta\phi}$:

$$\frac{\delta L}{\delta\phi(x,t)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(L\left[\{\phi(x',t)+\varepsilon\delta^3(x'-x)\}_{x'\in\mathbb{R}^3}\right] - L\left[\{\phi(x',t)\}_{x'\in\mathbb{R}^3}\right]\right)$$

E.g.: $F[u] := \int \sin(x)\left(\frac{d}{dx}u(x)\right)dx$ Is $\frac{\delta F}{\delta u(x)} = 0$? No:

$$= -\int \cos(x)\, u(x)\, dx$$ (We assume $u(x)\to 0$ at boundaries)

$$\Rightarrow \frac{\delta F}{u(x)} = -\cos(x)$$

⟹ L1, L2 will contain nontrivial time and space derivatives.

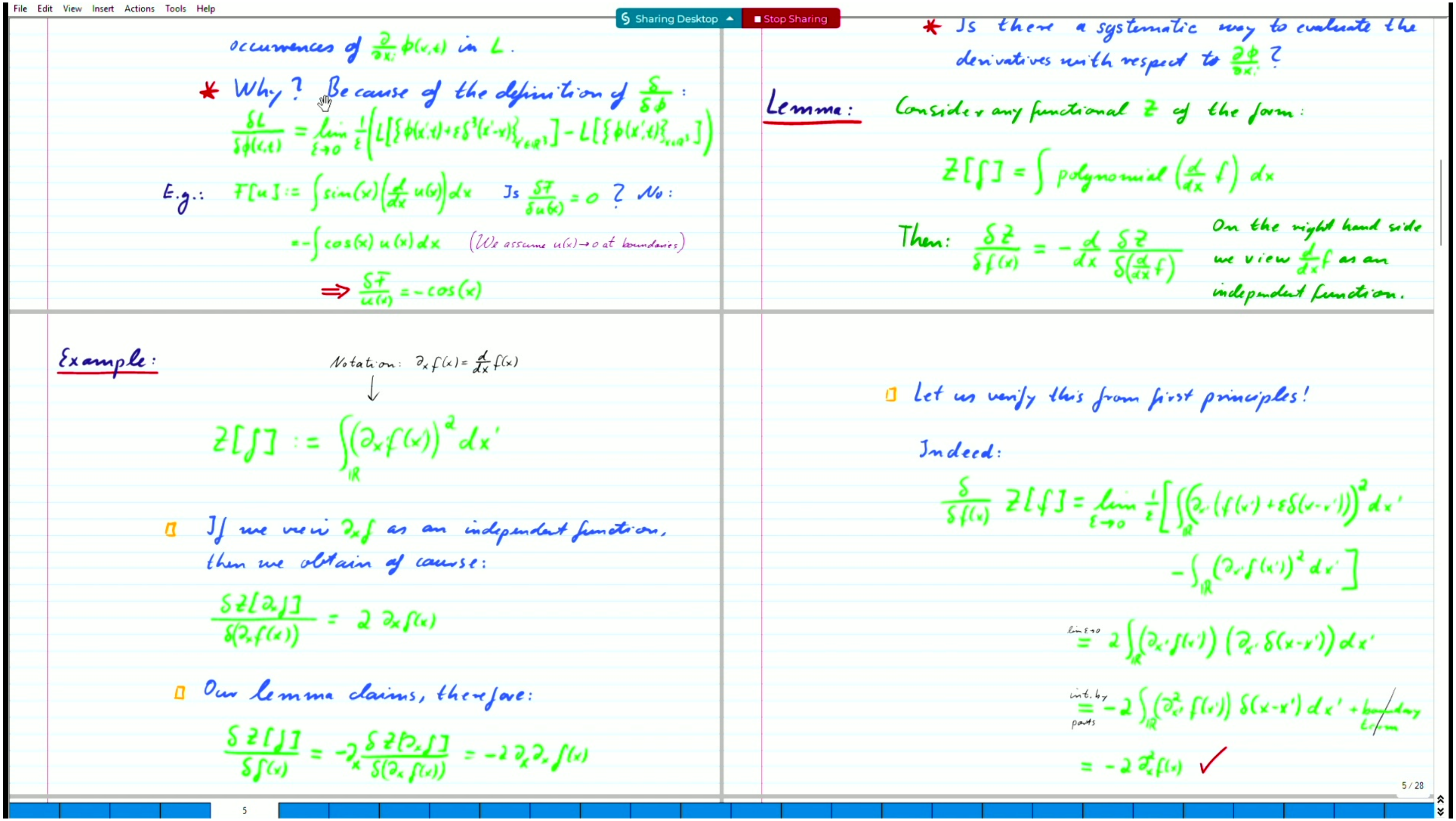
✱ Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial\phi}{\partial x_i}$?

Lemma: Consider any functional Z of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx}f\right)dx$$

Then: $$\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx}\frac{\delta Z}{\delta\left(\frac{d}{dx}f\right)}$$

On the right hand side we view $\frac{d}{dx}f$ as an independent function.



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On the right hand side we view $\frac{d}{dx}f$ as an independent function.

Example:

Notation: $\partial_x f(x) = \frac{d}{dx}f(x)$

$$Z[f] := \int_{\mathbb{R}}\left(\partial_{x'}f(x')\right)^2 dx'$$

- If we view $\partial_x f$ as an independent function, then we obtain of course:

$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2\,\partial_x f(x)$$

- Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -\partial_x\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2\,\partial_x\partial_x f(x)$$

- Let us verify this from first principles!

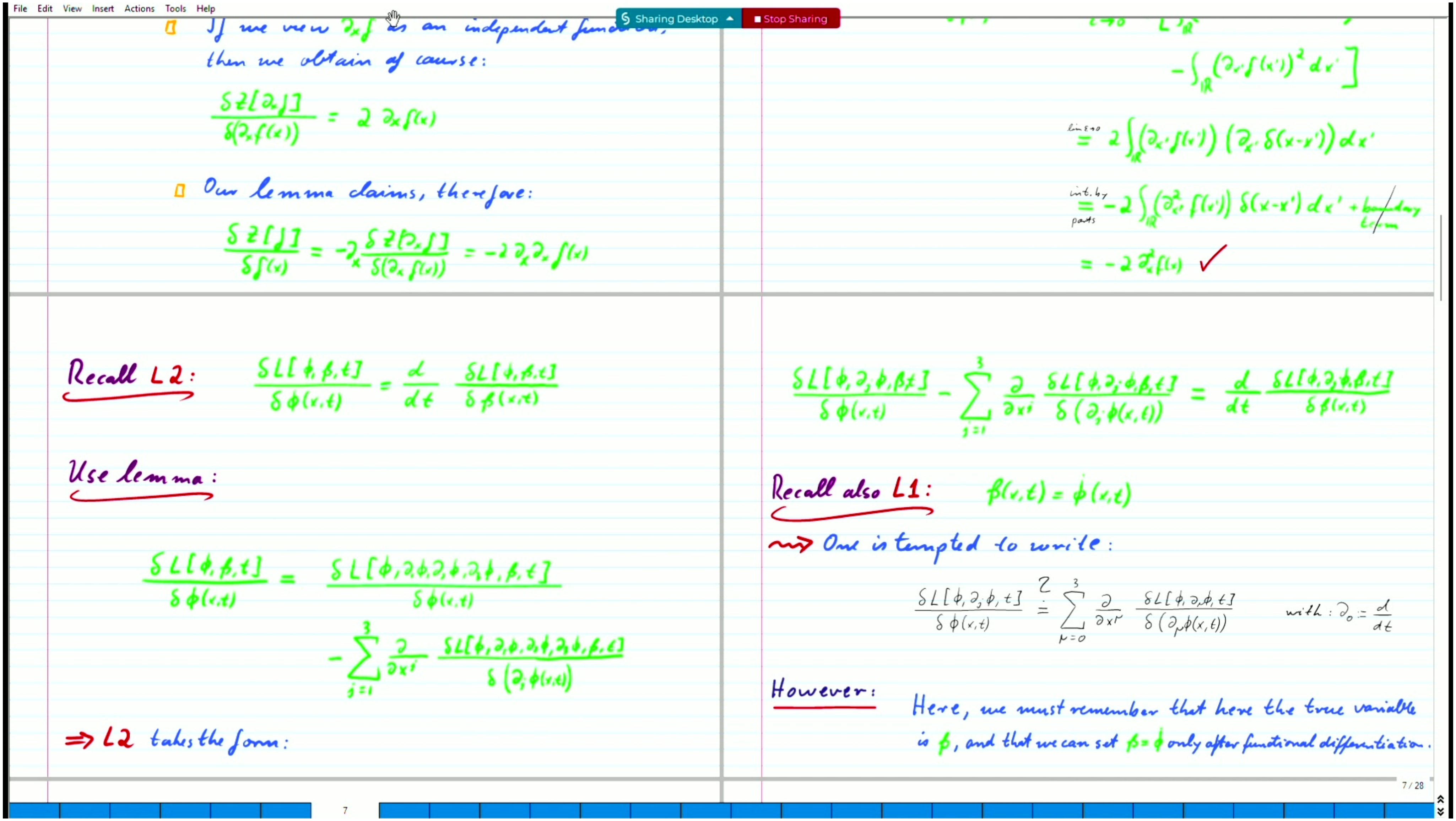
Indeed:

$$\frac{\delta}{\delta f(x)}Z[f] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{\mathbb{R}}\left(\partial_{x'}\left(f(x')+\varepsilon\delta(x-x')\right)\right)^2 dx' - \int_{\mathbb{R}}\left(\partial_{x'}f(x')\right)^2 dx'\right]$$

$$\overset{\lim\varepsilon\to 0}{=} 2\int_{\mathbb{R}}\left(\partial_{x'}f(x')\right)\left(\partial_{x'}\delta(x-x')\right)dx'$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -2\int_{\mathbb{R}}\left(\partial_{x'}^2 f(x')\right)\delta(x-x')\,dx' + \cancel{\text{boundary term}}$$

$$= -2\,\partial_x^2 f(x) \;\checkmark$$



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$$-\int_{\mathbb{R}} (\partial_{x'} f(x'))^2 \, dx' \Big]$$

$$\overset{\lim \varepsilon \to 0}{=} 2 \int_{\mathbb{R}} (\partial_{x'} f(x'))\,(\partial_{x'} \delta(x-x'))\, dx'$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -2 \int_{\mathbb{R}} (\partial^2_{x'} f(x'))\, \delta(x-x')\, dx' + \cancel{\text{boundary term}}$$

$$= -2\,\partial^2_x f(x) \quad \checkmark$$

Recall L2:

$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L[\phi, \beta, t]}{\delta \beta(x,t)}$$

Use lemma:

$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x,t)} = \frac{\delta L[\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi, \beta, t]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi, \beta, t]}{\delta(\partial_j \phi(x,t))}$$

⇒ L2 takes the form:

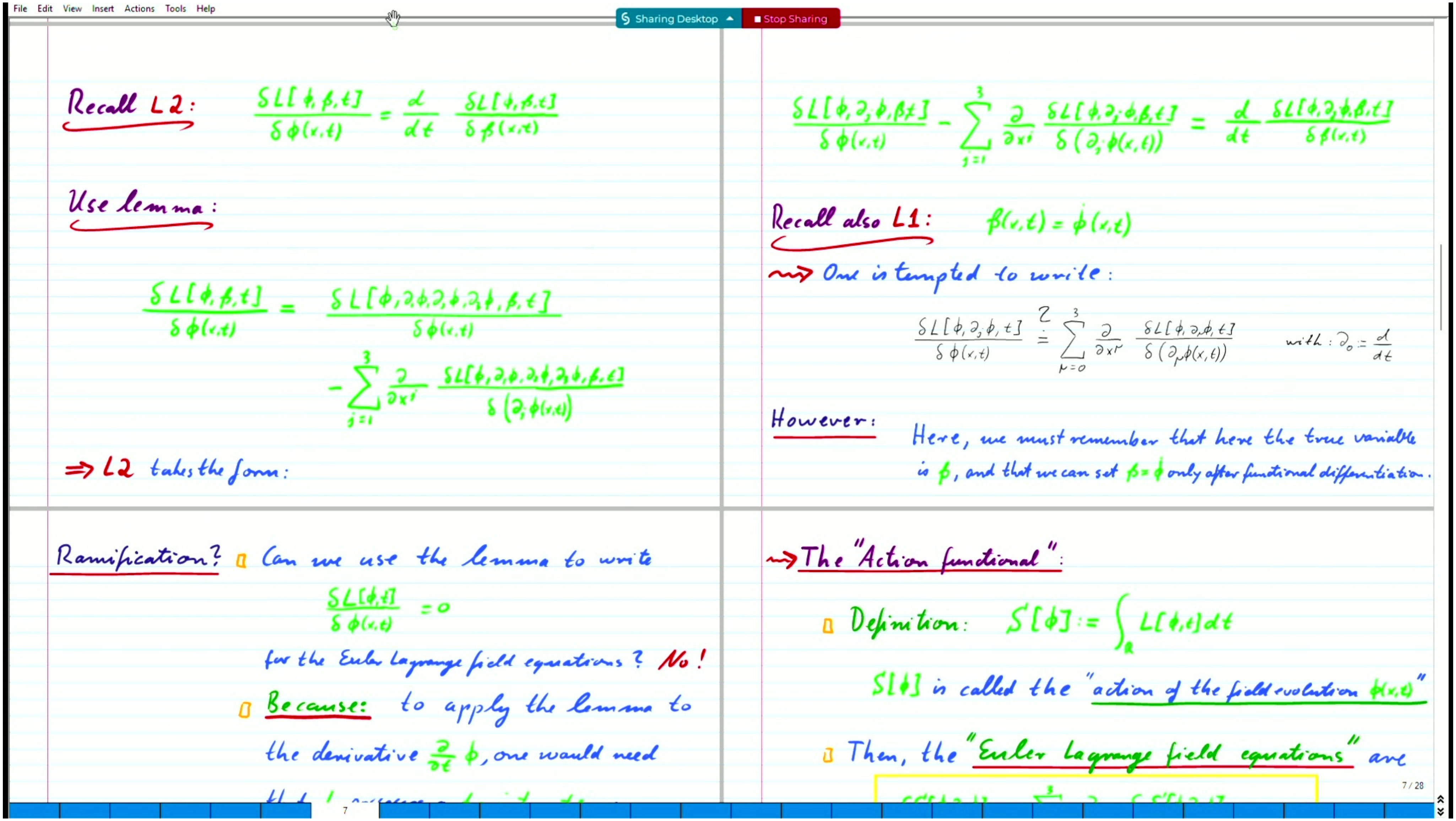
$$\frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta(\partial_j \phi(x,t))} = \frac{d}{dt} \frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta \beta(x,t)}$$

Recall also L1: $\beta(x,t) = \dot{\phi}(x,t)$

⇝ One is tempted to write:

$$\frac{\delta L[\phi, \partial_j\phi, t]}{\delta \phi(x,t)} \overset{?}{=} \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta L[\phi, \partial_\mu\phi, t]}{\delta(\partial_\mu \phi(x,t))} \qquad \text{with}: \partial_0 := \frac{d}{dt}$$

However: Here, we must remember that here the true variable is ϕ, and that we can set $\beta = \dot{\phi}$ only after functional differentiation.



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$\Rightarrow$ L2 takes the form:

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Ramification? ☐ Can we use the lemma to write

$$\frac{\delta L[\phi,t]}{\delta\phi(x,t)} = 0$$

for the Euler Lagrange field equations? **No!**

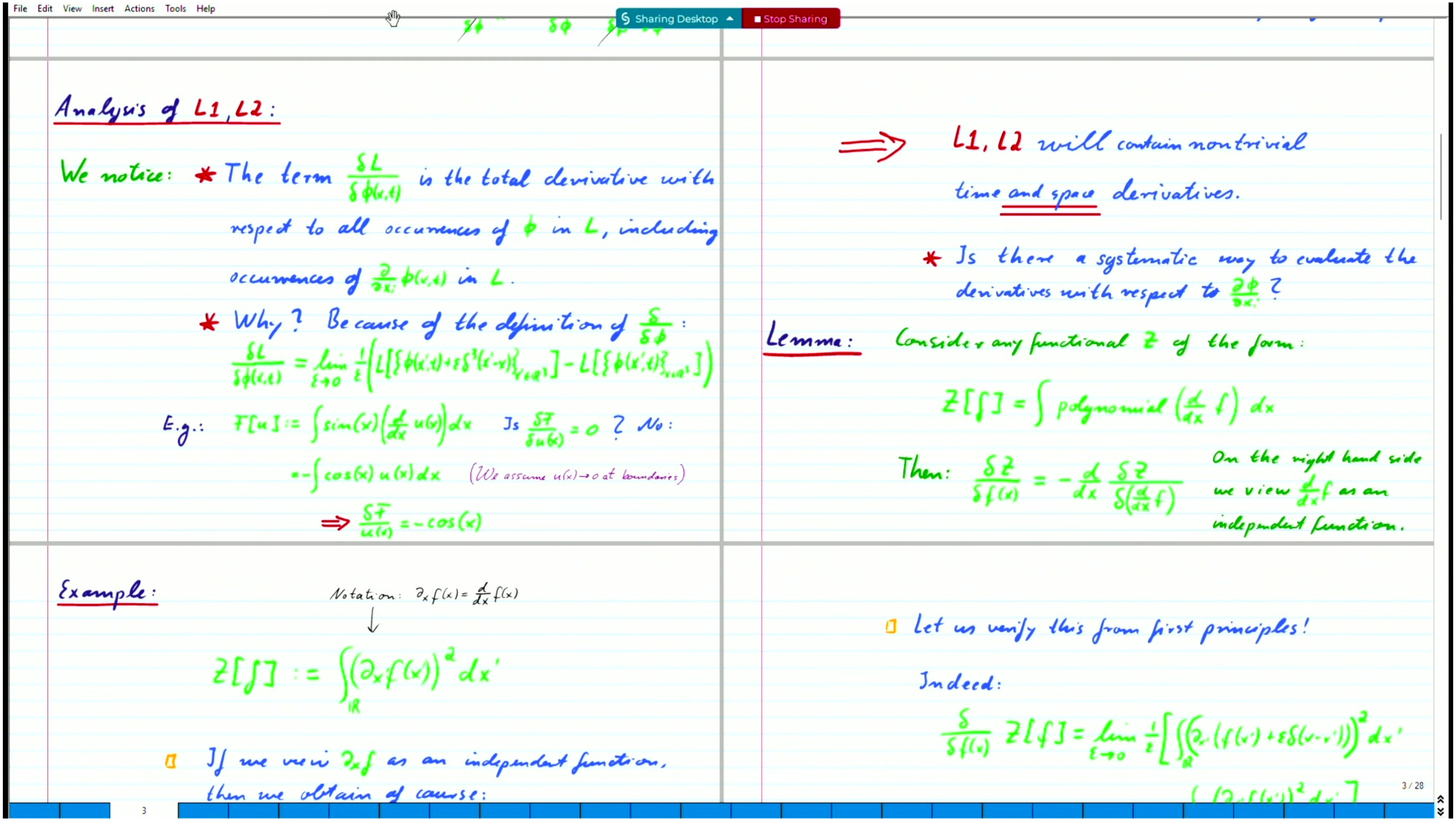
☐ **Because:** to apply the lemma to the derivative $\frac{\partial}{\partial t}\phi$, one would need

$\rightsquigarrow$ **The "Action functional":**

☐ Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi,t]\,dt$

$S[\phi]$ is called the "action of the field evolution $\phi(x,t)$"

☐ Then, the "Euler Lagrange field equations" are



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$\Longrightarrow$ L1, L2 will contain nontrivial time and space derivatives.

* Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial \phi}{\partial x_i}$?

Lemma: Consider any functional Z of the form:

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On the right hand side we view $\frac{d}{dx}f$ as an independent function.

Example:

Notation: $\partial_x f(x) = \frac{d}{dx}f(x)$

$$\downarrow$$

$$Z[f] := \int_{\mathbb{R}} \left(\partial_x f(x)\right)^2 dx'$$

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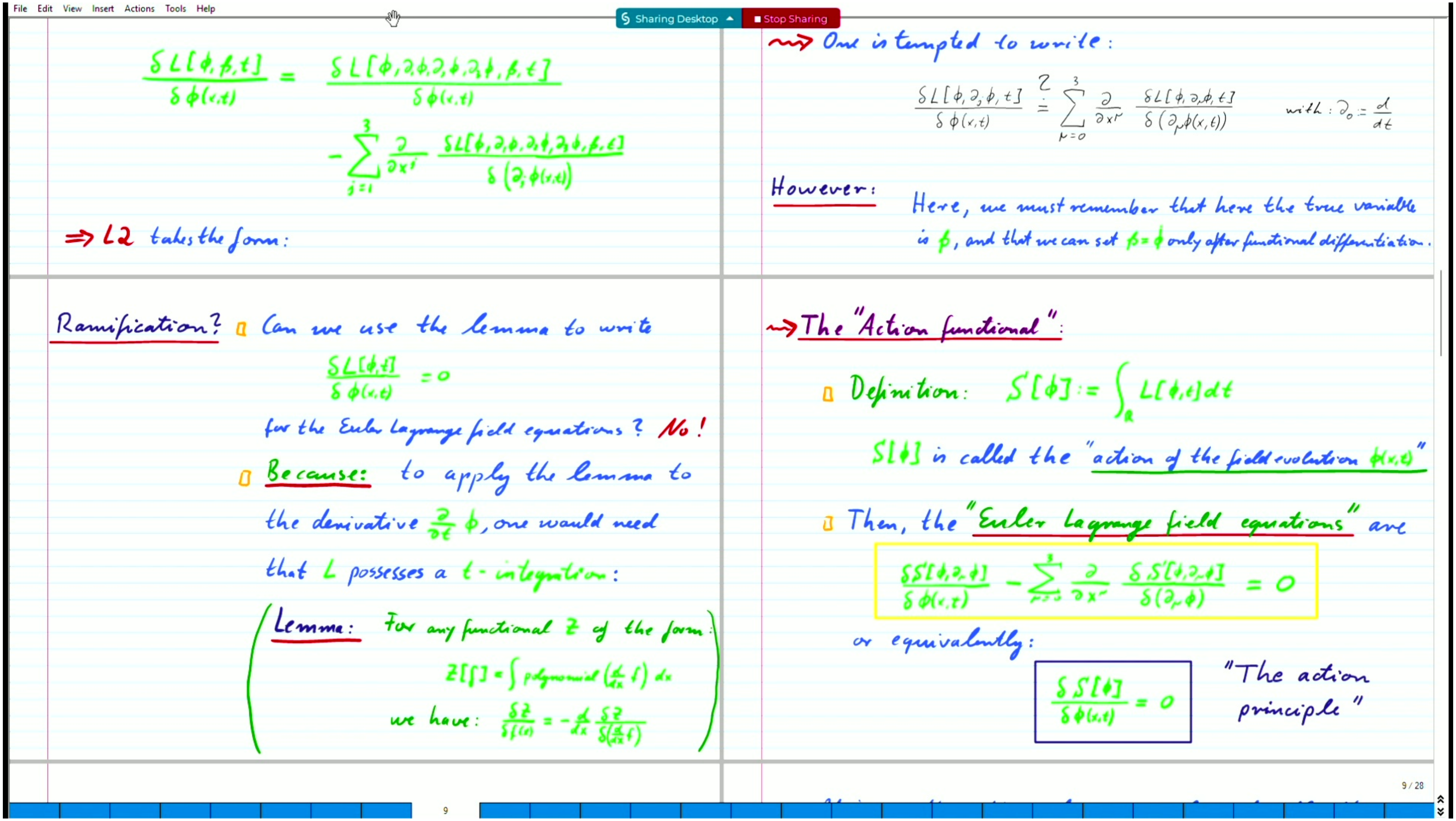

$$\frac{\delta L[\phi,\beta,t]}{\delta\phi(x,t)} = \frac{\delta L[\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi,\beta,t]}{\delta\phi(x,t)}$$

$$-\sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\delta L[\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi,\beta,t]}{\delta(\partial_j\phi(x,t))}$$

⇒ L2 takes the form:

Ramification? ▯ Can we use the lemma to write

$$\frac{\delta L[\phi,t]}{\delta\phi(x,t)} = 0$$

for the Euler Lagrange field equations? No!

▯ **Because:** to apply the lemma to the derivative $\frac{\partial}{\partial t}\phi$, one would need that L possesses a t-integration:

$$\left(\begin{array}{l}\textbf{Lemma:}\text{ For any functional } Z \text{ of the form:} \\ \qquad Z[f] = \int \text{polynomial}\left(\frac{d}{dx}f\right)dx \\ \quad \text{we have: } \frac{\delta Z}{\delta f(x)} = -\frac{d}{dx}\frac{\delta Z}{\delta\left(\frac{d}{dx}f\right)}\end{array}\right)$$

⇝ One is tempted to write:

$$\frac{\delta L[\phi,\partial_j\phi,t]}{\delta\phi(x,t)} \overset{?}{=} \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\delta L[\phi,\partial_\mu\phi,t]}{\delta(\partial_\mu\phi(x,t))} \qquad \text{with}: \partial_0 := \frac{d}{dt}$$

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▯ Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi,t]\,dt$

$S[\phi]$ is called the "action of the field evolution $\phi(x,t)$"

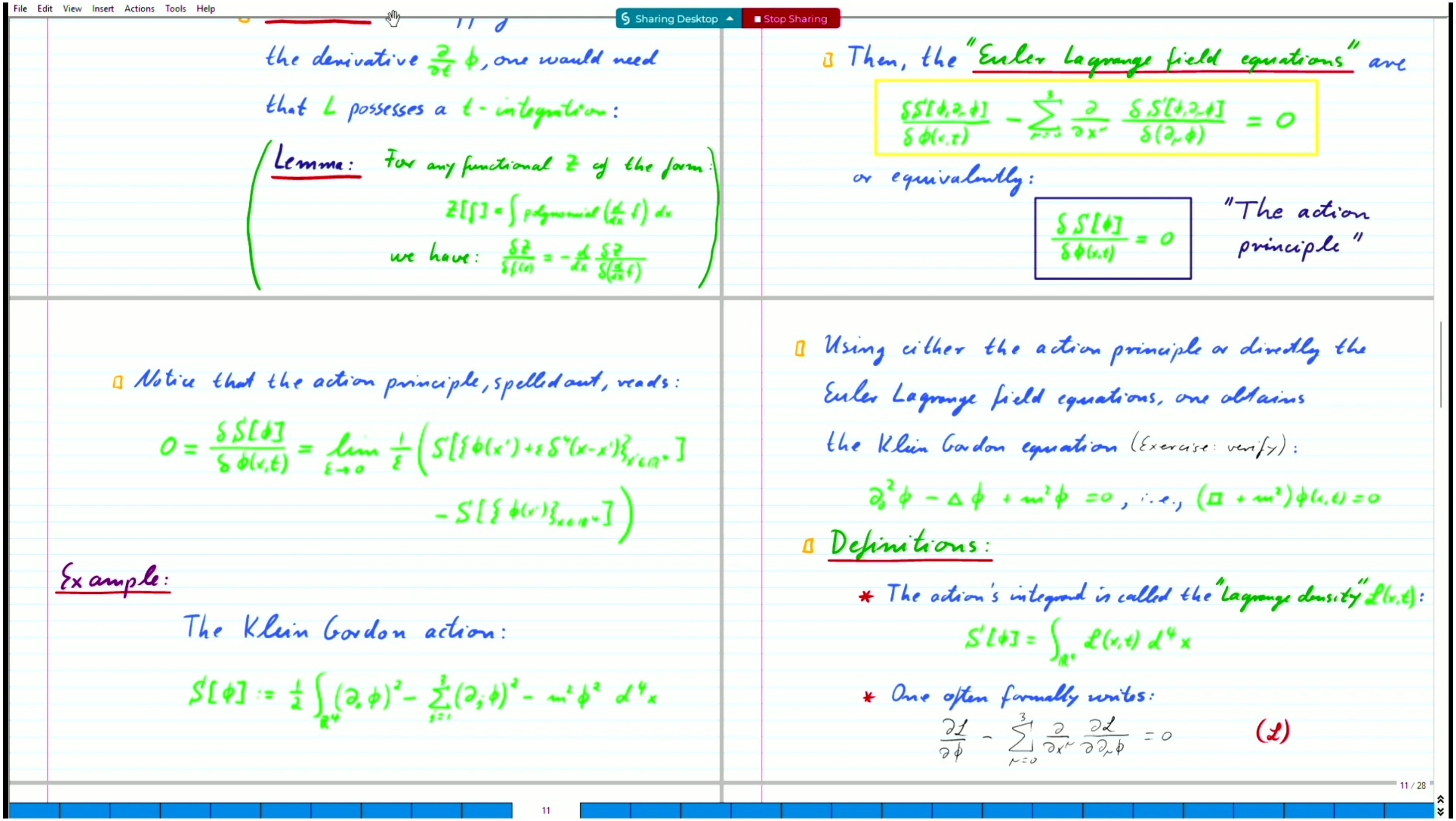
▯ Then, the "Euler Lagrange field equations" are

$$\boxed{\frac{\delta S[\phi,\partial_\mu\phi]}{\delta\phi(x,t)} - \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\delta S[\phi,\partial_\mu\phi]}{\delta(\partial_\mu\phi)} = 0}$$

or equivalently:

$$\boxed{\frac{\delta S[\phi]}{\delta\phi(x,t)} = 0}$$

"The action principle"



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- Then, the "**Euler Lagrange field equations**" are

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or equivalently:

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"The action principle"

- Notice that the action principle, spelled out, reads:

$$0 = \frac{\delta S[\phi]}{\delta\phi(x,t)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(S[\{\phi(x') + \varepsilon\,\delta^4(x-x')\}_{x'\in\mathbb{R}^4}] - S[\{\phi(x')\}_{x\in\mathbb{R}^4}]\Big)$$

Example:

The Klein Gordon action:

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}(\partial_0\phi)^2 - \sum_{j=1}^{3}(\partial_j\phi)^2 - m^2\phi^2\, d^4x$$

- Using either the action principle or directly the Euler Lagrange field equations, one obtains the Klein Gordon equation (Exercise: verify):

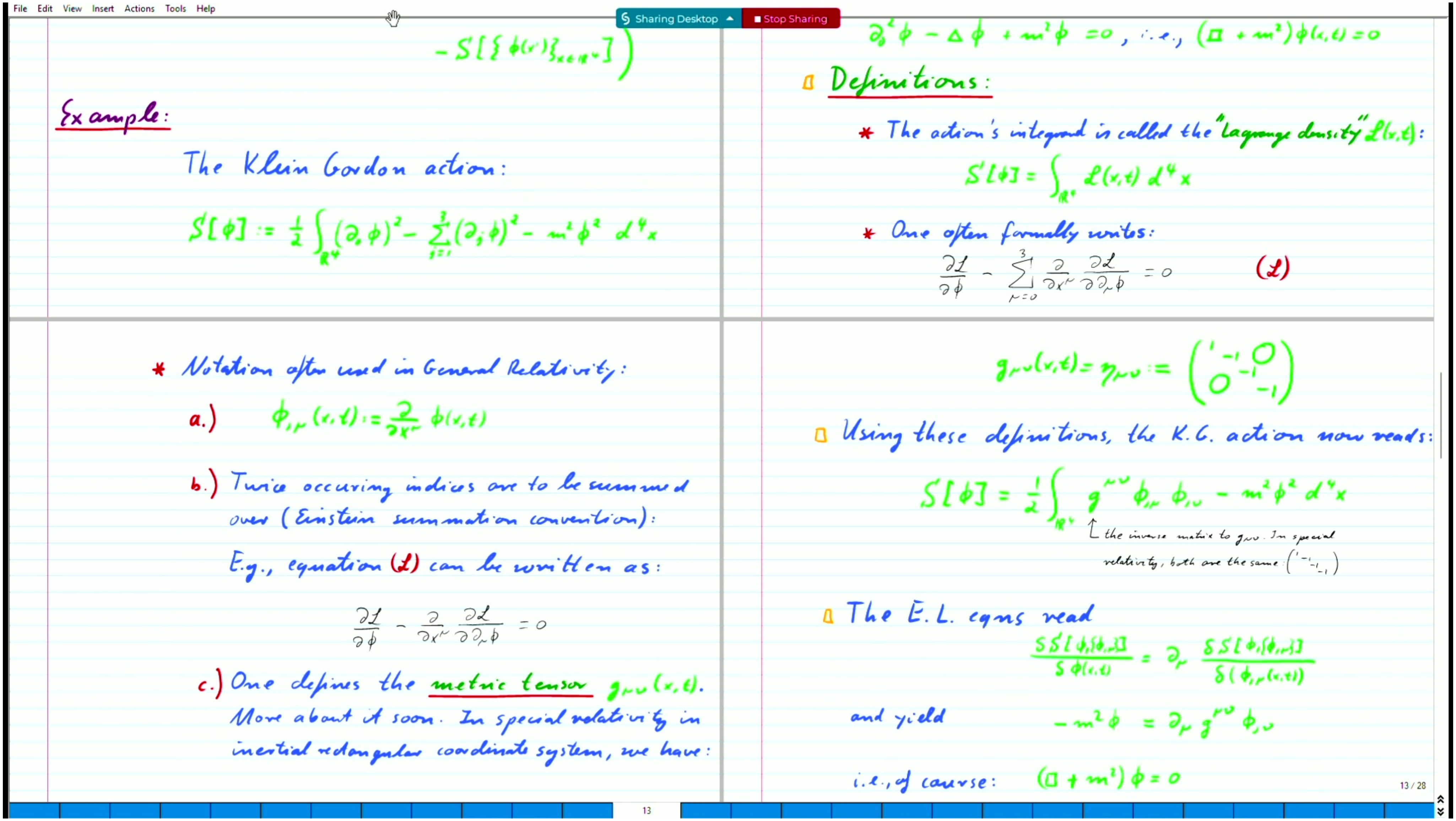
$$\partial_0^2\phi - \Delta\phi + m^2\phi = 0, \text{ i.e., } (\Box + m^2)\phi(x,t) = 0$$

- **Definitions:**
  - The action's integrand is called the "**Lagrange density**" $\mathcal{L}(x,t)$:

$$S[\phi] = \int_{\mathbb{R}^4}\mathcal{L}(x,t)\, d^4x$$

  - One often formally writes:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0 \qquad (\mathcal{L})$$



$$- S[\{\phi(x')\}_{x\in\mathbb{R}^4}]\Big)$$

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- Notation often used in General Relativity:

a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu}\phi(x,t)$

b.) Twice occurring indices are to be summed over (Einstein summation convention):

E.g., equation $(\mathcal{L})$ can be written as:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0$$

c.) One defines the metric tensor $g_{\mu\nu}(x,t)$. More about it soon. In special relativity in inertial rectangular coordinate system, we have:

$$g_{\mu\nu}(x,t) = \eta_{\mu\nu} := \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

☐ Using these definitions, the K.G. action now reads:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\,\phi_{,\nu} - m^2\phi^2\, d^4x$$

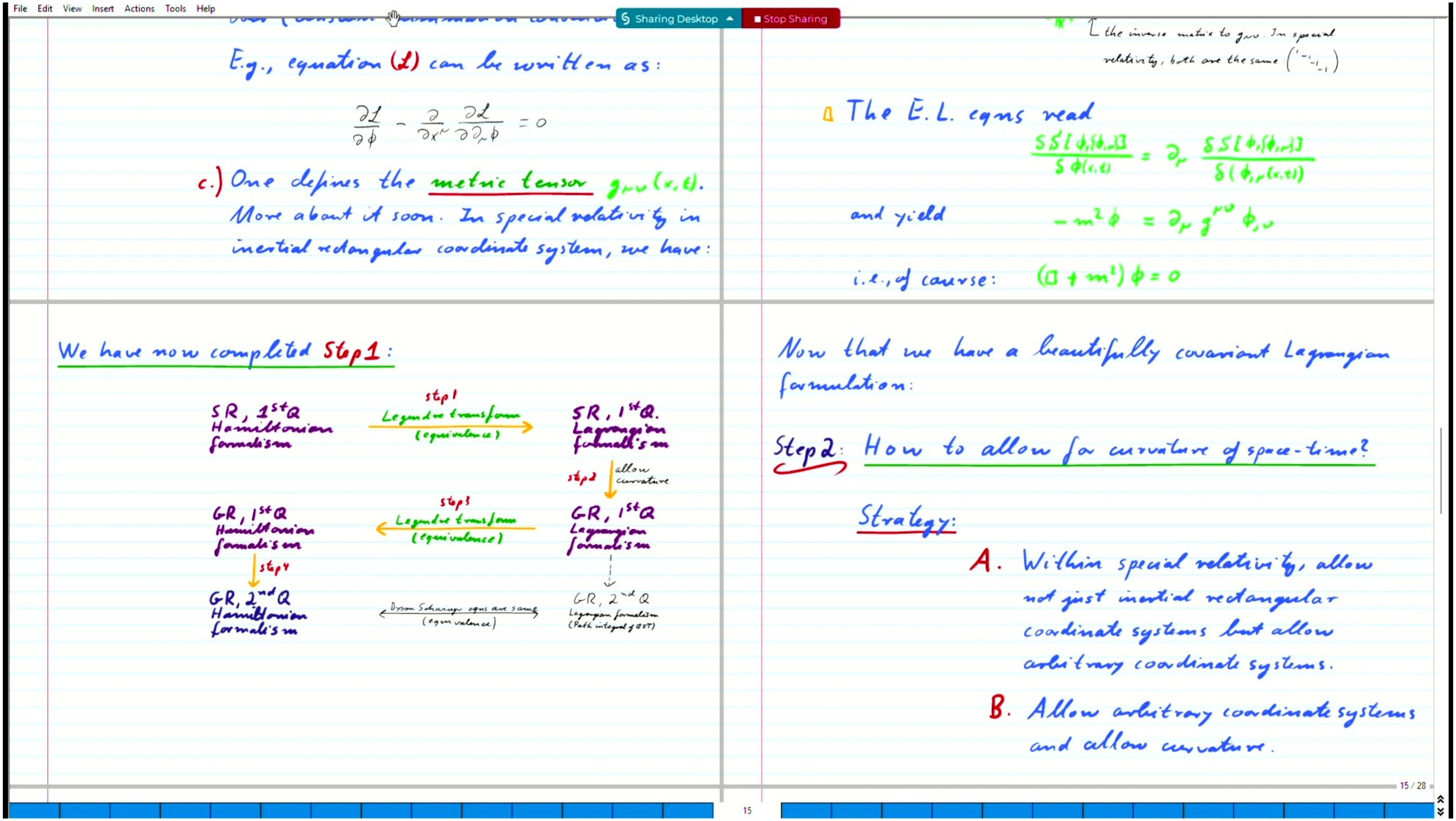
$\uparrow$ the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same: $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

☐ The E.L. eqns read

$$\frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta\phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta(\phi_{,\mu}(x,t))}$$

and yield
$$-m^2\phi = \partial_\mu g^{\mu\nu}\phi_{,\nu}$$

i.e., of course: $(\Box + m^2)\phi = 0$



E.g., equation $(\mathcal{L})$ can be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

c.) One defines the metric tensor $g_{\mu\nu}(x,t)$. More about it soon. In special relativity in inertial rectangular coordinate system, we have:

We have now completed **Step 1**:

$$\begin{tikzcd}
\text{SR, } 1^{st}Q \text{ Hamiltonian formalism} \arrow[r, "\text{step 1 Legendre transform}", "\text{(equivalence)}"'] & \text{SR, } 1^{st}Q. \text{ Lagrangian formalism} \arrow[d, "\text{allow curvature}", "\text{step 2}"'] \\
\text{GR, } 1^{st}Q \text{ Hamiltonian formalism} \arrow[d, "\text{step 4}"] & \text{GR, } 1^{st}Q \text{ Lagrangian formalism} \arrow[l, "\text{step 3 Legendre transform}"', "\text{(equivalence)}"] \arrow[d, dashed] \\
\text{GR, } 2^{nd}Q \text{ Hamiltonian formalism} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \text{GR, } 2^{nd}Q \text{ Lagrangian formalism (Path integral of QFT)}
\end{tikzcd}$$

↑ the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

The E.L. eqns read

$$\frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta \phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta(\phi_{,\mu}(x,t))}$$

and yield $\quad -m^2\phi = \partial_\mu g^{\mu\nu} \phi_{,\nu}$

i.e., of course: $\quad (\Box + m^2)\phi = 0$

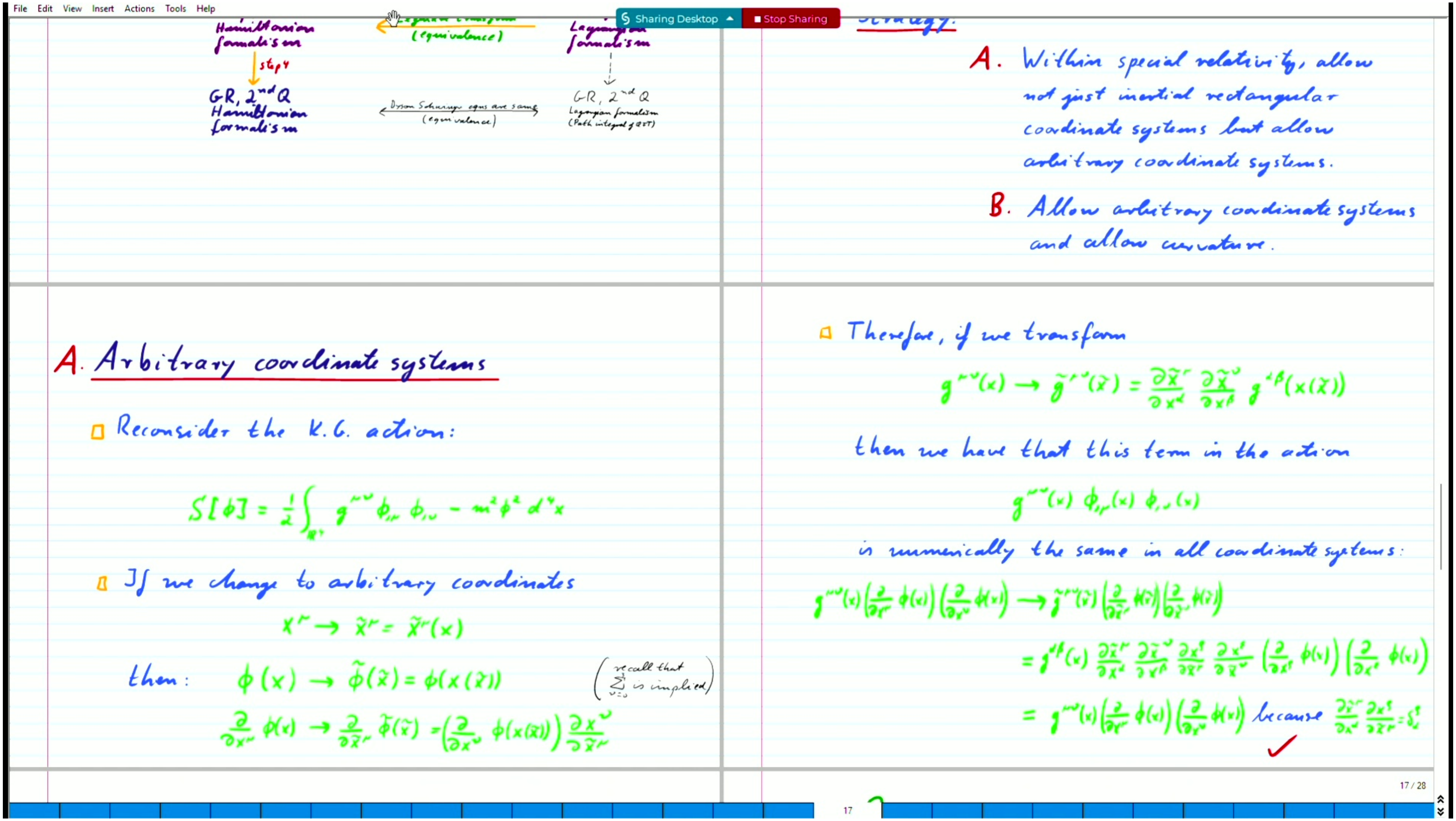
Now that we have a beautifully covariant Lagrangian formulation:

Step 2: How to allow for curvature of space-time?

Strategy:

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate systems and allow curvature.



Hamiltonian formalism

step 4

GR, 2nd Q Hamiltonian formalism

(equivalence)

Lagrangian formalism

Dyson Schwinger eqns are same (equivalence)

GR, 2nd Q Lagrangian formalism (Path integral of QFT)

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate systems and allow curvature.

A. Arbitrary coordinate systems

- Reconsider the K.G. action:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 \, d^4x$$

- If we change to arbitrary coordinates

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then:

$$\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$$

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu}\phi(x(\tilde{x}))\right)\frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

(recall that $\sum_{\nu=0}^{3}$ is implied)

- Therefore, if we transform

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x(\tilde{x}))$$

then we have that this term in the action

$$g^{\mu\nu}(x)\,\phi_{,\mu}(x)\,\phi_{,\nu}(x)$$

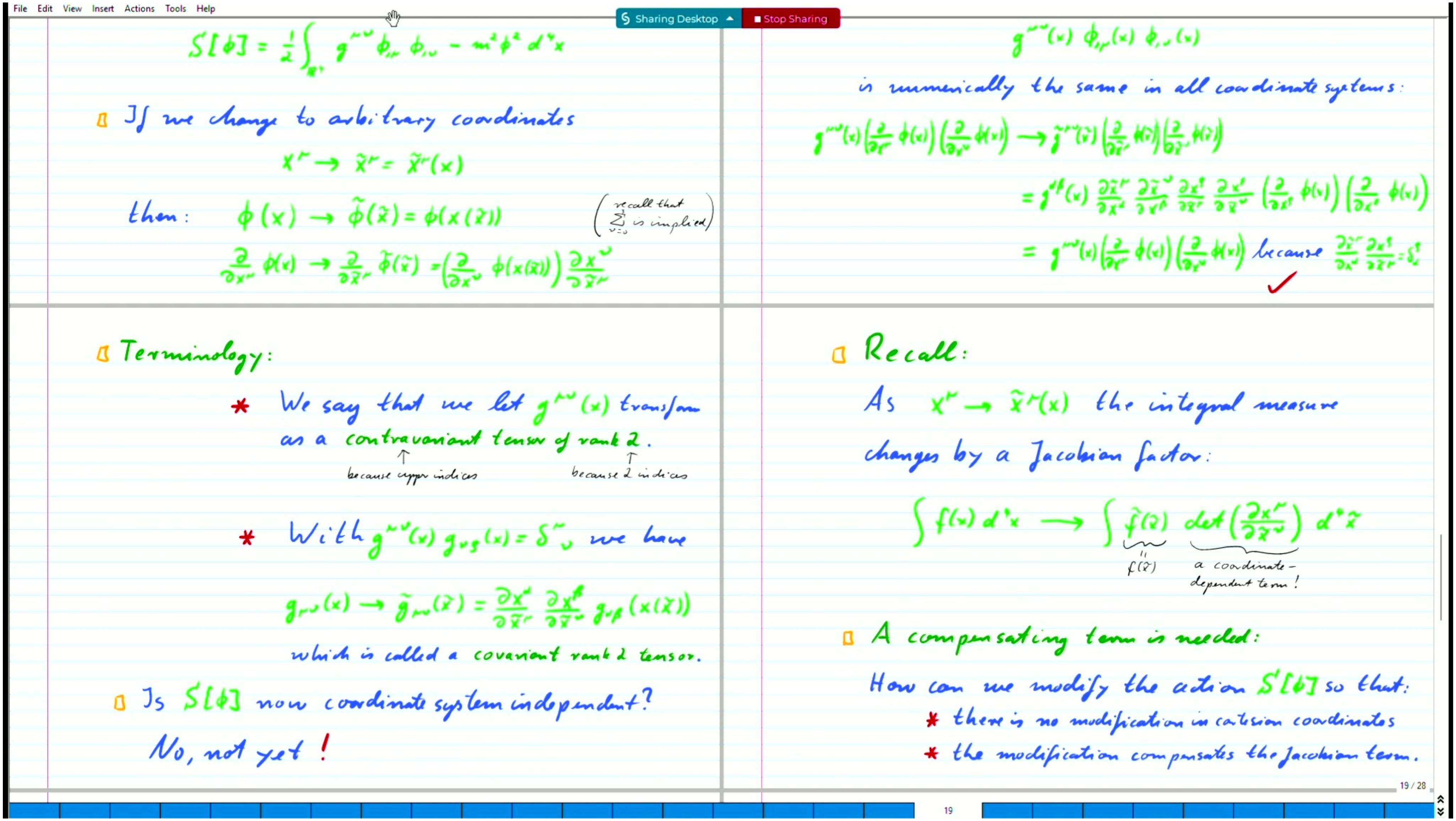
is numerically the same in all coordinate systems:

$$g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \to \tilde{g}^{\mu\nu}(\tilde{x})\left(\frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x})\right)\left(\frac{\partial}{\partial \tilde{x}^\nu}\tilde{\phi}(\tilde{x})\right)$$

$$= g^{\alpha\beta}(x)\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\left(\frac{\partial}{\partial x^\rho}\phi(x)\right)\left(\frac{\partial}{\partial x^\sigma}\phi(x)\right)$$

$$= g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \text{ because } \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\alpha$$

✓



$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 \, d^4x$$

- If we change to arbitrary coordinates

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then: $\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$ (recall that $\sum_{\nu=0}^{3}$ is implied)

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu}\phi(x(\tilde{x}))\right)\frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

$$g^{\mu\nu}(x)\,\phi_{,\mu}(x)\,\phi_{,\nu}(x)$$

is numerically the same in all coordinate systems:

$$g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \longrightarrow \tilde{g}^{\mu\nu}(\tilde{x})\left(\frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x})\right)\left(\frac{\partial}{\partial \tilde{x}^\nu}\tilde{\phi}(\tilde{x})\right)$$

$$= g^{\alpha\beta}(x)\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\left(\frac{\partial}{\partial x^\rho}\phi(x)\right)\left(\frac{\partial}{\partial x^\sigma}\phi(x)\right)$$

$$= g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \text{ because } \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\alpha$$

✓

- Terminology:
  - * We say that we let $g^{\mu\nu}(x)$ transform as a contravariant tensor of rank 2.

    $\uparrow$ because upper indices — $\uparrow$ because 2 indices
  - * With $g^{\mu\nu}(x)\,g_{\nu\rho}(x) = \delta^\mu{}_\rho$ we have

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\,g_{\alpha\beta}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

- Is $S[\phi]$ now coordinate system independent?

No, not yet !

- Recall:

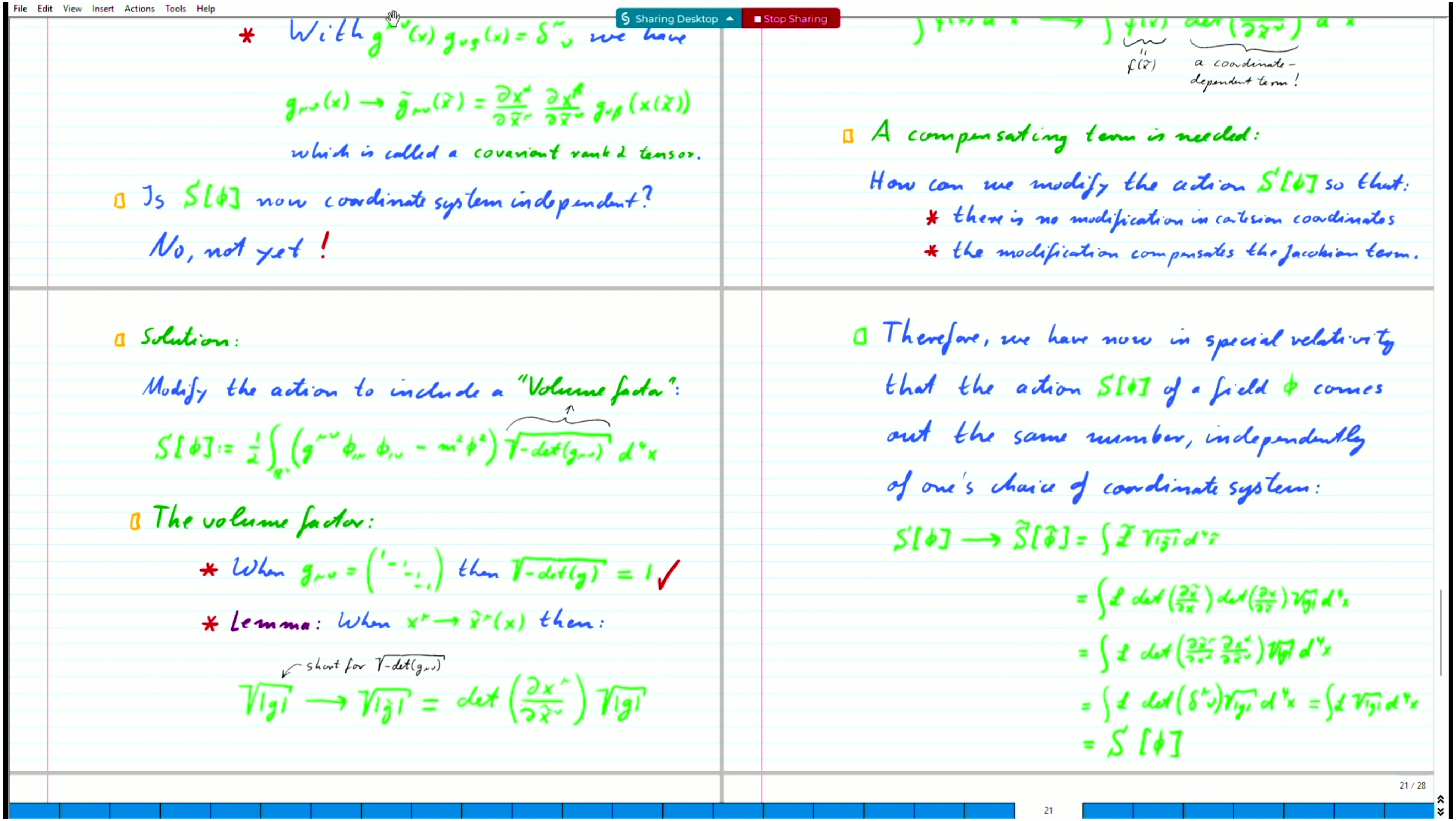
As $x^\mu \to \tilde{x}^\mu(x)$ the integral measure changes by a Jacobian factor:

$$\int f(x)\,d^4x \longrightarrow \int \underbrace{\tilde{f}(\tilde{x})}_{=f(\tilde{x})}\ \underbrace{\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)}_{\text{a coordinate-dependent term!}}\,d^4\tilde{x}$$

- A compensating term is needed:

How can we modify the action $S'[\phi]$ so that:

- * there is no modification in cartesian coordinates
- * the modification compensates the Jacobian term.



- With $g^{\mu\nu}(x)\, g_{\nu\beta}(x) = \delta^\mu{}_\nu$ we have

$$g_{\mu\nu}(x) \longrightarrow \tilde g_{\mu\nu}(\tilde x) = \frac{\partial x^\alpha}{\partial \tilde x^\mu} \frac{\partial x^\beta}{\partial \tilde x^\nu} g_{\alpha\beta}(x(\tilde x))$$

which is called a covariant rank 2 tensor.

☐ Is $S[\phi]$ now coordinate system independent?

No, not yet !

☐ Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \right) \overbrace{\sqrt{-\det(g_{\mu\nu})}}^{\uparrow}\, d^4x$$

☐ The volume factor:

- When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓
- Lemma: When $x^\mu \longrightarrow \tilde x^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde g|} = \det\left(\frac{\partial x^\mu}{\partial \tilde x^\nu} \right) \sqrt{|g|}$$

$$\int f(x)\, d^4x \longrightarrow \int \underbrace{f(x)}_{= f(\tilde x)} \underbrace{\det\left(\frac{\partial}{\partial \tilde x^\nu} \right)}_{\text{a coordinate-dependent term!}} d^4\tilde x$$

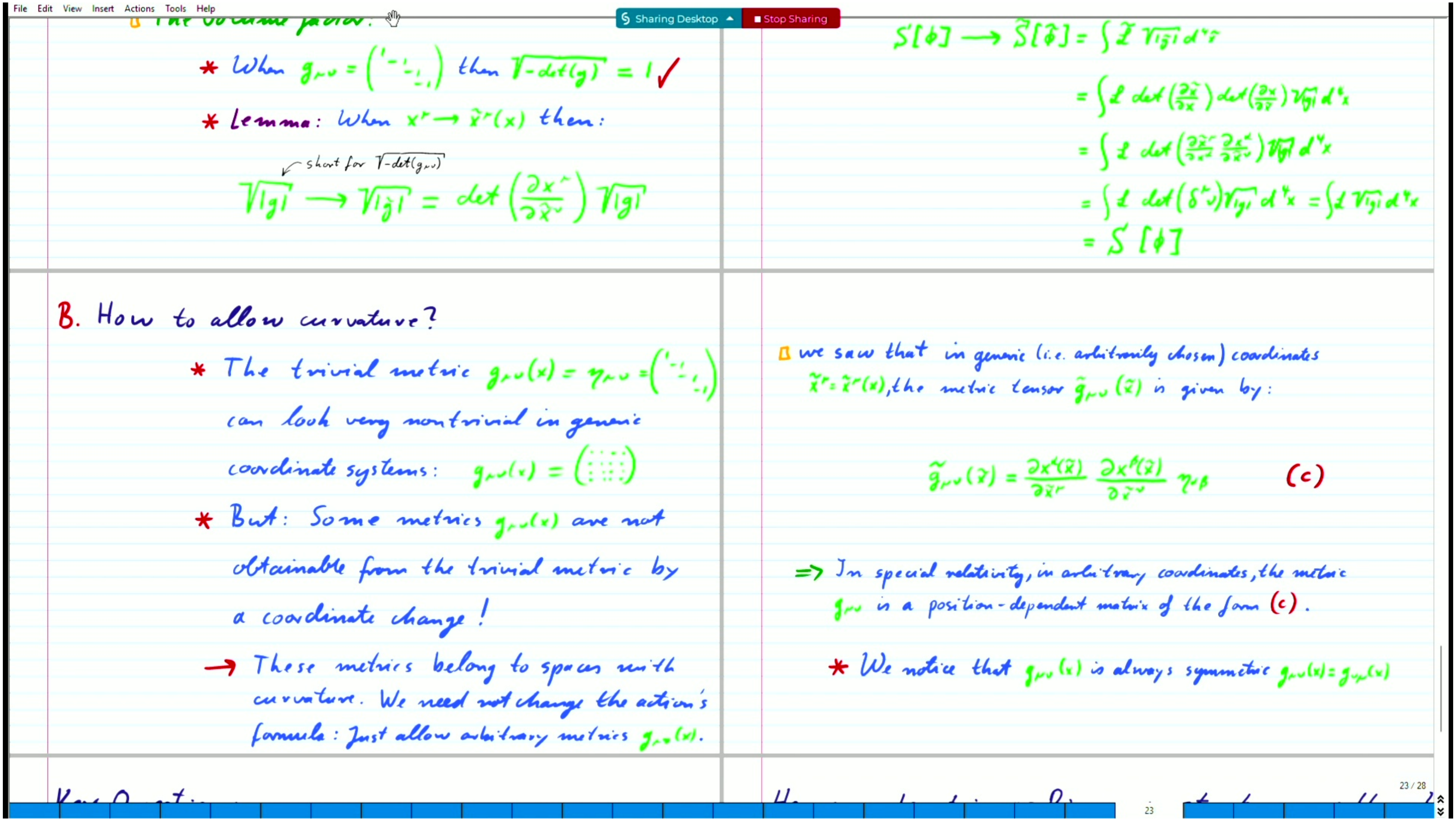
☐ A compensating term is needed:

How can we modify the action $S'[\phi]$ so that:

- there is no modification in catesian coordinates
- the modification compensates the Jacobian term.

☐ Therefore, we have now in special relativity that the action $S[\phi]$ of a field ϕ comes out the same number, independently of one's choice of coordinate system:

$$\begin{aligned} S[\phi] \longrightarrow \tilde S[\tilde\phi] &= \int \tilde{\mathcal{L}} \sqrt{|\tilde g|}\, d^4\tilde x \\ &= \int \mathcal{L} \det\left(\frac{\partial \tilde x}{\partial x}\right) \det\left(\frac{\partial x}{\partial \tilde x}\right) \sqrt{|g|}\, d^4x \\ &= \int \mathcal{L} \det\left(\frac{\partial \tilde x^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \tilde x^\nu} \right) \sqrt{|g|}\, d^4x \\ &= \int \mathcal{L} \det(\delta^\mu{}_\nu) \sqrt{|g|}\, d^4x = \int \mathcal{L} \sqrt{|g|}\, d^4x \\ &= S[\phi] \end{aligned}$$



☐ The volume factor:

- When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓
- Lemma: When $x^\mu \rightarrow \tilde{x}^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde{g}|} = \det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) \sqrt{|g|}$$

$$S[\phi] \longrightarrow \tilde{S}[\tilde{\phi}] = \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|}\, d^4\tilde{x}$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}}{\partial x}\right) \det\left(\frac{\partial x}{\partial \tilde{x}}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial \tilde{x}^\nu}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det(\delta^\mu{}_\nu) \sqrt{|g|}\, d^4x = \int \mathcal{L} \sqrt{|g|}\, d^4x$$

$$= S[\phi]$$

B. How to allow curvature?

- The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
- But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

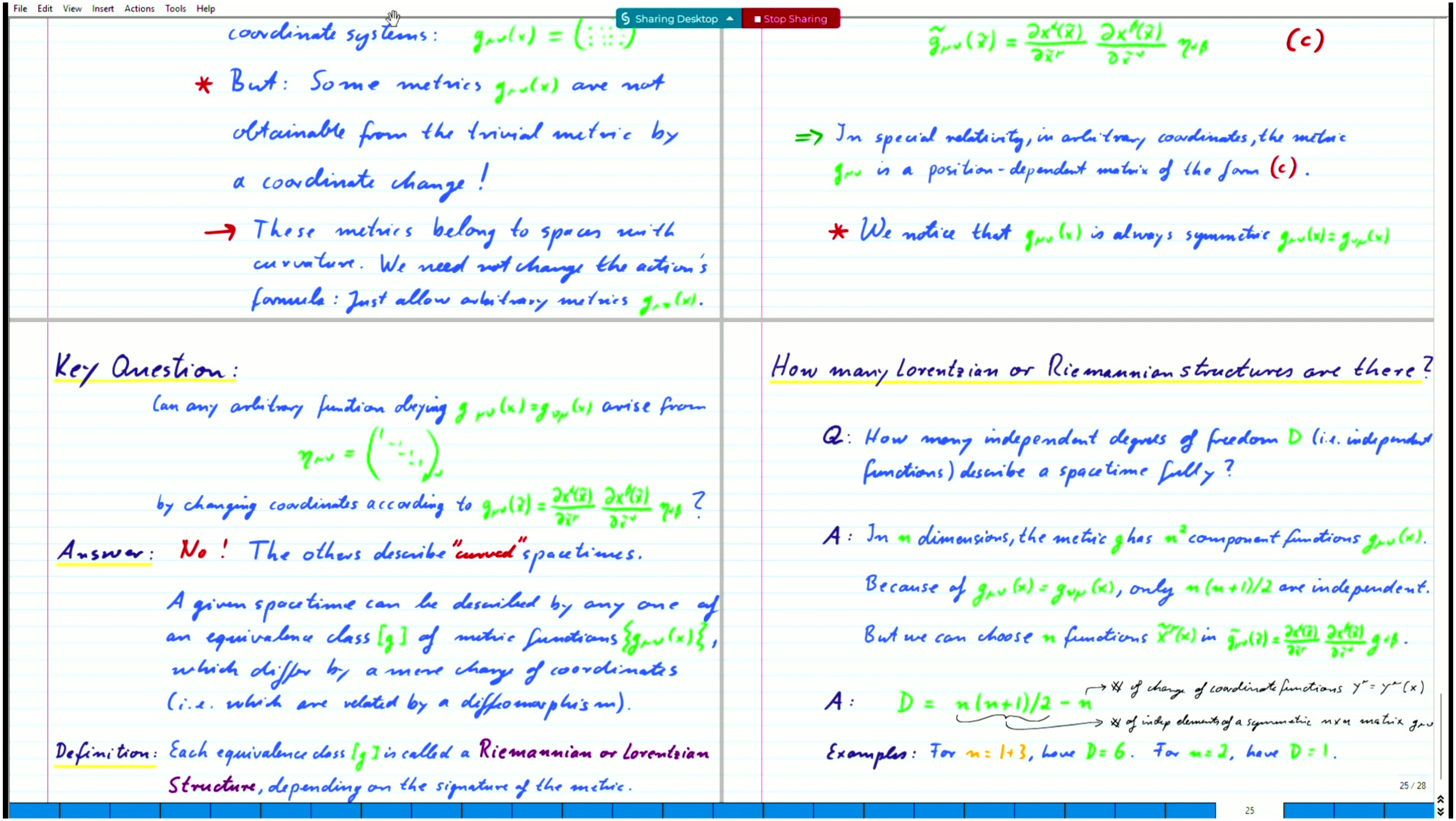
→ These metrics belong to spaces with curvature. We need not change the action's formula: Just allow arbitrary metrics $g_{\mu\nu}(x)$.

☐ we saw that in generic (i.e. arbitrarily chosen) coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

⇒ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

- We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$



coordinate systems: $g_{\mu\nu}(x) = \left(\vdots \ \vdots \right)$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

→ These metrics belong to spaces with curvature. We need not change the action's formula: Just allow arbitrary metrics $g_{\mu\nu}(x)$.

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

=> In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

Key Question:

Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

by changing coordinates according to $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta}$?

Answer: No ! The others describe "curved" spacetimes.

A given spacetime can be described by any one of an equivalence class $[g]$ of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

Definition: Each equivalence class $[g]$ is called a Riemannian or Lorentzian Structure, depending on the signature of the metric.

How many Lorentzian or Riemannian structures are there?

Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: In n dimensions, the metric g has n^2 component functions $g_{\mu\nu}(x)$.

Because of $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, only $n(n+1)/2$ are independent.

But we can choose n functions $\tilde{x}^\mu(x)$ in $\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} g_{\alpha\beta}$.

A: $$D = \underbrace{n(n+1)/2}_{\text{\# of indep elements of a symmetric } n\times n \text{ matrix } g_{\mu\nu}} - \overset{\text{\# of change of coordinate functions } y^\mu = y^\mu(x)}{n}$$

Examples: For $n = 1+3$, have $D = 6$. For $n = 2$, have $D = 1$.