Title: Advanced General Relativity - 240228 (afternoon)

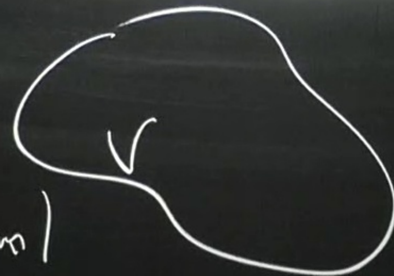
Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

Date: February 28, 2024 - 1:30 PM

URL: https://pirsa.org/24020007

4D Integration



$$\int_V f\, dV$$

$$dV = \sqrt{-g}\, d^4x$$

Levi-Civita tensor (volume form)

$$\underbrace{\varepsilon_{\alpha\beta\gamma\delta}}_{\text{tensor}} \equiv \sqrt{-g}\, \underbrace{[\alpha\beta\gamma\delta]}_{\text{permutation symbol}}$$

$$\text{permutation symbol} = \begin{cases} +1 & \underline{\text{even}} \text{ permutation of } 0123 \\ -1 & \underline{\text{odd}} \quad '' \quad 0123 \\ 0 & \text{if any two indices agree} \end{cases}$$

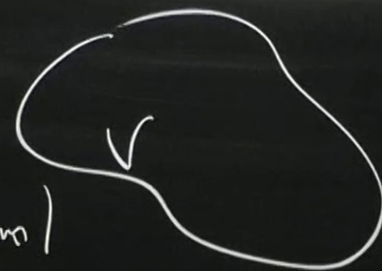
$$[0123] = 1$$

$$[1320] = -[0321] = +[0123] = 1$$

Matrix $M^\alpha{}_\beta$

$$\rightarrow \det(M^\alpha{}_\beta) = [\alpha\beta\gamma\delta]\, M^\alpha{}_0 M^\beta{}_1 M^\gamma{}_2 M^\delta{}_3 = \begin{vmatrix} M^0{}_0 & M^0{}_1 & M^0{}_2 & M^0{}_3 \\ M^1{}_0 & M^1{}_1 & M^1{}_2 & M^1{}_3 \\ & \vdots & & \end{vmatrix}$$

4D Integration



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Coordinate transf: $x^\alpha \longrightarrow z^\mu$

$$g_{\mu\nu} = g_{\alpha\beta}\,\frac{\partial x^\alpha}{\partial z^\mu}\,\frac{\partial x^\beta}{\partial z^\nu} \;\to\; \det(g_{\mu\nu}) = \det(g_{\alpha\beta})\left[\det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right)\right]^2$$

$$\sqrt{-\det(g_{\mu\nu})} = \sqrt{-\det(g_{\alpha\beta})}\,\det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right)$$

$$dV[z] = \sqrt{-\det(g_{\mu\nu})}\,d^4z$$

$$= \sqrt{-\det(g_{\alpha\beta})}\,\underbrace{\det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right) d^4z}_{d^4x}$$

$$= dV[x]$$

Integrate using z-coordinates: $dV[z] = \sqrt{-\det(g_{\mu\nu})}\, d^4 z$

Consider:

$$\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\alpha}{\partial z^0} \frac{\partial x^\beta}{\partial z^1} \frac{\partial x^\gamma}{\partial z^2} \frac{\partial x^\delta}{\partial z^3}\, dz^0 dz^1 dz^2 dz^3$$

$$= \sqrt{-\det(g_{\alpha\beta})}\, \underbrace{[\alpha\beta\gamma\delta] \frac{\partial x^\alpha}{\partial z^0} \frac{\partial x^\beta}{\partial z^1} \frac{\partial x^\gamma}{\partial z^2} \frac{\partial x^\delta}{\partial z^3}}_{\det\left(\partial x^\alpha / \partial z^\nu\right) \;=\; \text{Jacobian}}\, dz^0 dz^1 dz^2 dz^3$$

$$= \underbrace{\sqrt{-\det(g_{\alpha\beta})}\, \det\left(\partial x^\alpha / \partial z^\nu\right)}_{\sqrt{-\det(g_{\mu\nu})}}\, dz^0 dz^1 dz^2 dz^3 = dV[z]$$

Consider:

$$\boxed{\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\alpha}{\partial z^0} \frac{\partial x^\beta}{\partial z^1} \frac{\partial x^\gamma}{\partial z^2} \frac{\partial x^\delta}{\partial z^3} \, dz^0 dz^1 dz^2 dz^3}$$

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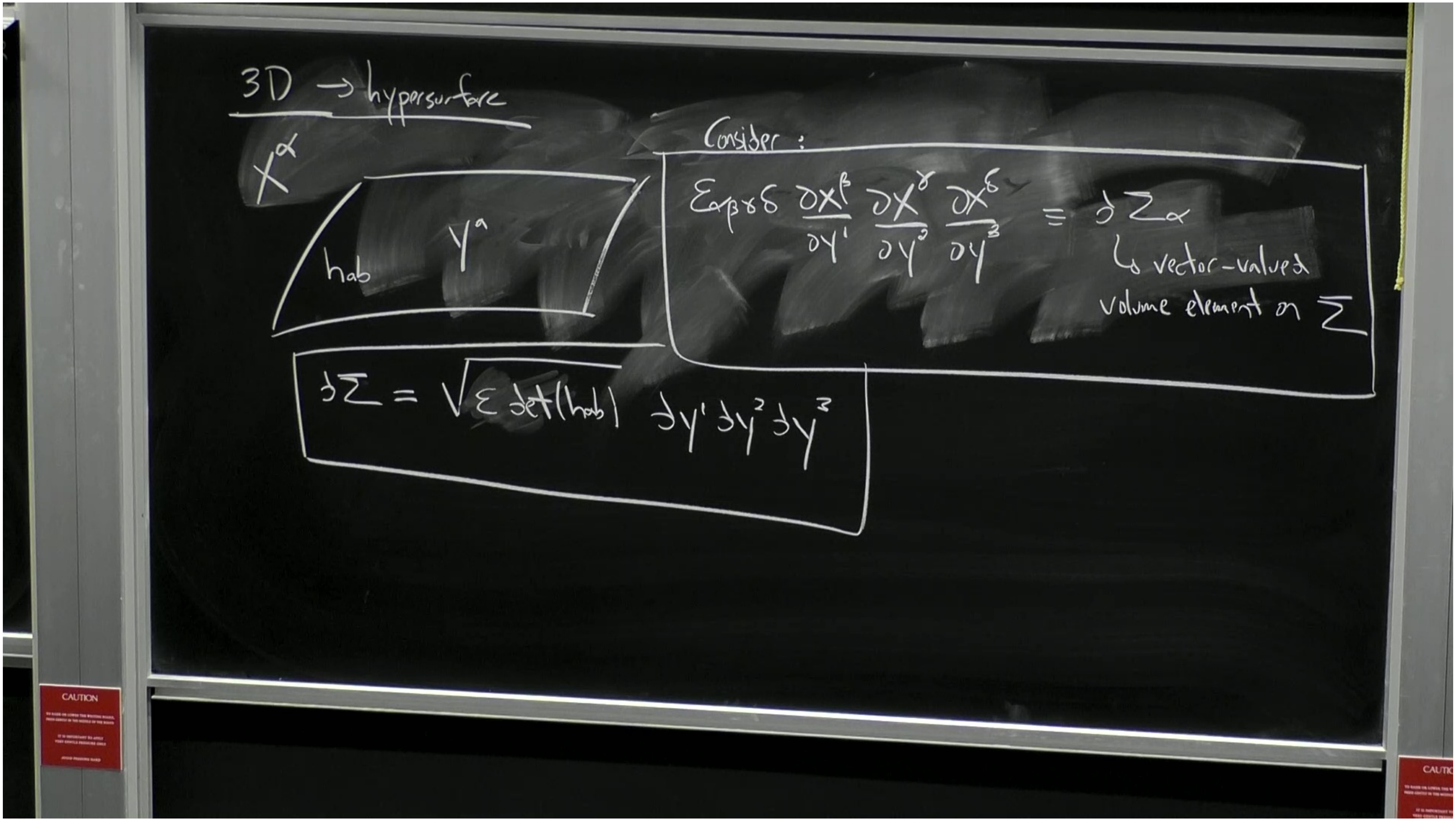
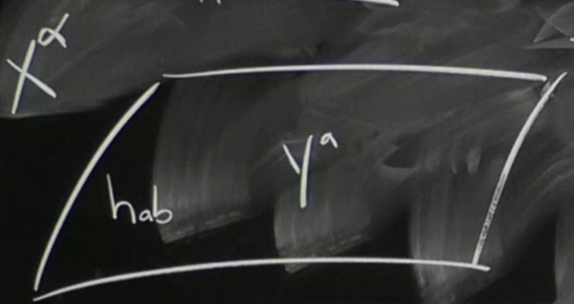

3D → hypersurface
X^α
y^a
h_{ab}
Consider :
$\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial X^\beta}{\partial y^1} \frac{\partial X^\gamma}{\partial y^2} \frac{\partial X^\delta}{\partial y^3} \equiv d\Sigma_\alpha$
↳ vector-valued volume element on Σ
$d\Sigma = \sqrt{\varepsilon \det(h_{ab})}\, dy^1 dy^2 dy^3$
CAUTION

3D → hypersurface



Consider:

$$\boxed{\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial X^\beta}{\partial y^1} \frac{\partial X^\gamma}{\partial y^2} \frac{\partial X^\delta}{\partial y^3} = d\Sigma_\alpha}$$

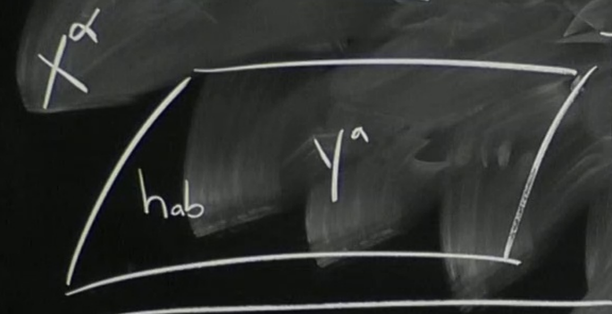
↳ vector-valued volume element on Σ

$$\boxed{d\Sigma = \sqrt{\varepsilon \det(h_{ab})}\, dy^1 dy^2 dy^3}$$

undirected surface element

→ $n_\alpha d\Sigma$ directed surface element

3D → hypersurface



Consider:

$$\boxed{\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\beta}{\partial y^1}\frac{\partial x^\gamma}{\partial y^2}\frac{\partial x^\delta}{\partial y^3} \equiv d\Sigma_\alpha}$$

↳ vector-valued volume element on Σ

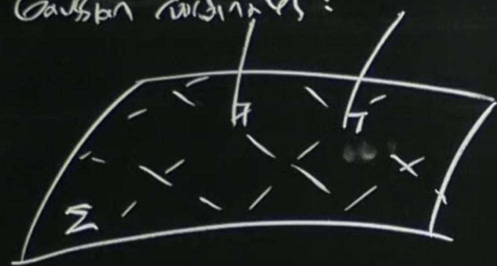
$$\boxed{d\Sigma = \sqrt{\varepsilon \det(h_{ab})}\, dy^1\, dy^2\, dy^3}$$

undirected surface element

→ $n_\alpha\, d\Sigma$ directed surface element

$$\boxed{d\Sigma_\alpha = \varepsilon\, n_\alpha\, d\Sigma}$$

Gaussian coordinates:



$$x^\alpha \stackrel{*}{=} (\ell, y^a) \qquad n^\alpha \stackrel{*}{=} (1,0,0,0)$$

$$n_\alpha \stackrel{*}{=} (\varepsilon,0,0,0)$$

$$e^\alpha_1 \stackrel{*}{=} (0,1,0,0), \dots$$

$$\sqrt{-g} \stackrel{*}{=} \sqrt{-\varepsilon h}$$

$$d\Sigma_\alpha = \varepsilon_{\alpha\beta\gamma\delta}\, e^\beta_1 e^\gamma_2 e^\delta_3\, d^3y = \sqrt{-g}\,[\alpha\beta\gamma\delta]\, e^\beta_1 e^\gamma_2 e^\delta_3\, d^3y$$

$$\stackrel{*}{=} \sqrt{-\varepsilon h}\,[\alpha\, 1\, 2\, 3]\, d^3y \stackrel{*}{=} \sqrt{-\varepsilon h}\, d^4y\, \delta^\ell_\alpha$$

$$n_\alpha\, d\Sigma \stackrel{*}{=} \varepsilon\, \delta^\ell_\alpha$$

h_{ab} y^a

$$\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x}{\partial y^1} \frac{\partial x}{\partial y^2} \frac{\partial x}{\partial y^3} = d\Sigma_\alpha$$

↳ vector-valued volume element on Σ

$$\boxed{d\Sigma = \sqrt{-\varepsilon \det(h_{ab})}\, dy^1 dy^2 dy^3}$$

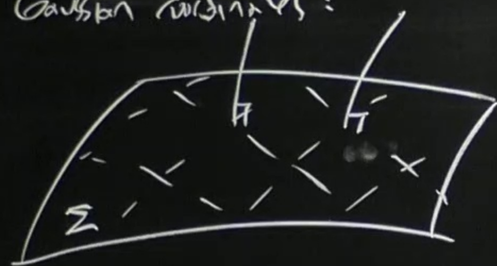
undirected surface element

$\rightarrow$ $n_\alpha \, d\Sigma$ directed surface element

$$\sqrt{-\varepsilon h} = \begin{cases} \sqrt{h} & \text{spacelike} \\ \sqrt{-h} & \text{timelike} \end{cases}$$

$$\boxed{d\Sigma_\alpha = \varepsilon \, n_\alpha \, d\Sigma}$$

Gaussian coordinates:



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$$\sqrt{-g} \overset{*}{=} \sqrt{-\varepsilon h}$$

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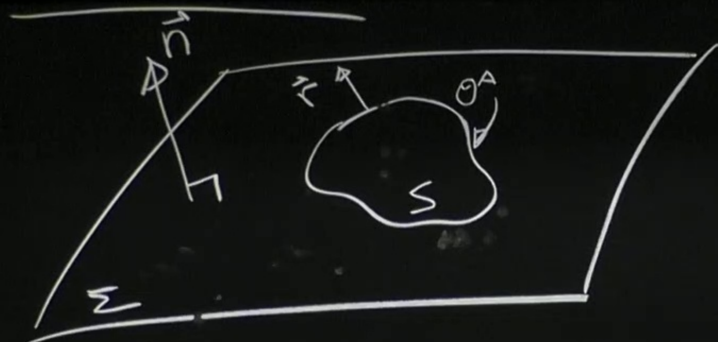
$$\overset{*}{=} \sqrt{-\varepsilon h}\,[\alpha\, 1\, 2\, 3]\, d^3y \overset{*}{=} \sqrt{-\varepsilon h}\, d^4y\, \delta^\ell_\alpha$$

$$n_\alpha d\Sigma \overset{*}{=} \varepsilon\, \delta^\ell_\alpha \sqrt{-\varepsilon h}\; d^3y$$

$$d\Sigma_\alpha \overset{*}{=} \varepsilon\, n_\alpha\, d\Sigma$$

$$\boxed{d\Sigma_\alpha = \varepsilon\, n_\alpha\, d\Sigma}$$

2D surface



two normals: n^α, r^α ↳ normal to S, in Σ

↓ normal to Σ

$$n_\alpha n^\alpha = \varepsilon$$

$$r_\alpha r^\alpha = -\varepsilon$$

surface element: $$dS_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\gamma}{\partial \theta^2} \frac{\partial x^\delta}{\partial \theta^3}$$ = antisymmetric in α, β

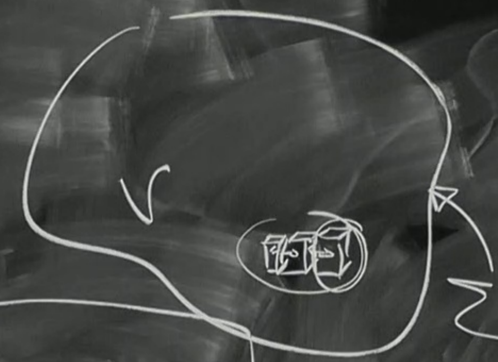
$$= (r_\alpha n_\beta - n_\alpha r_\beta)\, dS$$

$$dS = \sqrt{\det(g_{AB})}\, d^2\theta$$

↳ induced metric in θ^A coordinates.

Gauss's theorem

$$\boxed{\int_V \nabla_\alpha A^\alpha \, dV = \oint_\Sigma A^\alpha \, d\Sigma_\alpha}$$



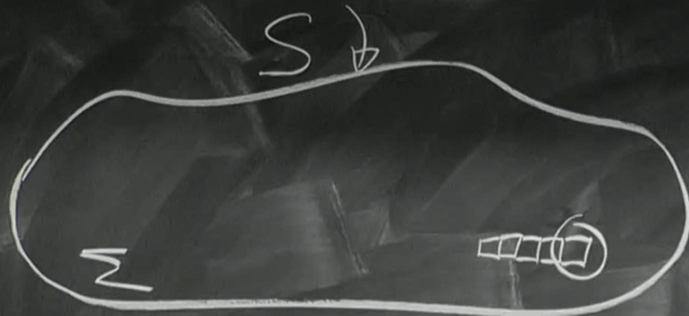
V : 4D region in spacetime

Σ : ∂V ≡ boundary of V

≡ hypersurface

A^α – arbitrary vector field

Stokes's Theorem



Σ : hypersurface (3D)

S : boundary $= \partial\Sigma$ (2D)

$B^{\alpha\beta}$: antisymmetric tensor fields

$$\boxed{\int_\Sigma \nabla_\beta B^{\alpha\beta}\, d\Sigma_\alpha = \frac{1}{2}\oint_S B^{\alpha\beta}\, dS_{\alpha\beta}}$$