Title: Advanced General Relativity - 240214 (afternoon)

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Collection: Advanced General Relativity (PHYS7840)

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$K_\alpha = -\nabla_\alpha S \,,\quad k_\alpha k^\alpha = 0 \,,\quad k^\beta \nabla_\beta k^\alpha = 0$

$\Omega^\alpha{}_\beta = g^\alpha{}_\beta + k^\alpha N_\beta + N^\alpha k_\beta$

$\nabla_\alpha k_\beta = B\, k_\alpha k_\beta + k_\alpha B^\perp_\beta + B^\perp_\alpha k_\beta + B^{\parallel}_{\alpha\beta}$

$B^\parallel_{\alpha\beta} = \frac{1}{2}\Omega_{\alpha\beta}\,\Theta + \sigma_{\alpha\beta} \;;\quad \Omega^{\alpha\beta}\sigma_{\alpha\beta} = 0$

$\checkmark\ \Theta = \Omega^{\alpha\beta} B^\parallel_{\alpha\beta} = \nabla_\alpha k^\alpha$

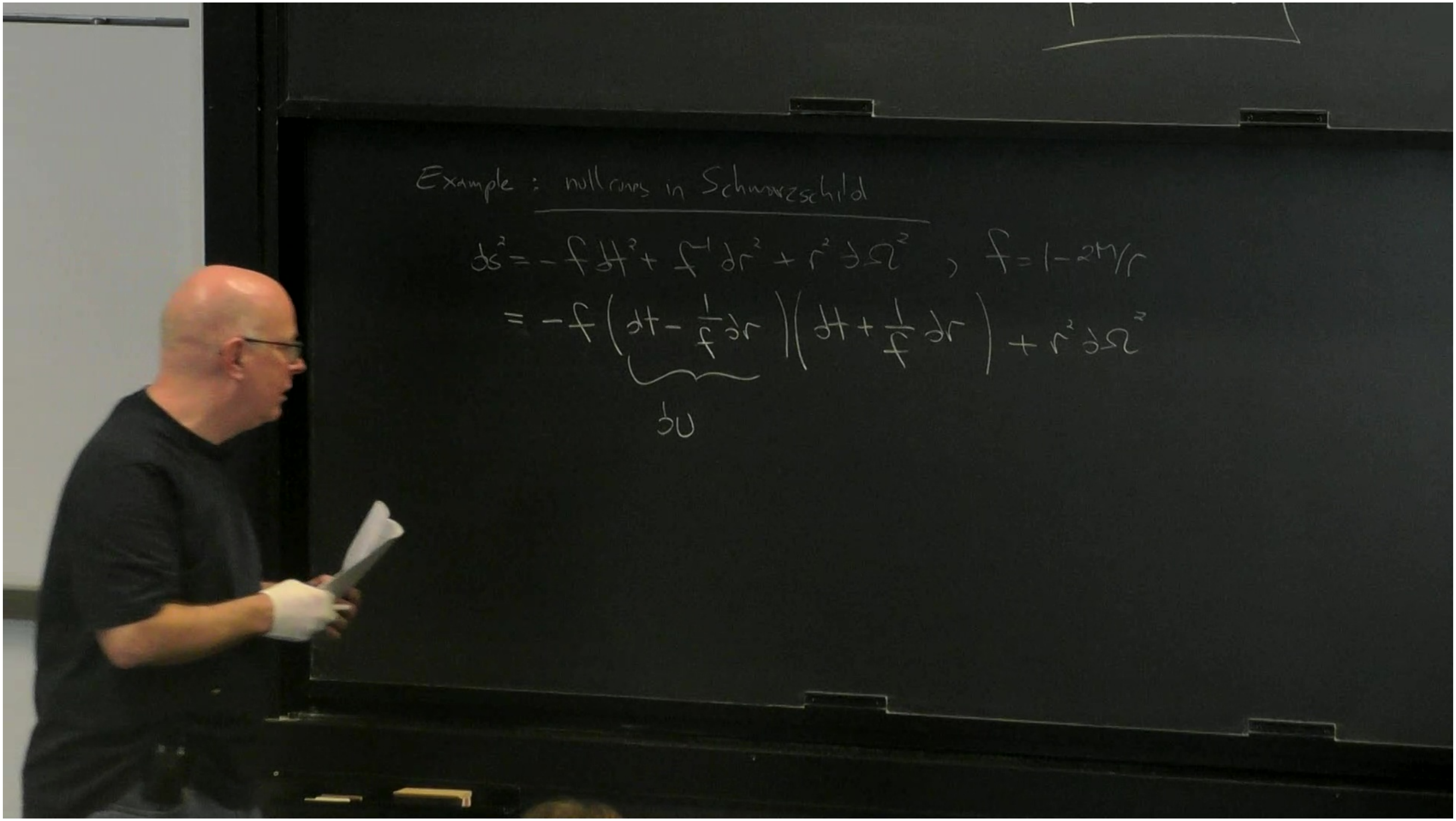
$\sigma_{\alpha\beta} = B^\parallel_{\alpha\beta} - \frac{1}{2}\Omega_{\alpha\beta}\,\Theta$

$\nabla_\alpha k^\alpha = g^{\alpha\beta}\nabla_\alpha k_\beta = \left(-k^\alpha N^\beta - N^\alpha k^\beta + \Omega^{\alpha\beta}\right)\nabla_\alpha k_\beta$

$= \Omega^{\alpha\beta}\,\nabla_\alpha k_\beta$

$= \Omega^{\alpha\beta}\left(B\, k_\alpha k_\beta + k_\alpha B^\perp_\beta + B^\perp_\alpha k_\beta + B^\parallel_{\alpha\beta}\right)$

$= \Omega^{\alpha\beta} B^\parallel_{\alpha\beta} \equiv \Theta$


Example : null cones in Schwarzschild
$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$, $f = 1 - 2M/r$
$= -f\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\left(dt + \frac{1}{f}dr\right) + r^2 d\Omega^2$

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2, \quad f = 1 - 2M/r$$

$$= -f\Big(\underbrace{dt - \tfrac{1}{f}dr}_{dU}\Big)\Big(\underbrace{dt + \tfrac{1}{f}dr}_{dV}\Big) + r^2\,d\Omega^2$$

$$\left.\begin{aligned} U &= t - r^* \\ V &= t + r^* \end{aligned}\right\} \quad r^* = \int \frac{dr}{f} = r + 2M\ln\left(r/2M - 1\right)$$

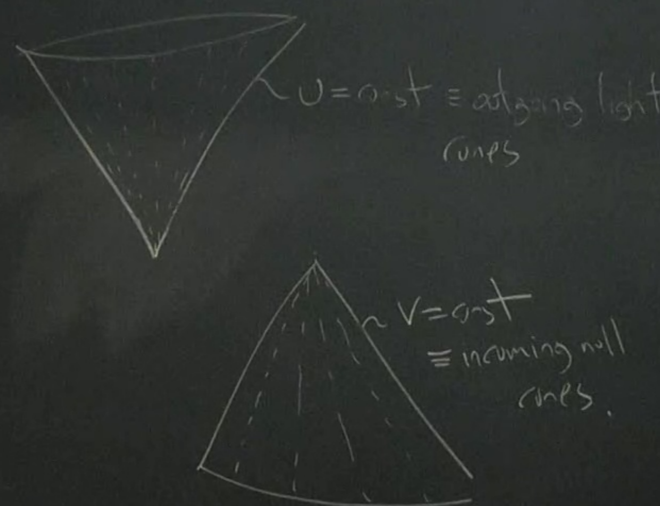
$$\boxed{\textcircled{H} = \nabla_a K^*}$$

Example : null cones in Schwarzschild

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f = 1 - 2M/r$$

$$= -f\Big(\underbrace{dt - \tfrac{1}{f}dr}_{du}\Big)\Big(\underbrace{dt + \tfrac{1}{f}dr}_{dv}\Big) + r^2 d\Omega^2$$

$$\left.\begin{aligned} u &= t - r^* \\ v &= t + r^* \end{aligned}\right\} \quad r^* = \int \frac{dr}{f} = r + 2M\ln\left(r/2M - 1\right)$$



$$\nabla_\alpha K_\beta = B K_\alpha K_\beta + K_\alpha B^{\perp}_\beta + B^{\perp}_\alpha K_\beta + B^{\amalg}_{\alpha\beta}$$

$$B^{\amalg}_{\alpha\beta} = \tfrac{1}{2}\Omega_{\alpha\beta}\, H + \sigma_{\alpha\beta} \;;\quad \Omega^{\alpha\beta}\sigma_{\alpha\beta} = 0$$

$$\checkmark\; H = \Omega^{\alpha\beta} B^{\amalg}_{\alpha\beta} = \nabla_\alpha K^\alpha$$

$$\sigma_{\alpha\beta} = B^{\amalg}_{\alpha\beta} - \tfrac{1}{2}\Omega_{\alpha\beta}\, H$$

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$

$$\nabla_\alpha K^\alpha = g^{\alpha\beta}\nabla_\alpha K_\beta = \left(-K^\alpha N^\beta - N^\alpha K^\beta + \Omega^{\alpha\beta}\right)\nabla_\alpha K_\beta$$

$$= \Omega^{\alpha\beta}\nabla_\alpha K_\beta$$

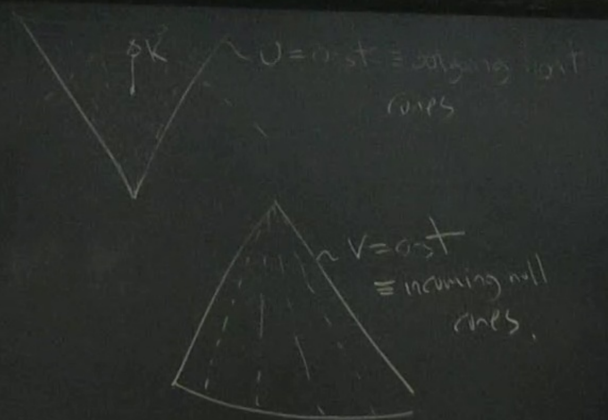
$$= \Omega^{\alpha\beta}\left(B K_\alpha K_\beta + K_\alpha B^{\perp}_\beta + B^{\perp}_\alpha K_\beta + B^{\amalg}_{\alpha\beta}\right)$$

$$= \Omega^{\alpha\beta} B^{\amalg}_{\alpha\beta} \equiv H$$

$$\boxed{H = \nabla_\alpha K^\alpha}$$

$$= -f\,\underbrace{\left(dt - \tfrac{1}{f}dr\right)}_{dU}\,\underbrace{\left(dt + \tfrac{1}{f}dr\right)}_{dV} + r^2 d\Omega^2$$

$$\left.\begin{aligned} U &= t - r^* \\ V &= t + r^* \end{aligned}\right\}\quad r^* = \int \frac{dr}{f} = r + 2M\ln(r/2M - 1)$$



$K_\alpha = -\nabla_\alpha U = \left(-1, \frac{1}{f}, 0, 0\right)$

$= -\nabla_\alpha (t - r^*)$

$K^\alpha = \left(\frac{1}{f}, 1, 0, 0\right) \longrightarrow \frac{\partial t}{\partial \lambda} = \frac{1}{f} \longrightarrow \frac{dt}{dr} = \frac{1}{f} \Rightarrow dt = \frac{dr}{f}$

$\frac{\partial r}{\partial \lambda} = 1 \qquad \Rightarrow t = r^* + \text{const}$

$\theta = \text{const}, \ \phi = \text{const}.$

$$\nabla_\alpha K_\beta = \begin{pmatrix} -\frac{M}{r^2} & \frac{M}{r^2 f} & 0 & 0 \\ \frac{M}{r^2 f} & -\frac{M}{r^2 f^2} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{pmatrix}$$

$N_\alpha \propto -\nabla_\alpha V = \left(-1, -\frac{1}{f}, 0, 0\right)$

$$\boxed{N_\alpha = \frac{1}{2} f \left(-1, -\frac{1}{f}, 0, 0\right)}$$

$B = -M/r^2$

$B^\perp_\alpha = 0 \qquad B^{\parallel}_{\alpha\beta} = \frac{1}{2} \Theta \Omega_{\alpha\beta} \qquad \sigma_{\alpha\beta} = 0$

$\Theta = \nabla_\alpha K^\alpha = \frac{2}{r} = \frac{1}{4\pi r^2} \frac{\partial}{\partial \lambda} 4\pi r^2$

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$B^{\perp}_{\alpha} = 0 \qquad B^{\parallel}_{\alpha\beta} = \frac{1}{2}\Theta\,\Omega_{\alpha\beta} \qquad \sigma_{\alpha\beta} = 0$

$$\Theta = \nabla_\alpha k^\alpha = \frac{2}{r} = \frac{1}{4\pi r^2}\frac{d}{d\lambda}\left(4\pi r^2\right)$$

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Example — anisotropic cosmology

$$ds^2 = -dt^2 + a^2(t)\,dx^2 + b^2(t)\,dy^2 + c^2(t)\,dz^2$$

$$= -dt^2 + c^2(t)\,dz^2 + \cdots$$

$$ds^2 = -c^2\Big(\underbrace{\tfrac{1}{c}dt - dz}_{du}\Big)\Big(\underbrace{\tfrac{1}{c}dt + dz}_{dv}\Big) + \cdots$$

$$u = \int\frac{dt}{c} - z\,, \quad v = \int\frac{dt}{c} + z$$

↳ right-moving light surfaces   ↳ right moving

Kasner spacetime — vacuum solution

$$a(t) = t^\alpha \qquad \alpha + \beta + \gamma = 1$$
$$b(t) = t^\beta \qquad \alpha^2 + \beta^2 + \gamma^2 = 1$$
$$c(t) = t^\gamma$$

$$K_\alpha = -\nabla_\alpha U = \left(-\frac{1}{c}, 0, 0, 1\right)$$

$$K^\alpha = \left(\frac{1}{c}, 0, 0, \frac{1}{c^2}\right)$$

$$N_\alpha \propto -\nabla_\alpha V = \left(-\frac{1}{c}, 0, 0, -1\right)$$

$$\boxed{N_\alpha = \frac{c^2}{2}\left(-\frac{1}{c}, 0, 0, -1\right)}$$

$$\left.\begin{aligned} \frac{dt}{d\lambda} &= \frac{1}{c} \\ \frac{dz}{d\lambda} &= \frac{1}{c^2} \end{aligned}\right\} \quad \frac{dt}{dz} = c \Rightarrow dz = \frac{dt}{c}$$

$$\Rightarrow z = \int \frac{dt}{c} + \text{cnst} \checkmark$$

$N_a N^a = 0$

$N_a K^\alpha = -1$

$= \ldots B_{\alpha\beta} \equiv \Theta$

$$\boxed{\Theta = \nabla_\alpha K^\alpha}$$

$N_\alpha \alpha - \nabla_\alpha V = \left(-\frac{1}{c}, 0, 0, -1\right)$

$$\boxed{N_\alpha = \frac{c^2}{2}\left(-\frac{1}{c}, 0, 0, -1\right)}$$

$\nabla_\alpha K_\beta = B K_\alpha K_\beta + K_\alpha B^\perp_\beta + B^\perp_\alpha K_\beta + \frac{1}{2}\Theta\, \omega_{\alpha\beta} + \sigma_{\alpha\beta}$

$B = \dot{c}$

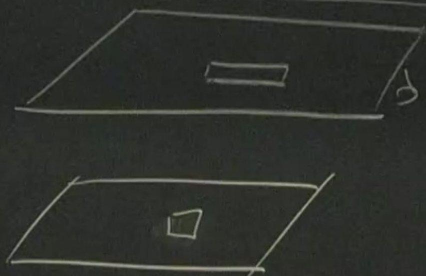
$B^\alpha_\perp = 0$

$\Theta = \nabla_\alpha K^\alpha = \frac{1}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)$

$\sigma_{xx} = \frac{a^2}{2c}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)$

$\sigma_{yy} = \frac{b^2}{2c}\left(-\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)$

Example — anisotropic cosmology



$$ds^2 = -dt^2 + a^2(t)\,dx^2 + b^2(t)\,dy^2 + c^2(t)\,dz^2$$

$$= -dt^2 + c^2(t)\,dz^2 + \dots$$

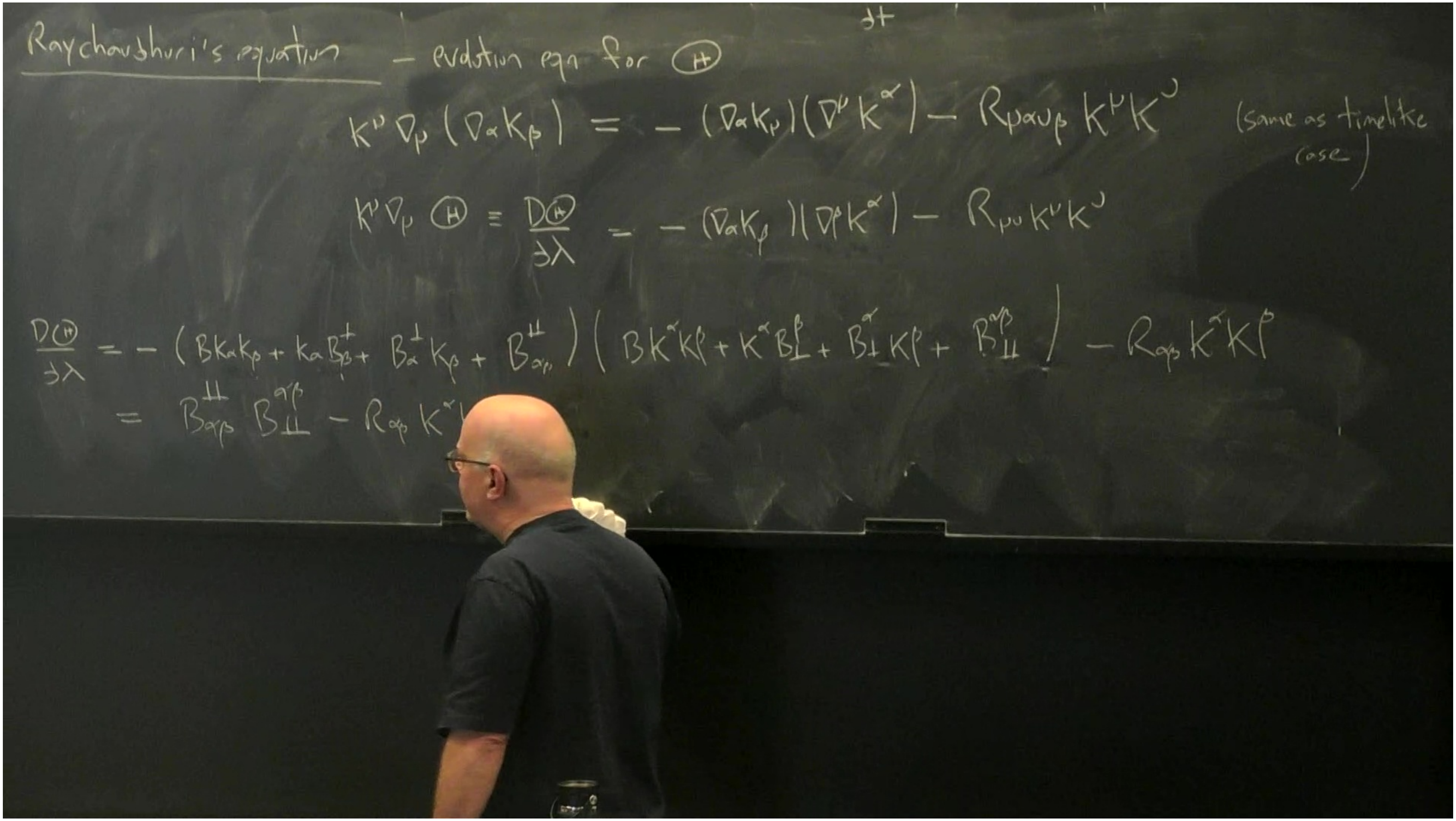
$$ds^2 = -c^2\underbrace{\left(\frac{1}{c}dt - dz\right)}_{dU}\underbrace{\left(\frac{1}{c}dt + dz\right)}_{dV} + \dots$$

$$U = \int \frac{dt}{c} - z \,, \quad V = \int \frac{dt}{c} + z$$

→ right-moving light surfaces → right moving.

Kasner spacetime —

$$a(t) = t^{\alpha}$$
$$b(t) = t^{\beta}$$
$$c(t) = t^{\gamma}$$


Raychaudhuri's equation – evolution eqn for Θ
$k^\mu \nabla_\mu (\nabla_\alpha k_\beta) = -(\nabla_\alpha k_\mu)(\nabla^\mu k^\alpha) - R_{\mu\alpha\nu\beta} k^\mu k^\nu$
(same as timelike case)
$k^\mu \nabla_\mu \Theta \equiv \frac{D\Theta}{d\lambda} = -(\nabla_\alpha k_\beta)(\nabla^\beta k^\alpha) - R_{\mu\nu} k^\mu k^\nu$
$\frac{D\Theta}{d\lambda} = -(B k_\alpha k_\beta + k_\alpha B^\perp_\beta + B^\perp_\alpha k_\beta + B^{\perp\!\perp}_{\alpha\beta})(B k^\alpha k^\beta + k^\alpha B^\beta_\perp + B^\alpha_\perp k^\beta + B^{\alpha\beta}_{\perp\!\perp}) - R_{\alpha\beta} k^\alpha k^\beta$
$= B^{\perp\!\perp}_{\alpha\beta} B^{\alpha\beta}_{\perp\!\perp} - R_{\alpha\beta} k^\alpha k$

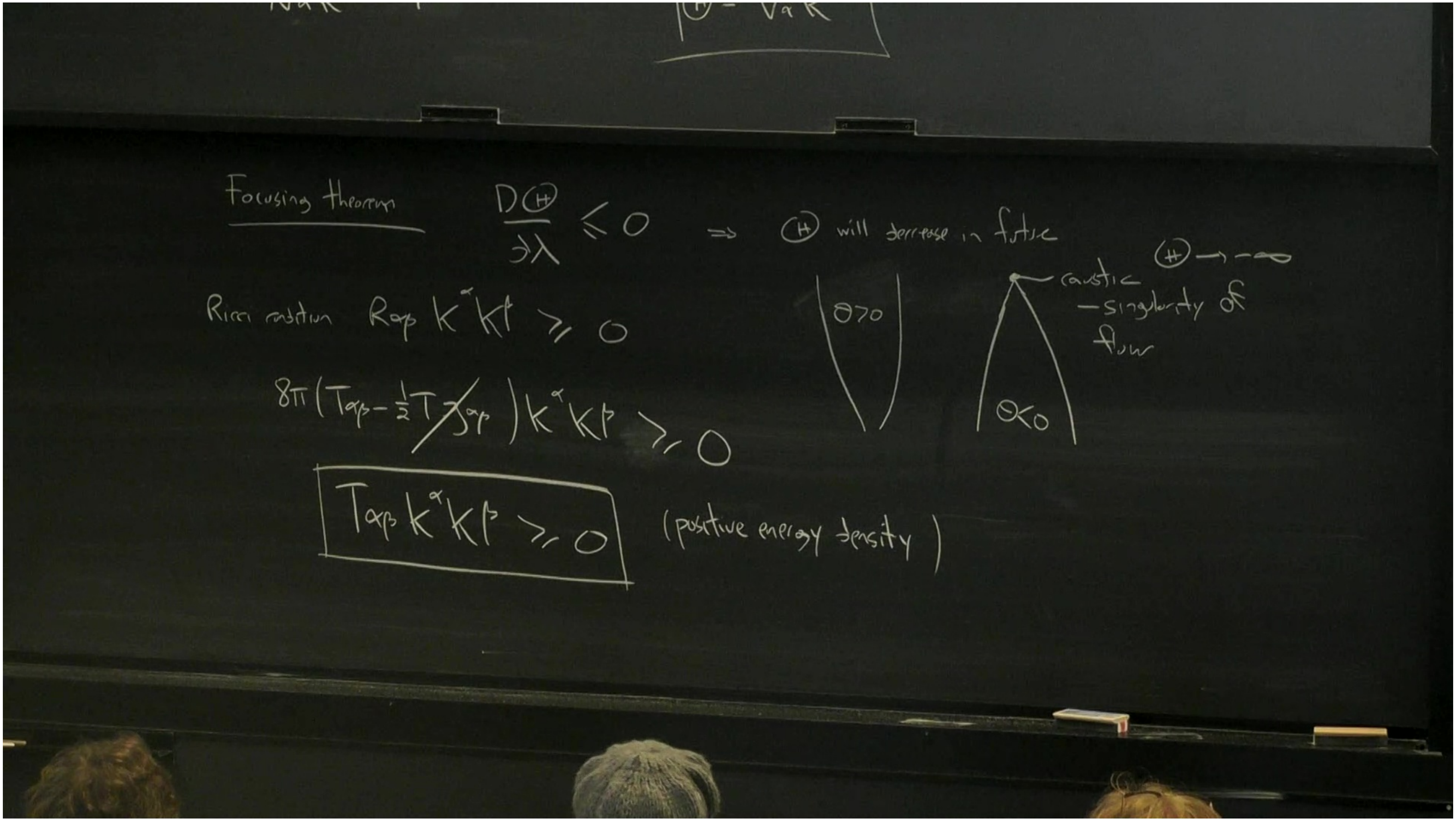
$$= B^{\perp\!\!\perp}_{\alpha\beta} B^{\alpha\beta}_{\perp\!\!\perp} - R_{\alpha\beta} K^{\alpha} K^{\beta}$$

$$= -\left(\frac{1}{2} \Theta \, \Omega_{\alpha\beta} + \sigma_{\alpha\beta} \right) \left(\frac{1}{2} \Theta \, \Omega^{\alpha\beta} + \sigma^{\alpha\beta} \right) - R_{\alpha\beta} K^{\alpha} K^{\beta}$$

$$= -\frac{1}{2} \Theta^{2} - \sigma_{\alpha\beta} \sigma^{\alpha\beta}$$

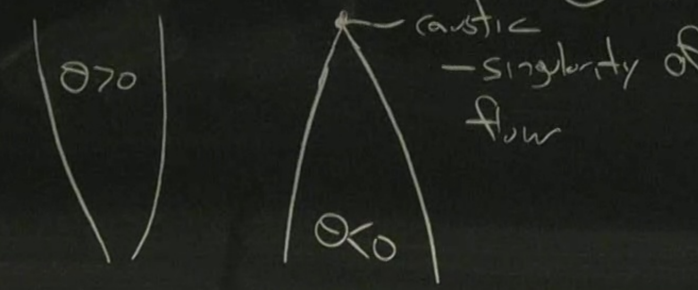
$$\boxed{\frac{D\Theta}{d\lambda} = -\left(\underbrace{\frac{1}{2} \Theta^{2}}_{\geq 0} + \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\geq 0} + \underbrace{R_{\alpha\beta} K^{\alpha} K^{\beta}}_{?} \right)}$$

Raychaudhuri


$\Theta = \nabla_a k$
Focusing theorem
$\frac{D\Theta}{d\lambda} \le 0 \Rightarrow \Theta$ will decrease in future
$\Theta \to -\infty$
caustic
— singularity of flow
$\theta > 0$
$\theta < 0$
Ricci condition $R_{\alpha\beta} k^\alpha k^\beta \ge 0$
$8\pi (T_{\alpha\beta} - \frac{1}{2} T \cancel{g_{\alpha\beta}}) k^\alpha k^\beta \ge 0$
$\boxed{T_{\alpha\beta} k^\alpha k^\beta \ge 0}$ (positive energy density)

Focusing theorem $\quad \frac{D\Theta}{d\lambda} \leq 0 \Rightarrow \Theta$ will decrease in future $\rightsquigarrow$ gravity attractive

$\Theta \to -\infty$

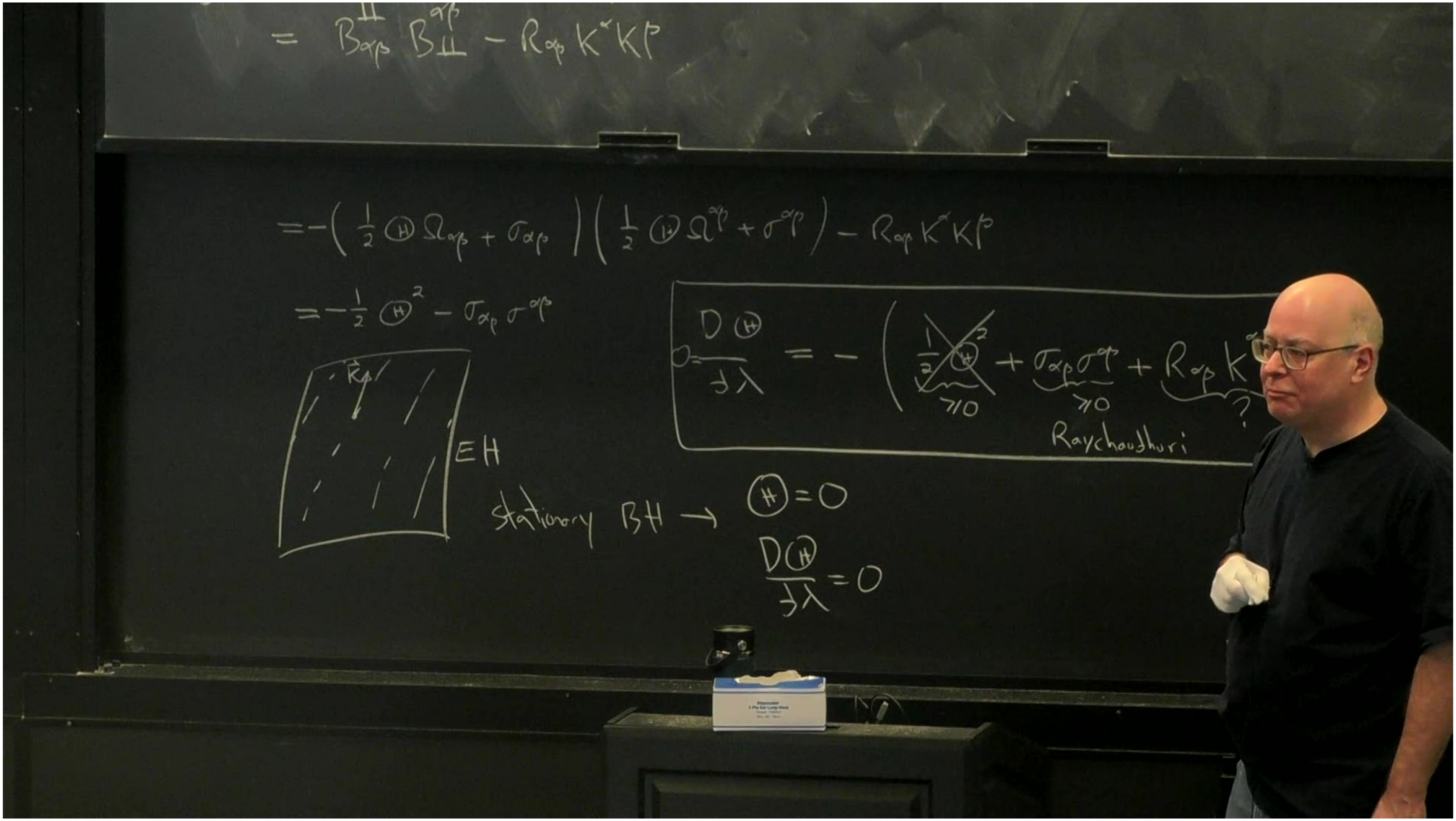


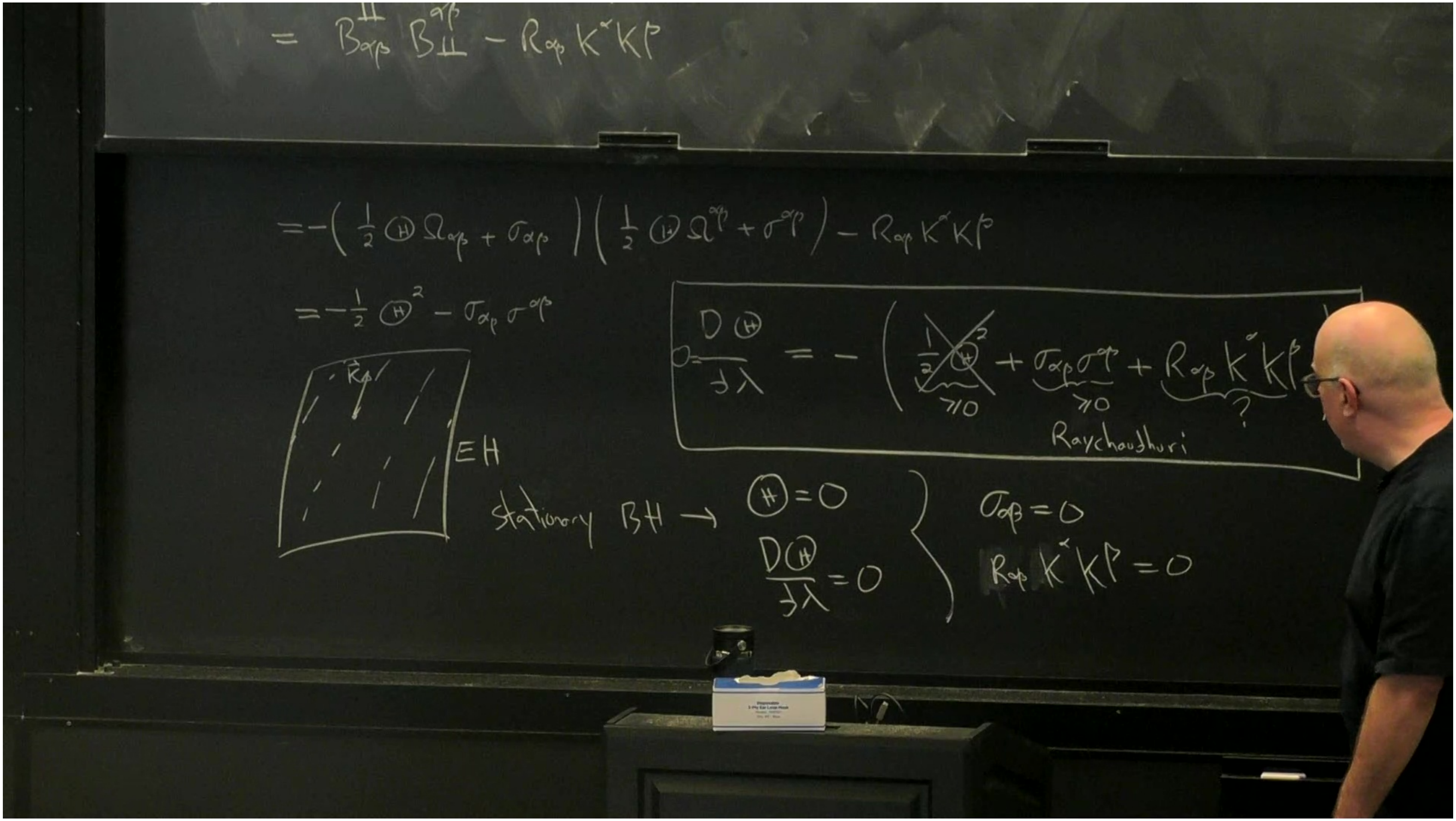
Ricci condition $\quad R_{\alpha\beta} k^\alpha k^\beta \geq 0$

EFE

$$8\pi\left(T_{\alpha\beta} - \cancel{\tfrac{1}{2} T g_{\alpha\beta}}\right) k^\alpha k^\beta \geq 0$$

$$\boxed{T_{\alpha\beta} k^\alpha k^\beta \geq 0}$$ (positive energy density)


$= B_{\alpha\beta} B_{\perp}^{\alpha\beta} - R_{\alpha\beta} K^\alpha K^\beta$
$= -\left(\frac{1}{2}\Theta\Omega_{\alpha\beta} + \sigma_{\alpha\beta}\right)\left(\frac{1}{2}\Theta\Omega^{\alpha\beta} + \sigma^{\alpha\beta}\right) - R_{\alpha\beta} K^\alpha K^\beta$
$= -\frac{1}{2}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta}$
$\frac{D\Theta}{d\lambda} = -\left(\frac{1}{2}\Theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + R_{\alpha\beta}K^\alpha K^\beta\right)$
≥ 0
≥ 0
?
Raychaudhuri
$\vec{K}_A$
EH
stationary BH $\rightarrow$ $\Theta = 0$
$\frac{D\Theta}{d\lambda} = 0$


$= B^{\perp\perp}_{\alpha\beta} B_{\perp\perp}^{\alpha\beta} - R_{\alpha\beta} k^\alpha k^\beta$
$= -\left(\frac{1}{2}\Theta \Omega_{\alpha\beta} + \sigma_{\alpha\beta}\right)\left(\frac{1}{2}\Theta\Omega^{\alpha\beta} + \sigma^{\alpha\beta}\right) - R_{\alpha\beta}k^\alpha k^\beta$
$= -\frac{1}{2}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta}$
$\frac{D\Theta}{d\lambda} = -\left(\underbrace{\cancel{\frac{1}{2}\Theta^2}}_{\geq 0} + \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\geq 0} + \underbrace{R_{\alpha\beta}k^\alpha k^\beta}_{?}\right)$
Raychoudhuri
$\vec{k}$
EH
stationary BH → $\Theta = 0$
$\frac{D\Theta}{d\lambda} = 0$
$\sigma_{\alpha\beta} = 0$
$R_{\alpha\beta}k^\alpha k^\beta = 0$