Title: Advanced General Relativity - 240207 (afternoon)

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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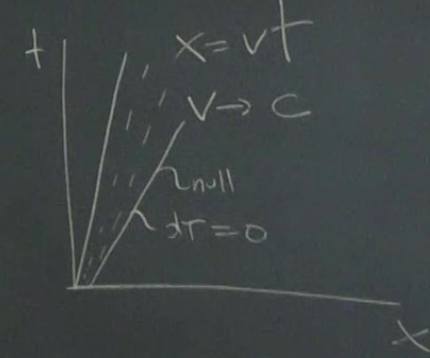
URL: https://pirsa.org/24020004

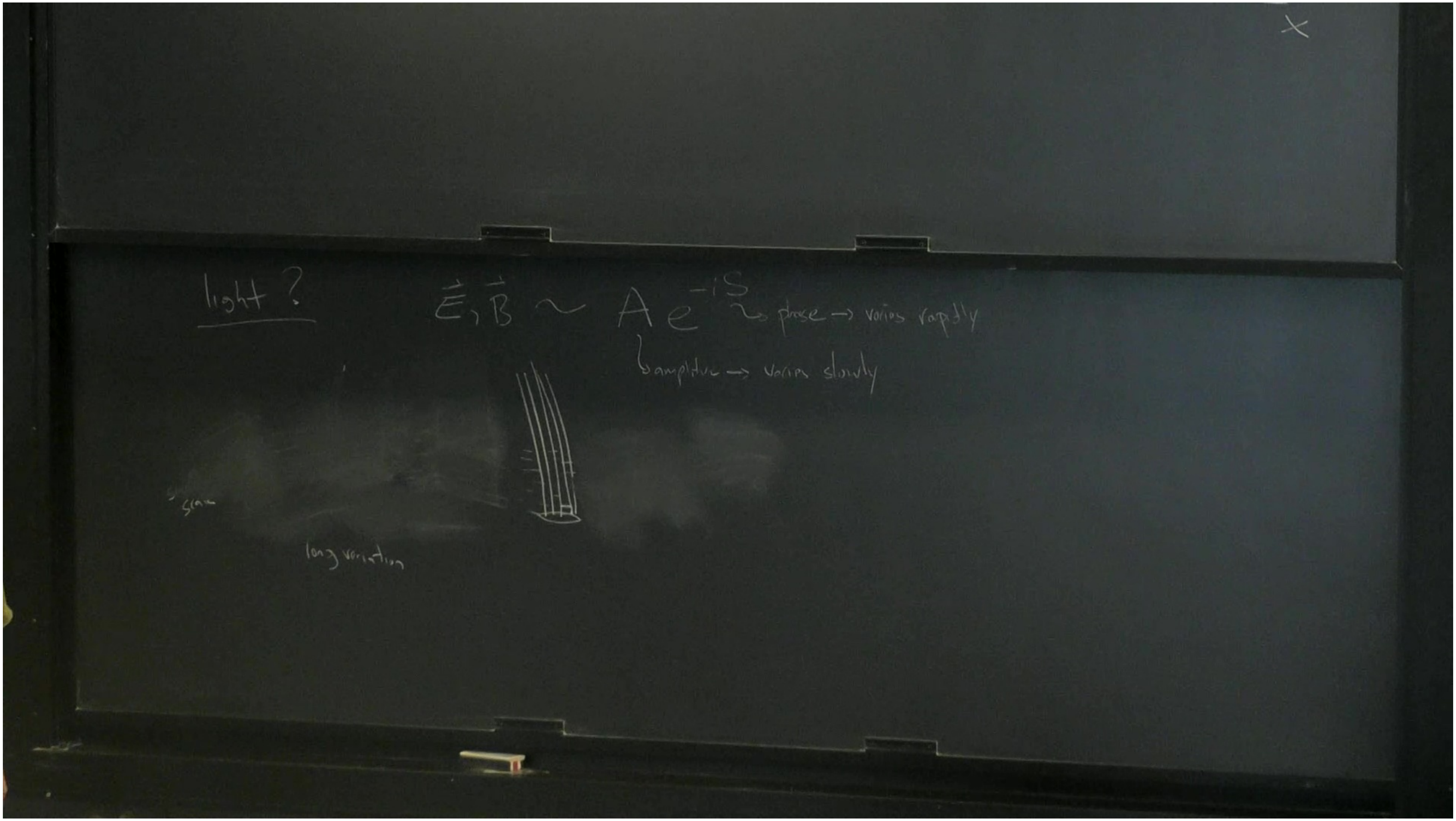
Null geodesics

$$\frac{Dt^\alpha}{d\lambda} = k\,t^\alpha \qquad t_\alpha t^\alpha < 0$$

$$\xrightarrow{\text{null}}$$

$$\frac{Dt^\alpha}{d\lambda} = k\,t^\alpha \qquad t_\alpha t^\alpha = 0$$




light ?
E, B ~ Ae^{-iS}
phase → varies rapidly
amplitude → varies slowly
long variation

Dipole scalar field

scalar fields Φ, $\Box \Phi = 0$ (flat spacetime)

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\boxed{\Phi = \underbrace{\frac{1}{r}\left(1 + \frac{i}{\omega r}\right)}_{\substack{\text{amplitude} \\ \equiv A}} \cos\theta \; e^{-i\underbrace{\omega(t-r)}_{\text{phase} \equiv S}}}$$

$\omega \gg 1$

$$\delta r = \lambda = \frac{2\pi}{\omega} \Rightarrow \delta S = 2\pi \quad \text{(large variation)}$$

$$\delta A \sim \frac{\delta r}{r^2} \rightarrow \frac{\delta A}{A} \sim r\frac{\delta r}{r^2} = \frac{\lambda}{r}$$

wave zone: $r \gg \lambda$

$\delta A / A \ll 1$

$$S = \omega(t-r)$$

$$\boxed{K_\alpha \equiv -\nabla_\alpha(S/\omega)} = \text{normal to phase fronts } S = \text{const.}$$

$$= -\nabla_\alpha(t-r) = \begin{array}{cccc} t & r & \theta & \phi \\ (-1, & +1, & 0, & 0) \end{array}$$

$$\left.\begin{array}{l} K_\alpha = (-1,1,0,0) \\ K^\alpha = (1,1,0,0) \end{array}\right\} \boxed{K_\alpha K^\alpha = 0}$$

$$K^\beta \nabla_\beta K_\alpha = -K^\beta \nabla_\beta \nabla_\alpha (S/\omega) = -K^\beta \nabla_{\beta\alpha}(S/\omega)$$

$$= -K^\beta \nabla_{\alpha\beta}(S/\omega) = +K^\beta \nabla_\alpha K_\beta = \frac{1}{2}\nabla_\alpha(K^\beta K_\beta) = 0$$

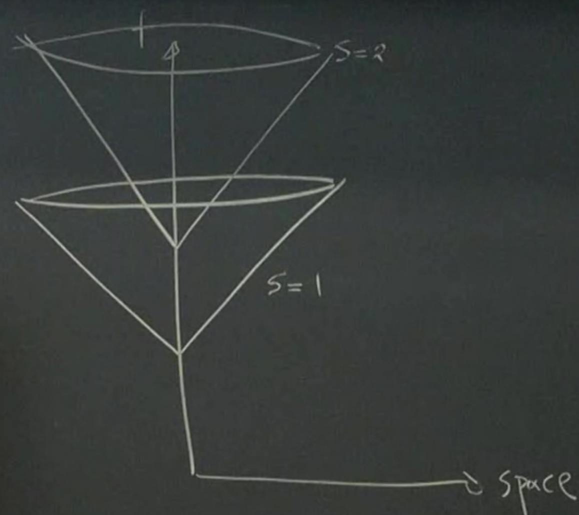
$$\boxed{K^\beta \nabla_\beta K^\alpha = 0}$$

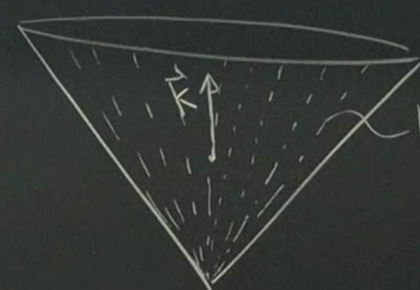
$$= -K^\beta \nabla_{\alpha\beta}(S/\omega) = +K^\beta \nabla_\alpha K_\beta = \frac{1}{2}\nabla_\alpha\left(K^\beta K_\beta\right) = 0 \qquad \boxed{K^\beta \nabla_\beta K^\alpha = 0}$$

<u>Geodesics</u>

$$k^\alpha = \frac{\partial x^\alpha}{\partial \lambda} = (1,1,0,0) \quad \left\{ \frac{\partial t}{\partial\lambda} = 1, \ \frac{\partial r}{\partial\lambda} = 1, \ \frac{\partial\theta}{\partial\lambda} = 0, \ \frac{\partial\phi}{\partial\lambda} = 0 \right.$$

$$\boxed{\begin{array}{ll} t = \lambda + \text{const} & \theta = \text{const} \\ r = \lambda + \text{const} & \phi = \text{const} \end{array}} \qquad \boxed{\begin{array}{l} S/\omega = t - r \\ = \text{const} \end{array}}$$





light rays

K_α = normal to phasefront

K^α = tangent to each null geodesic

each null geodesic is tangent to surface

K^α is <u>normal</u> and <u>tangent</u>!

amplitude $\equiv A$

$$\delta r = \lambda = \frac{2\pi}{\omega} \Rightarrow \delta S = 2\pi \quad \text{(large variation)}$$

$$\delta A \sim \frac{\delta r}{r^2} \rightarrow \frac{\delta A}{A} \sim r\frac{\delta r}{r^2} = \frac{\lambda}{r}$$

wave zone: $r \gg \lambda$

$\delta A/A \ll 1$

Calculate $\nabla_\alpha k_\beta$:

$$\begin{array}{c|cccc} & t & r & \theta & \varphi \\ \hline t & & & & \\ r & & & & \\ \theta & & & & \\ \varphi & & & & \end{array}\quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{pmatrix}$$

$$\nabla_\alpha k^\alpha = g^{\alpha\beta}\nabla_\alpha k_\beta = \frac{1}{r^2}r + \frac{1}{r^2\cancel{\sin^2\theta}}r\cancel{\sin^2\theta} = \frac{2}{r} = \frac{1}{4\pi r^2}\frac{\partial}{\partial\lambda}4\pi r^2$$

amplitude $\equiv A$

$\delta r = \lambda = \frac{2\pi}{\omega} \Rightarrow \delta S = 2\pi$ (large variation)

$$\delta A \sim \frac{\delta r}{r^2} \to \frac{\delta A}{A} \sim r\,\frac{\delta r}{r^2} = \frac{\lambda}{r}$$

wave zone: $r \gg \lambda$

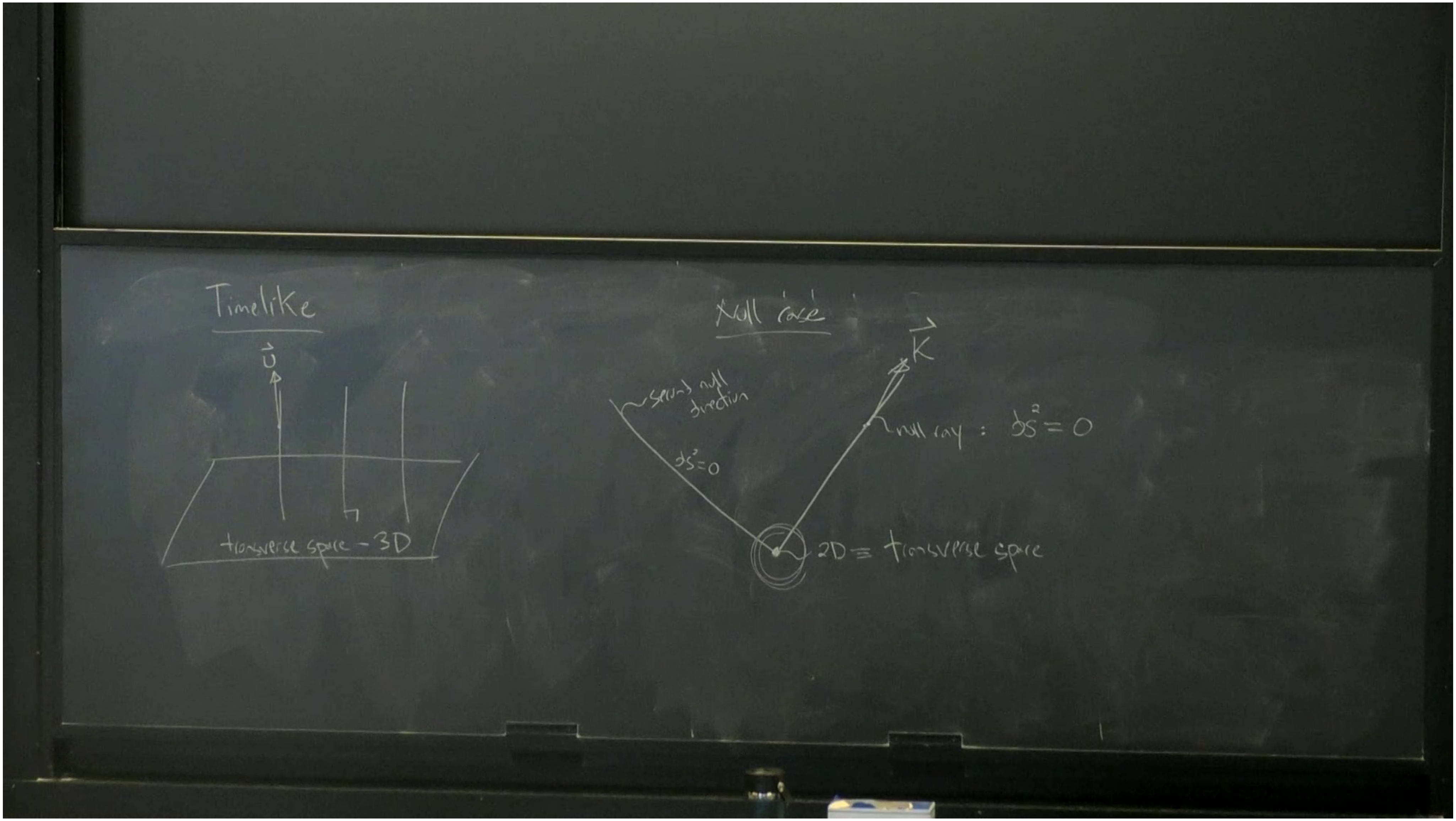
$\delta A/A \ll 1$

Calculate $\nabla_\alpha k_\beta$:

$$\begin{array}{c|cccc} & t & r & \theta & \varphi \\ \hline t & & & & \\ r & & & & \\ \theta & & & & \\ \varphi & & & & \end{array} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{pmatrix}$$

$$\nabla_\alpha k^\alpha = g^{\alpha\beta}\nabla_\alpha k_\beta = \frac{1}{r^2}\,r + \frac{1}{r^2\cancel{\sin^2\theta}}\,r\cancel{\sin^2\theta} = \frac{2}{r} = \frac{1}{4\pi r^2}\frac{\partial}{\partial\lambda}4\pi r^2$$

fractional change in cross-sectional area.



Timelike
$\vec{U}$
transverse space - 3D
Null case
$\vec{K}$
second null direction
$ds^2=0$
null ray : $ds^2 = 0$
2D ≡ transverse space

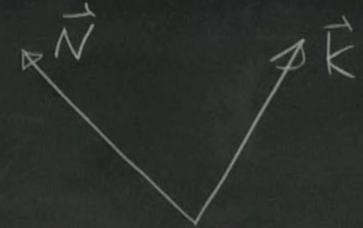
$r\sin\theta$ $\quad r \quad 4\pi r^2 \quad \partial\lambda$

fractional change in cross-sectional area.

<u>Transverse projection</u>

— Given null vector k^α

— Choose a second null direction: N^α

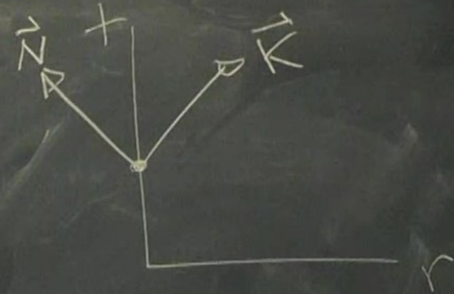


$$\boxed{\begin{aligned} k_\alpha k^\alpha &= 0 \\ N_\alpha N^\alpha &= 0 \\ N_\alpha k^\alpha &= -1 \end{aligned}}$$

(N^α is <u>not</u> unique)

Example: $k^\alpha = (1,1,0,0)$

$$k_\alpha = (-1,1,0,0)$$



$$\boxed{N_\alpha = \frac{1}{2}\left(-1,-1,0,0\right)}$$

$$N^\alpha = \frac{1}{2}\left(1,-1,0,0\right)$$

$$N^\mu = \left(\frac{1}{2}(1+\alpha^2+\beta^2),\ -\frac{1}{2}(1-\alpha^2-\beta^2),\ \frac{\alpha}{r},\ \frac{\beta}{r\sin\theta}\right) \longrightarrow \begin{array}{l} N_\mu N^\mu = 0 \\ N_\mu k^\mu = -1 \end{array}$$

for <u>any</u> (α,β)

Decomposition of vector

$$\underset{\overset{\uparrow}{4}}{A^\alpha} = \underset{\overset{\uparrow}{1}}{A_1} K^\alpha + \underset{\overset{\uparrow}{1}}{A_2} N^\alpha + \underset{\overset{\uparrow}{2}}{A_\perp^\alpha} \;; \qquad A_\perp^\alpha K_\alpha = 0, \quad A_\perp^\alpha N_\alpha = 0$$

$$K_\alpha A^\alpha = - A_2$$

$$N_\alpha A^\alpha = - A_1$$

Decomposition of vector

$$\boxed{\underset{4}{A^\alpha} = \underset{1}{A_1} K^\alpha + \underset{1}{A_2} N^\alpha + \underset{2}{A_\perp^\alpha}}\,;$$

$$\boxed{\begin{aligned} A_\perp^\alpha K_\alpha &= 0 \\ A_\perp^\alpha N_\alpha &= 0 \end{aligned}}$$

$$K_\alpha A^\alpha = -A_2$$

$$N_\alpha A^\alpha = -A_1$$

$$\boxed{\begin{aligned} A_1 &= -N_\alpha A^\alpha \\ A_2 &= -K_\alpha A^\alpha \end{aligned}}$$

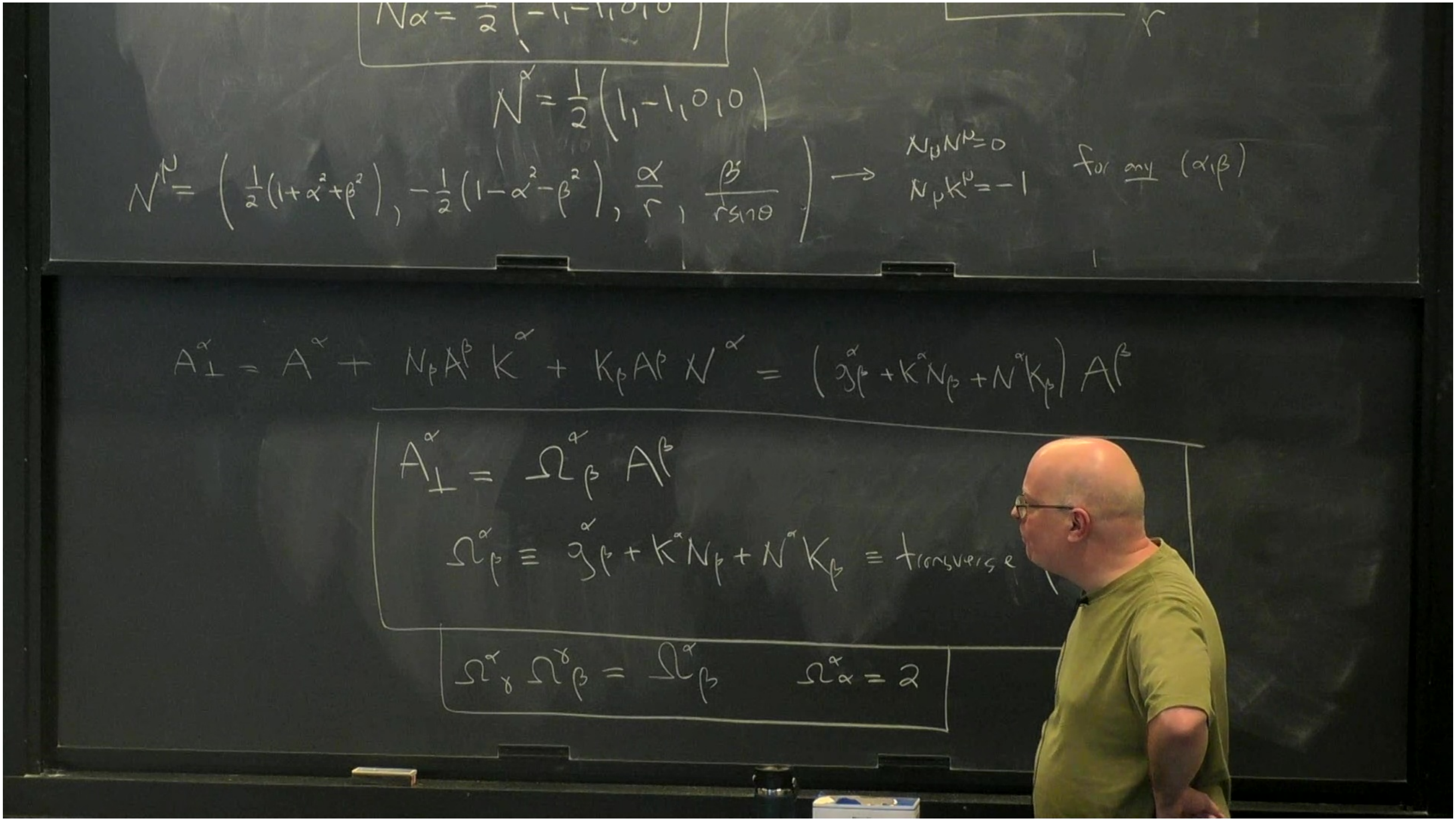
$$N_\alpha = \frac{1}{2}(-1,-1,0,0)$$

$$N^\alpha = \frac{1}{2}(1,-1,0,0)$$

$$N^\mu = \left(\frac{1}{2}(1+\alpha^2+\beta^2),\ -\frac{1}{2}(1-\alpha^2-\beta^2),\ \frac{\alpha}{r},\ \frac{\beta}{r\sin\theta}\right) \longrightarrow \begin{array}{l} N_\mu N^\mu = 0 \\ N_\mu K^\mu = -1 \end{array} \quad \text{for } \underline{\text{any}}\ (\alpha,\beta)$$

$$A_\perp^\alpha = A^\alpha + N_\beta A^\beta K^\alpha + K_\beta A^\beta N^\alpha = \left(g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta\right) A^\beta$$

$$\boxed{\begin{array}{l} A_\perp^\alpha = \Omega^\alpha{}_\beta A^\beta \\ \Omega^\alpha{}_\beta \equiv g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta \equiv \text{transverse projector.} \end{array}}$$


$N_\alpha = \frac{1}{2}(-1,-1,0,0)$
$N^\alpha = \frac{1}{2}(1,-1,0,0)$
$N^\mu = \left(\frac{1}{2}(1+\alpha^2+\beta^2), -\frac{1}{2}(1-\alpha^2-\beta^2), \frac{\alpha}{r}, \frac{\beta}{r\sin\theta}\right) \longrightarrow$
$N_\mu N^\mu = 0$
$N_\mu K^\mu = -1$
for any (α,β)
$A_\perp^\alpha = A^\alpha + N_\beta A^\beta K^\alpha + K_\beta A^\beta N^\alpha = (g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta) A^\beta$
$A_\perp^\alpha = \Omega^\alpha{}_\beta A^\beta$
$\Omega^\alpha{}_\beta \equiv g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta \equiv$ transverse
$\Omega^\alpha{}_\gamma \Omega^\gamma{}_\beta = \Omega^\alpha{}_\beta$
$\Omega^\alpha{}_\alpha = 2$