Title: Advanced General Relativity - 240228

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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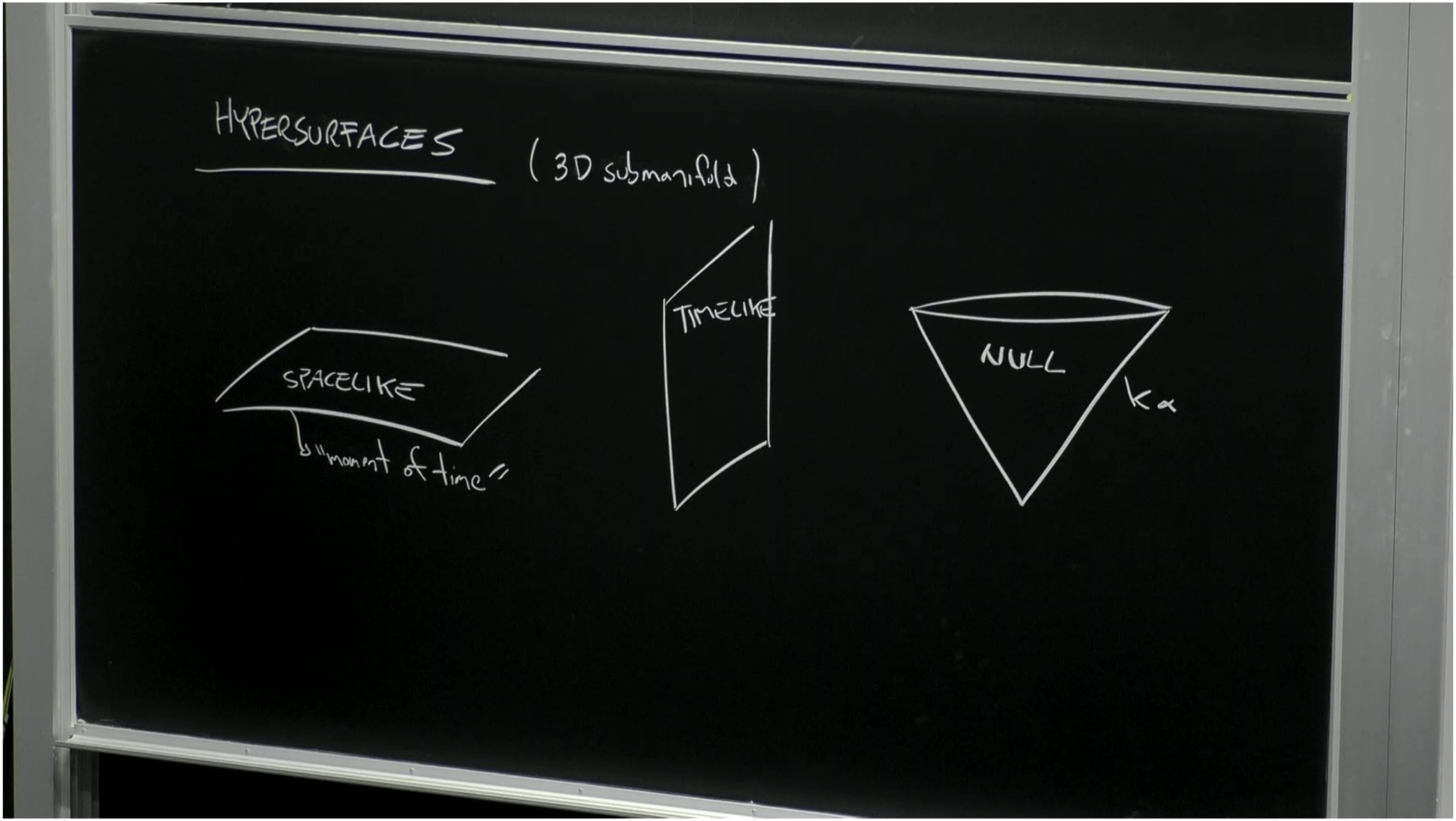
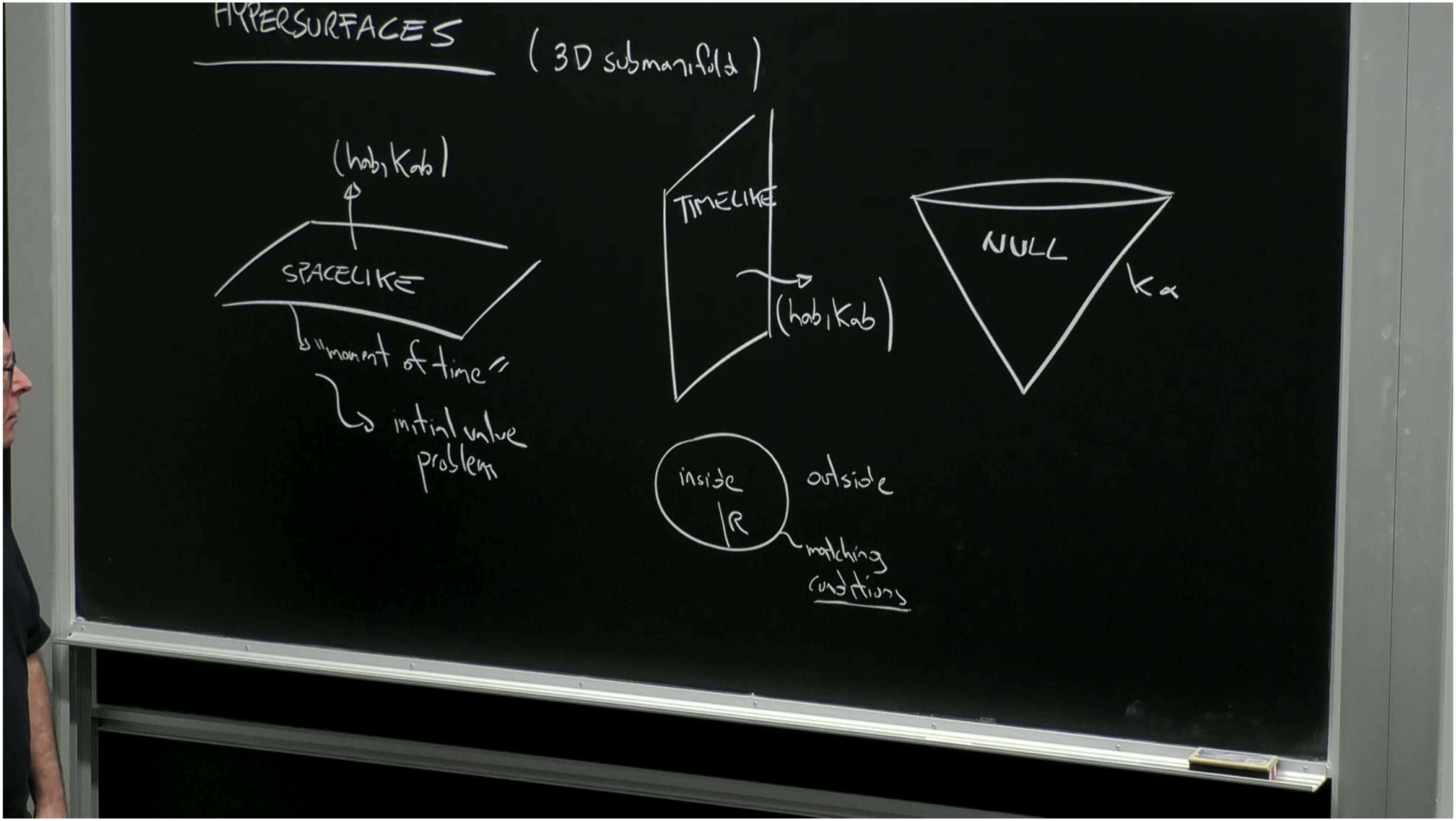

HYPERSURFACES
(3D submanifold)
SPACELIKE
"moment of time"
TIMELIKE
NULL
k_α

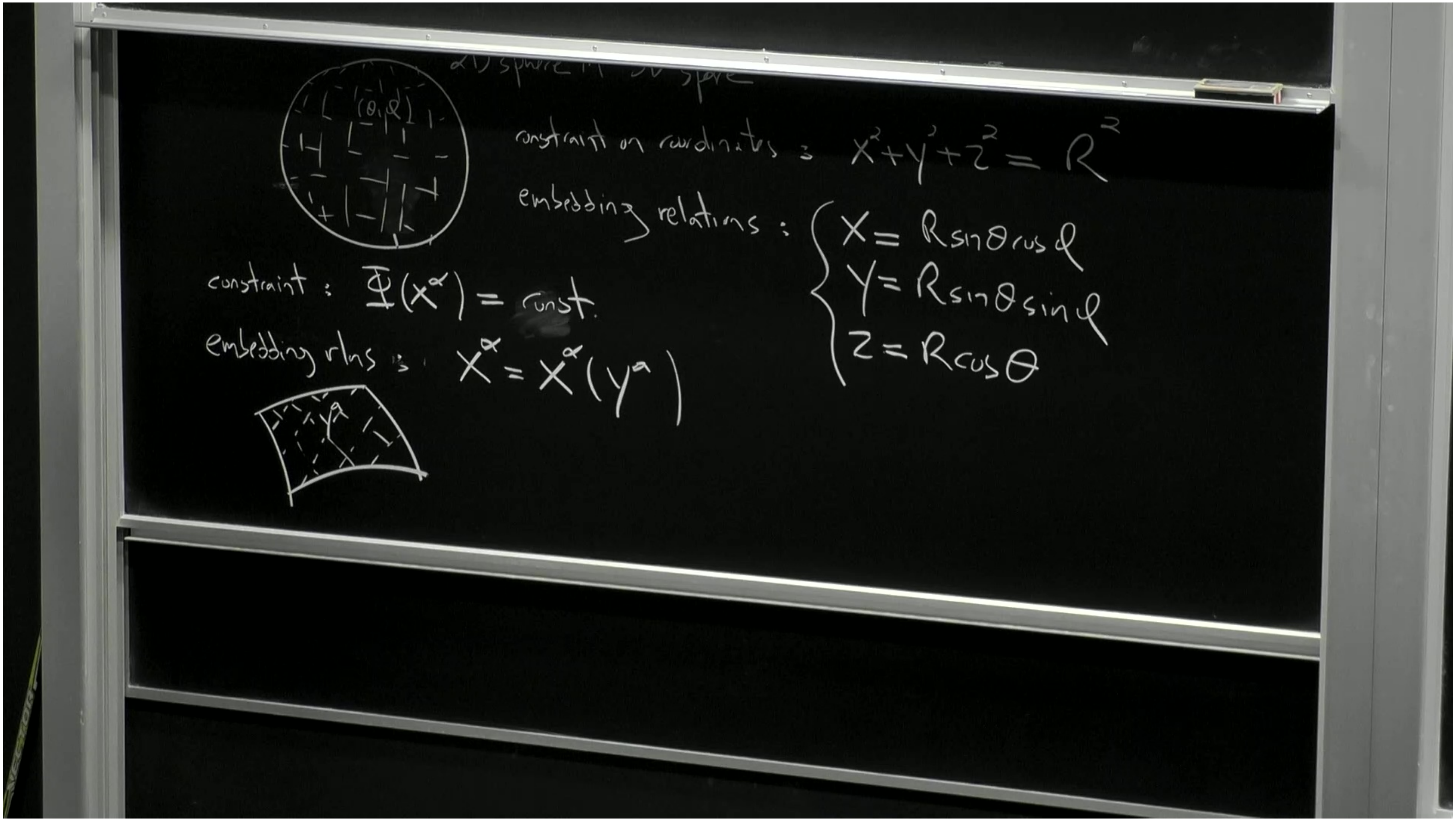

HYPERSURFACES
(3D submanifold)
(hab, Kab)
SPACELIKE
"moment of time"
initial value problem
TIMELIKE
(hab, Kab)
NULL
kα
inside
R
outside
matching conditions

2D sphere in 3D space

constraint on coordinates: $X^2+Y^2+Z^2=R^2$

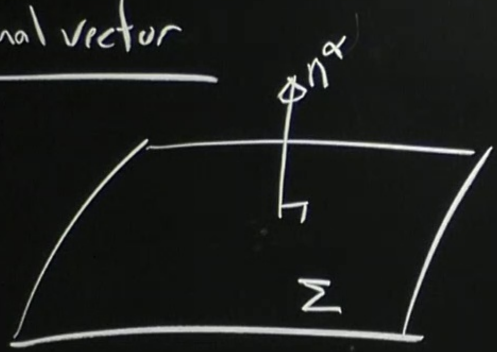
embedding relations:
$$\begin{cases} X = R\sin\theta\cos\varphi \\ Y = R\sin\theta\sin\varphi \\ Z = R\cos\theta \end{cases}$$

constraint: $\Phi(X^\alpha)=0$



2D sphere in 3D space
(θ,φ)
constraint on coordinates : $x^2+y^2+z^2=R^2$
embedding relations : $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$
constraint : $\Phi(x^\alpha) = \text{const.}$
embedding rlns : $x^\alpha = x^\alpha(y^a)$
y^a

Normal vector



Normal vector — unit vector $n_\alpha n^\alpha = \varepsilon = \begin{cases} -1 & \Sigma \text{ spacelike} \\ +1 & \Sigma \text{ timelike} \end{cases}$

— point in direction of increasing Φ (convention)

$$n_\alpha \propto \nabla_\alpha \Phi$$

$$\boxed{\begin{aligned} n_\alpha &= \varepsilon \rho \nabla_\alpha \Phi \\ \rho &= \left| g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \right|^{-1/2} \end{aligned}}$$

Ex: $t = $ const

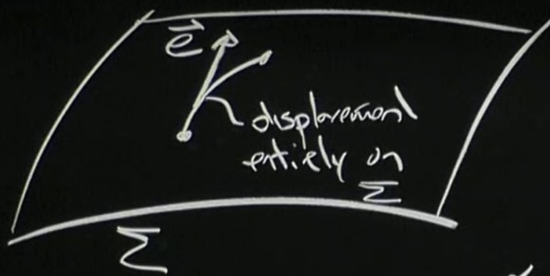
$\Phi = t = $ const

$\nabla_\alpha \Phi = (1, 0, 0, 0)$

$n_\alpha = (-1, 0, 0, 0)$

$n^\alpha = (1, 0, 0, 0)$

tangent vectors



Intrinsic displacement

$$x^\alpha = x^\alpha(y^a) \rightarrow \text{curves on } \Sigma$$

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right) dy^a = e^\alpha_a \, dy^a$$

$$e^\alpha_a \equiv \frac{\partial x^\alpha}{\partial y^a} \equiv \text{tangent vectors (basis on } \Sigma)$$

→ spacetime vector

hypersurface co-vector.

tangent vectors

Intrinsic displacement

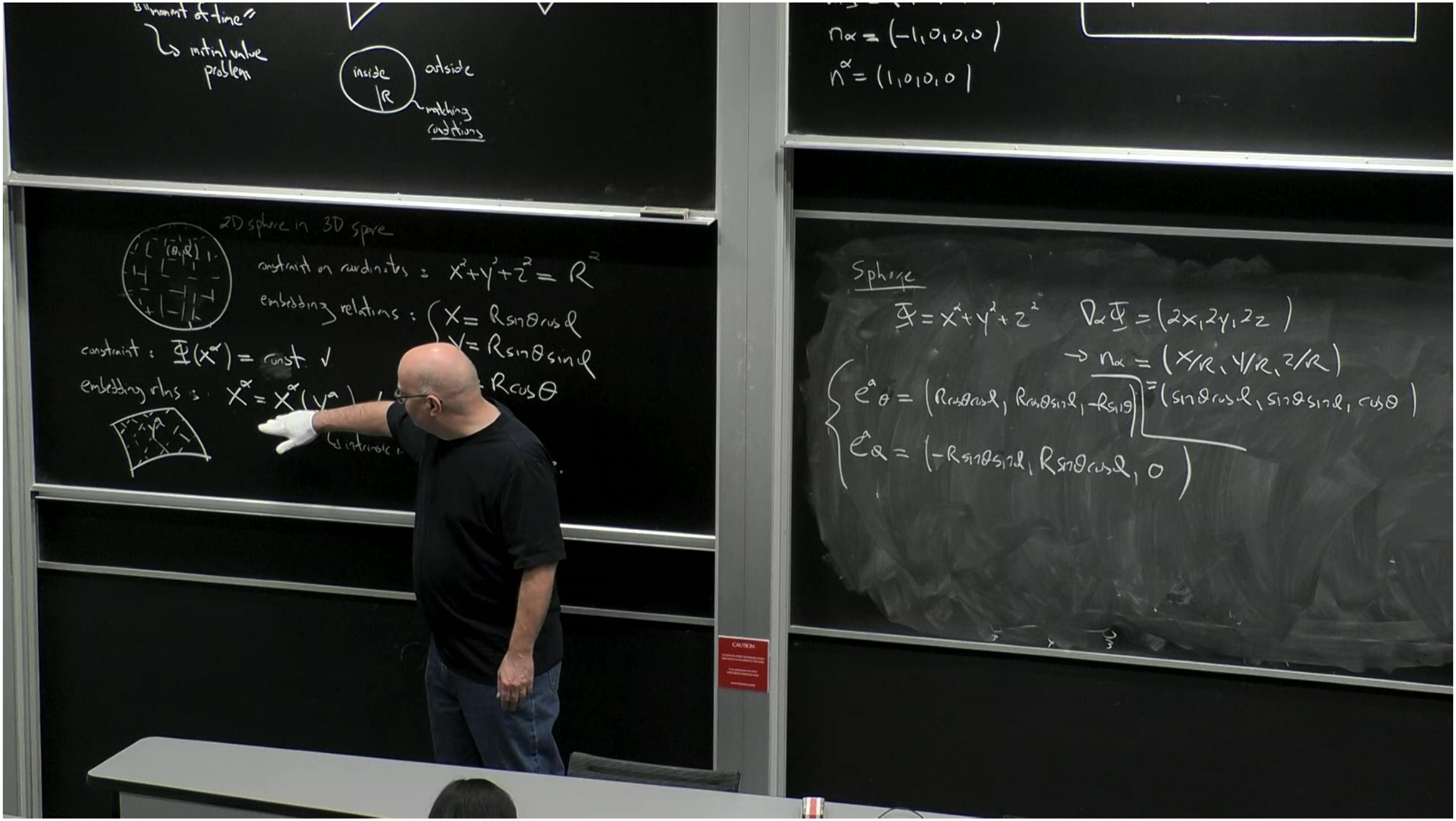
$$x^\alpha = x^\alpha(y^a) \rightarrow \text{curves on } \Sigma$$

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right) dy^a = e^\alpha_a \, dy^a$$

$$\boxed{e^\alpha_a \equiv \frac{\partial x^\alpha}{\partial y^a}} \equiv \text{tangent vectors (basis on } \Sigma)$$

$\rightarrow$ spacetime vector

hypersurface co-vector.


"moment of time"
initial value problem
inside
R
outside
matching conditions
$n_\alpha = (-1,0,0,0)$
$n^\alpha = (1,0,0,0)$
2D sphere in 3D space
constraint on coordinates : $x^2+y^2+z^2 = R^2$
embedding relations : $x = R\sin\theta\cos\varphi$
$y = R\sin\theta\sin\varphi$
$R\cos\theta$
constraint : $\Phi(x^\alpha) = \text{const}$ ✓
embedding rlns : $x^\alpha = x^\alpha(y^a)$
Sphere
$\Phi = x^2+y^2+z^2$
$\nabla_\alpha \Phi = (2x, 2y, 2z)$
$\rightarrow n_\alpha = (x/R, y/R, z/R)$
$e^a{}_\theta = (R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, -R\sin\theta)$
$= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$
$e^a{}_\varphi = (-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0)$
CAUTION

Sphere

$$\Phi = x^2 + y^2 + z^2 \qquad \nabla_\alpha \Phi = (2x, 2y, 2z)$$

$$\rightarrow n_\alpha = (x/R, y/R, z/R) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\begin{cases} e^a{}_\theta = (R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, -R\sin\theta) \\ e^a{}_\varphi = (-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0) \end{cases}$$

$$h_{\theta\theta} = R^2$$

$$h_{\varphi\varphi} = R^2\sin^2\theta$$

$$h_{\theta\varphi} = 0$$

$$ds^2\big|_{\text{sphere}} = R^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Intrinsic geometry : $h_{ab} \to \Gamma_{bc} \to R_{bcd}$

h^{ab} : inverse metric

$$h^{ab} h_{bc} = \delta^a{}_c$$

$$h = \det(h_{ab})$$

completeness relation :

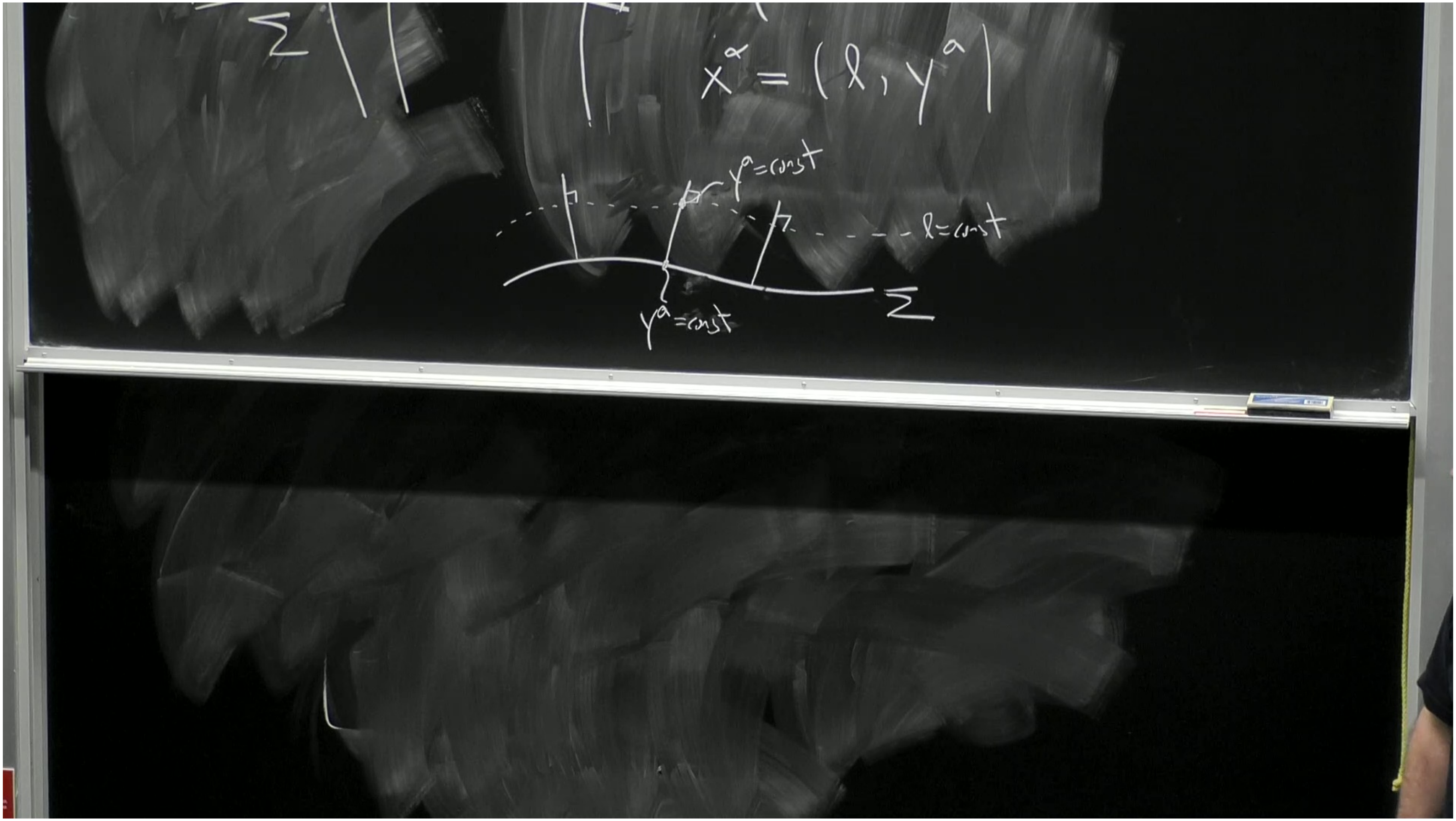
$$\boxed{g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + \underbrace{h^{ab} e^\alpha{}_a e^\beta{}_b}_{\text{"}p^{\alpha\beta}\text{"}}}$$

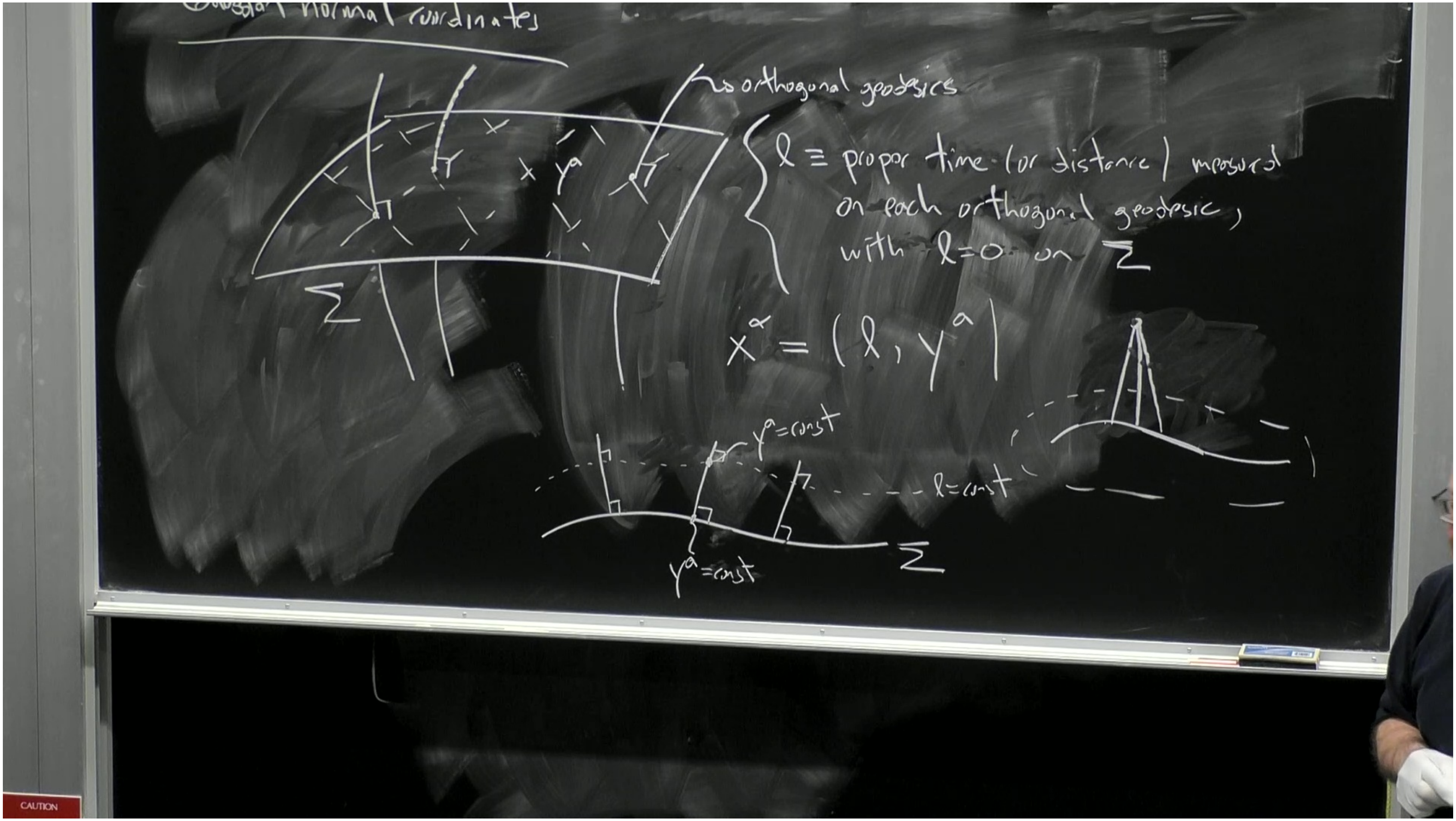
Check this!

... orthogonal geodesics

$\ell \equiv$ proper time (or distance) measured on each orthogonal geodesic, with $\ell = 0$ on Σ

$$x^\alpha = (\ell, y^a)$$


Σ
x^α = (l, y^a)
y^a=const
l=const
Σ
y^a=const


normal coordinates
orthogonal geodesics
ℓ ≡ proper time (or distance) measured on each orthogonal geodesic, with ℓ=0 on Σ
Σ
y^a
x^α = (ℓ, y^a)
y^a=const
ℓ=const
y^a=const
Σ

$$\boxed{\begin{gathered} ds^2 = \varepsilon\, dl^2 + g_{ab}(l, y)\, dy^a dy^b \\ h_{ab} = g_{ab}(l=0, y) \end{gathered}}$$

$$x^\alpha \overset{*}{=} (l, y^a)$$

$$n^\alpha \overset{*}{=} (1, 0, 0, 0)$$

$$\det(g_{\alpha\beta}) \overset{*}{=} \varepsilon \det(g_{ab})$$

$$n_\alpha \overset{*}{=} (\varepsilon, 0, 0, 0)$$

$$e^\alpha_1 \overset{*}{=} (0, 1, 0, 0)$$

$$dV = \sqrt{-g}\, d^4x$$

$$e^\alpha_2 \overset{*}{=} (0, 0, 1, 0)$$

$$e^\alpha_3 \overset{*}{=} (0, 0, 0, 1)$$

$$\boxed{\begin{gathered} ds^2 = \varepsilon\, dl^2 + g_{ab}(l, y)\, dy^a dy^b \\ h_{ab} = g_{ab}(l=0, y) \end{gathered}}$$

$$x^\alpha \overset{*}{=} (l, y^a)$$

$$n^\alpha \overset{*}{=} (1, 0, 0, 0)$$

$$n_\alpha \overset{*}{=} (\varepsilon, 0, 0, 0)$$

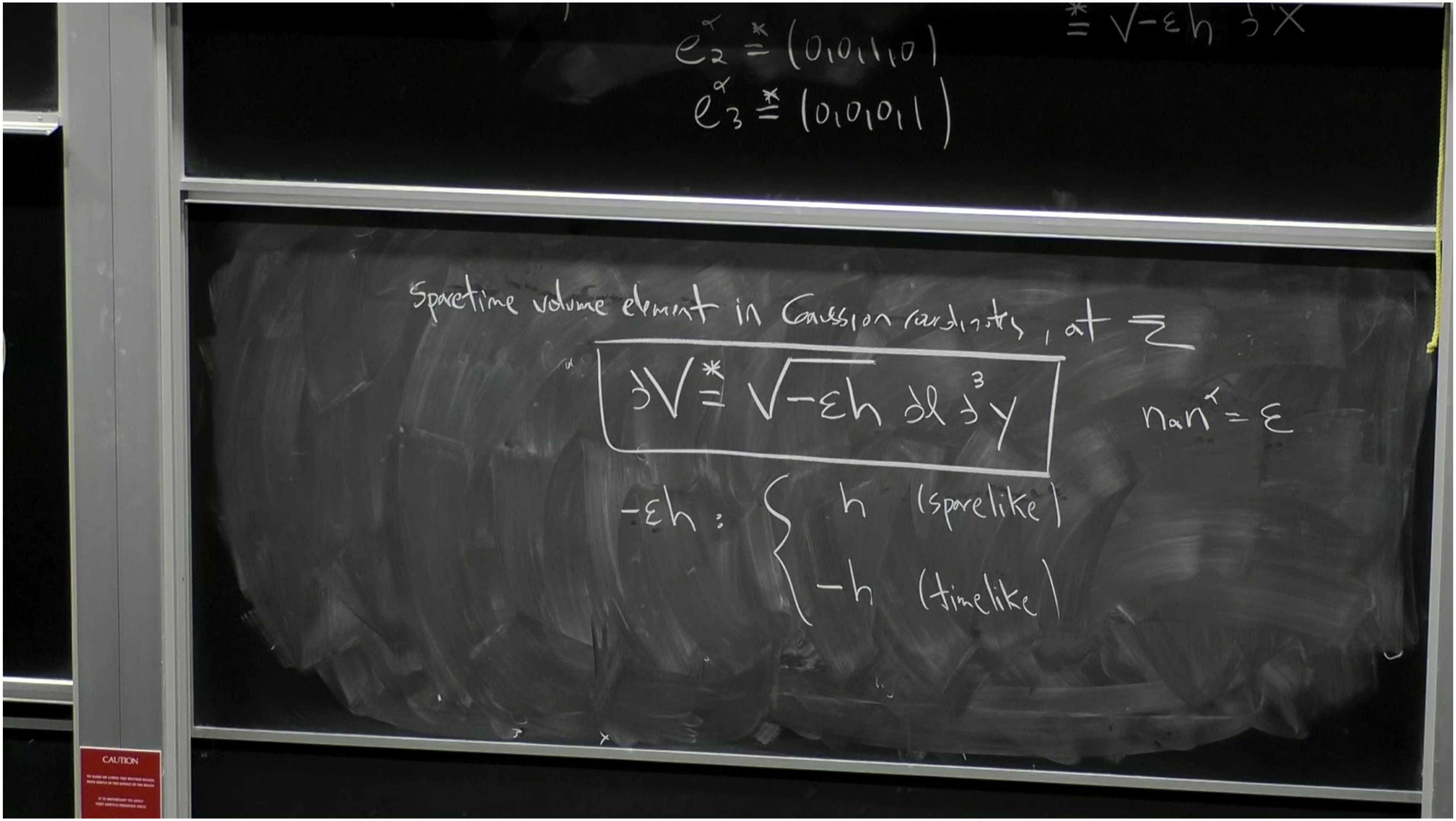
$$\det(g_{\alpha\beta}) \overset{*}{=} \varepsilon \det(g_{ab})$$

$$e^\alpha_1 \overset{*}{=} (0, 1, 0, 0)$$

$$e^\alpha_2 \overset{*}{=} (0, 0, 1, 0)$$

$$e^\alpha_3 \overset{*}{=} (0, 0, 0, 1)$$

$$dV = \sqrt{-g}\, d^4x \overset{*}{=} \sqrt{-\varepsilon h}\, d^4x$$


$e_2^\alpha \overset{*}{=} (0,0,1,0)$
$e_3^\alpha \overset{*}{=} (0,0,0,1)$
$\overset{*}{=} \sqrt{-\varepsilon h}\, d^3x$
Spacetime volume element in Gaussian coordinates, at Σ
$\boxed{dV \overset{*}{=} \sqrt{-\varepsilon h}\, d\ell\, d^3y}$
$n_\alpha n^\alpha = \varepsilon$
$-\varepsilon h :$ $\begin{cases} h & \text{(spacelike)} \\ -h & \text{(timelike)} \end{cases}$