Title: Advanced General Relativity - 240214

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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Null cones

$$S = t - r$$

$$K_\alpha = -\nabla_\alpha S = (-1, 1, 0, 0)$$

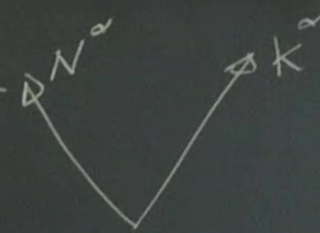
$$K^\beta \nabla_\beta K^\alpha = 0$$

$$K^\alpha = (1, 1, 0, 0)$$

$$\nabla_\alpha K_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{pmatrix}$$

$$\theta = \nabla_\alpha K^\alpha = \frac{2}{r} = \frac{1}{4\pi r^2}\frac{d}{d\lambda} 4\pi r^2$$

Decomposition



$$K_\alpha K^\alpha = 0$$

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$

$$A^\alpha = A_1 K^\alpha + A_2 N^\alpha + A_\perp^\alpha$$

$$A_1 = -N_\alpha A^\alpha$$

$$A_2 = -K_\alpha A^\alpha$$

$$A_\perp^\alpha = \Omega^\alpha{}_\beta A^\beta$$

$$\Omega^\alpha{}_\beta = g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta$$

Null cones

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$$K^\alpha = (1, 1, 0, 0)$$

$$\nabla_\alpha K_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{pmatrix}$$

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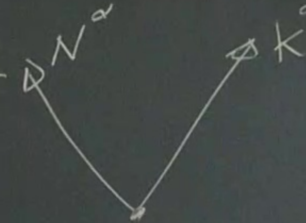
Decomposition



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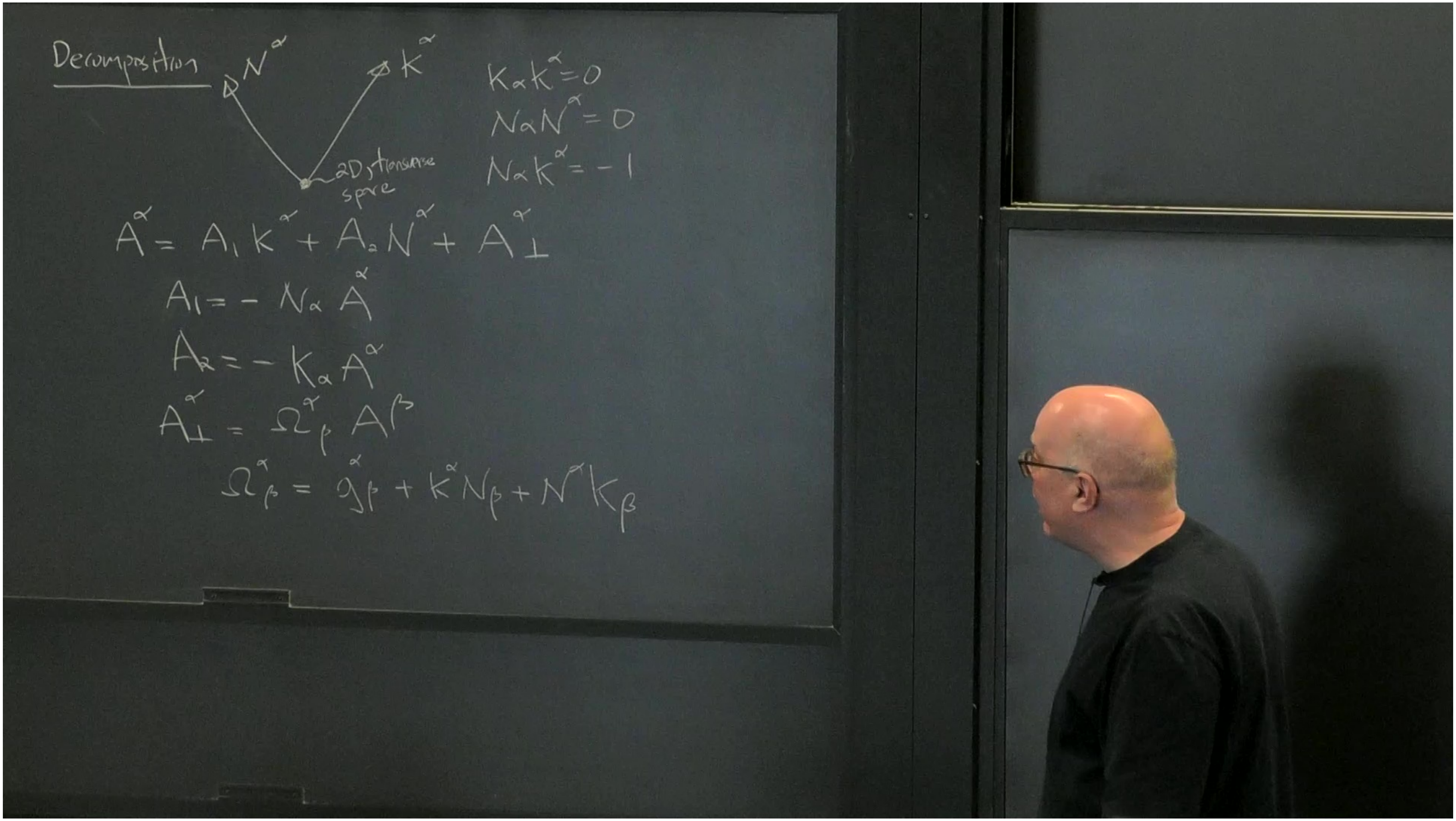

Decomposition
N^α
K^α
2D, transverse space
$K_\alpha K^\alpha = 0$
$N_\alpha N^\alpha = 0$
$N_\alpha K^\alpha = -1$
$A^\alpha = A_1 K^\alpha + A_2 N^\alpha + A^\alpha_\perp$
$A_1 = -N_\alpha A^\alpha$
$A_2 = -K_\alpha A^\alpha$
$A^\alpha_\perp = \Omega^\alpha{}_\beta A^\beta$
$\Omega^\alpha{}_\beta = g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta$

$$\ldots K^\alpha K^\beta + A_2 K^\alpha N^\beta + A_3 N^\alpha K^\beta + A_4 N^\alpha N^\beta$$

$$K^\alpha B^\beta_{1\perp} + N^\alpha B^\beta_{2\perp} + C^\alpha_{1\perp} K^\beta + C^\alpha_{2\perp} N^\beta$$

$$D^{\alpha\beta}_{\perp\perp}$$

$$A = \begin{array}{c} K \\ N \\ T \end{array} \begin{pmatrix} A_1 & A_2 & B_1^T \\ A_3 & A_4 & B_2^T \\ & & \end{pmatrix}$$

$$K_\beta B^\beta_{1,2\perp} = N_\beta B^\beta_{1,2\perp}$$

$$K_\beta C^\beta_{1,2\perp} = N_\beta C^\beta_{1,2\perp}$$

$$K_\alpha D^{\alpha\beta}_{\perp\perp} = N_\alpha D^{\alpha\beta}_{\perp\perp} = D^{\alpha\beta}_{\perp\perp} K_\beta = D^{\alpha\beta}_{\perp\perp} N_\beta = 0$$

$$A^{\alpha\beta} = A_1 K^\alpha K^\beta + A_2 K^\alpha N^\beta + A_3 N^\alpha K^\beta + A_4 N^\alpha N^\beta$$

$$+ K^\alpha B^\beta_{1\perp} + N^\alpha B^\beta_{2\perp} + C^\alpha_{1\perp} K^\beta + C^\alpha_{2\perp} N^\beta$$

$$+ D^{\alpha\beta}_{\perp\perp}$$

$$K_\beta B^\beta_{1,2\perp} = N_\beta B^\beta_{1,2\perp}$$

$$K_\beta C^\beta_{1,2\perp} = N_\beta C^\beta_{1,2\perp}$$

$$K_\alpha D^{\alpha\beta}_{\perp\perp} = N_\alpha D^{\alpha\beta}_{\perp\perp} = D^{\alpha\beta}_{\perp\perp} K_\beta = D^{\alpha\beta}_{\perp\perp} N_\beta = 0$$

$$A^{\alpha\beta} = g^{\alpha}{}_{\mu}\, g^{\beta}{}_{\nu}\, A^{\mu\nu} = \left(-K^{\alpha}N_{\mu} - N^{\alpha}K_{\mu} + \Omega^{\alpha}{}_{\mu}\right)\left(-K^{\beta}N_{\nu} - N^{\beta}K_{\nu} + \Omega^{\beta}{}_{\nu}\right) A^{\mu\nu}$$

$$A_1 = N_{\mu}N_{\nu}A^{\mu\nu}, \quad A_2 = N_{\mu}K_{\nu}A^{\mu\nu}, \quad A_3 = K_{\mu}N_{\nu}A^{\mu\nu}, \quad A_4 = K_{\mu}K_{\nu}A^{\mu\nu}$$

$$B^{\beta}_{1\perp} = -N_{\mu}\Omega^{\beta}{}_{\nu}A^{\mu\nu}, \quad B^{\beta}_{2\perp} = -K_{\mu}\Omega^{\beta}{}_{\nu}A^{\mu\nu}, \quad C^{\alpha}_{1\perp} = -\Omega^{\alpha}{}_{\mu}N_{\nu}A^{\mu\nu}, \quad C^{\alpha}_{2\perp} = -\Omega^{\alpha}{}_{\mu}K_{\nu}A^{\mu\nu}$$

$$D^{\alpha\beta}_{\perp\perp} = \Omega^{\alpha}{}_{\mu}\Omega^{\beta}{}_{\nu}A^{\mu\nu}$$

Geometric optics

Geometric optics

↳ Maxwell's eqns (in vacuum)

$$\nabla_\beta F^{\alpha\beta} = 0$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0 \checkmark$$

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$$

$$\bar{A}_\alpha = A_\alpha + \nabla_\alpha f \rightarrow \bar{F}_{\alpha\beta} = F$$

Lorenz gauge: $\nabla_\alpha A^\alpha = 0$

doesn't fix gauge uniquely

$$\bar{A}_\alpha = A_\alpha + \nabla_\alpha f \Rightarrow \nabla_\alpha \bar{A}^\alpha$$

$$\Box f = \nabla_\alpha \nabla^\alpha f = 0$$

s

Maxwell's eqns (in vacuum)

$= 0$

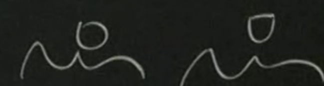
$+ \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0$ ✓

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$$

$$\bar{A}_\alpha = A_\alpha + \nabla_\alpha f \longrightarrow \bar{F}_{\alpha\beta} = F_{\alpha\beta}$$

Lorenz gauge: $\nabla_\alpha A^\alpha = 0$

doesn't fix gauge uniquely



$$\bar{A}_\alpha = A_\alpha + \nabla_\alpha f \Rightarrow \nabla_\alpha \bar{A}^\alpha = \nabla_\alpha A^\alpha + \nabla_\alpha \nabla^\alpha f$$

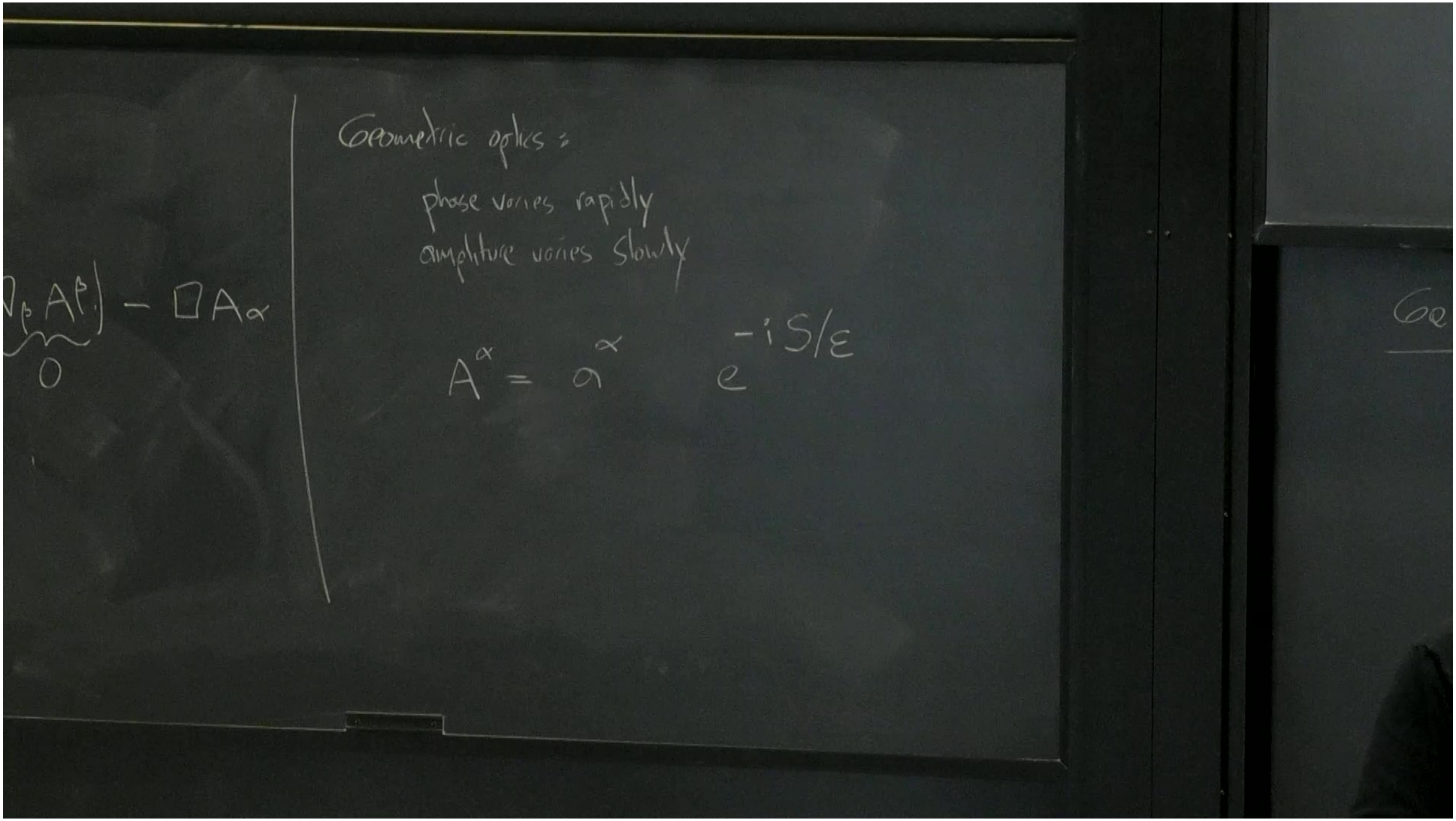
$$\Box f = \nabla_\alpha \nabla^\alpha f = 0$$

$$0 = \nabla_\beta \left(\nabla_\alpha A^\beta - \nabla^\beta A_\alpha \right)$$

$$= \nabla_\beta \nabla_\alpha A^\beta - \Box A_\alpha$$

$$= \underbrace{\left(\nabla_{\beta\alpha} - \nabla_{\alpha\beta} \right) A^\beta}_{R^\beta{}_{\mu\beta\alpha} A^\mu \,=\, R_{\mu\alpha} A^\mu} + \nabla_\alpha \big(\underbrace{\nabla_\beta A^\beta}_{0} \big) - \Box A_\alpha$$

$$\boxed{\Box A^\alpha - R^\alpha{}_\beta A^\beta = 0}$$


Geometric optics :
phase varies rapidly
amplitude varies slowly
$A^\alpha = a^\alpha e^{-iS/\varepsilon}$
$- \Box A_\alpha$

$$\nabla_\alpha A^\beta - \nabla^\beta A_\alpha)$$

$$_\alpha A^\beta - \Box A_\alpha$$

$$- \nabla_{\alpha\beta}) A^\beta + \nabla_\alpha (\underbrace{\nabla_\beta A^\beta}_{0}) - \Box A_\alpha$$

$$^\beta{}_{\mu\beta\alpha} A^\mu = R_{\mu\alpha} A^\mu$$

$$\boxed{{}^\alpha{}_\beta A^\beta = 0} \qquad \boxed{\nabla_\alpha A^\alpha = 0}$$

Geometric optics:

phase varies rapidly ✓

amplitude varies slowly ✓

$$\boxed{A^\alpha = \left[a^\alpha + O(\varepsilon) \right] e^{-iS/\varepsilon} \qquad \varepsilon \ll 1}$$

$$\boxed{k_\alpha \equiv -\nabla_\alpha S}$$

$$\nabla_\beta A^\alpha = -\frac{i}{\varepsilon}\nabla_\beta S \left(a^\alpha + \dots\right) e \qquad + O(1)$$

$$= \frac{i}{\varepsilon} k_\beta\, a^\alpha\, e^{-iS/\varepsilon} + O(1)$$

$$\nabla_\alpha A^\alpha = 0 \implies k_\alpha a^\alpha = 0$$

$$\Box A^\alpha = \nabla^\beta \nabla_\beta A^\alpha = \frac{i}{\varepsilon} \nabla^\beta \left[\left(k_\beta a^\alpha + \dots \right) e^{-iS/\varepsilon} \right] = \frac{i}{\varepsilon}\left(-\frac{i}{\varepsilon}\right) \nabla^\beta S \, k_\beta a^\alpha e^{-iS/\varepsilon} + O(1/\varepsilon)$$

$$= -\frac{1}{\varepsilon^2} k^\beta k_\beta \, a_\alpha \, e^{-iS/\varepsilon} + O(1/\varepsilon)$$

$$R_{\alpha\beta} A^\beta = O(1)$$

$$\Box A^\alpha - R^\alpha{}_\beta A^\beta = 0 \Longrightarrow$$

$$\boxed{k_\alpha k^\alpha = 0}$$

$$\nabla_\beta A^\alpha = -\frac{i}{\varepsilon}\nabla_\beta S\left(a^\alpha + \dots\right)e^{-iS/\varepsilon} + O(1)$$

$$= \frac{i}{\varepsilon} k_\beta\, a^\alpha\, e^{-iS/\varepsilon} + O(1)$$

$$\nabla_\alpha A^\alpha = 0 \implies k_\alpha a^\alpha = 0$$

$$k_\alpha a^\alpha = 0 = N_\alpha a^\alpha$$

$$a^\alpha = a^\alpha_\perp$$

Maxwell: $k_\alpha k^\alpha = 0$

$$k_\alpha = -\nabla_\alpha S$$

$$\to k^\beta \nabla_\beta k^\alpha = 0$$

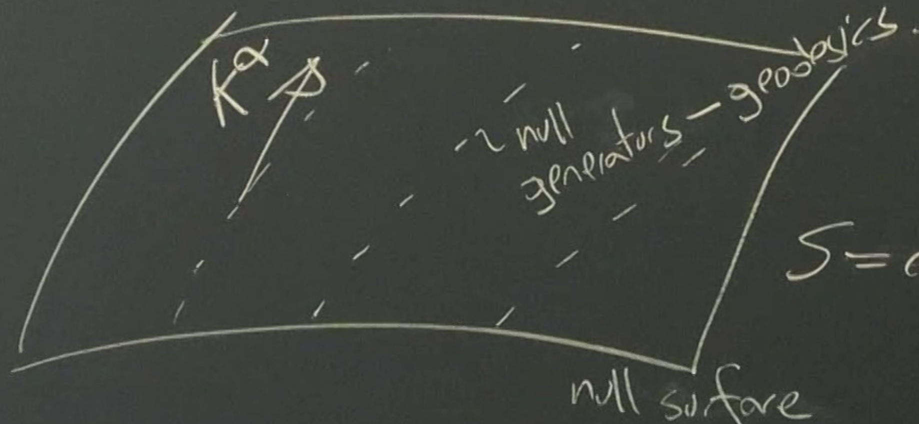
$S = \text{const} \to$ light surface

k^α tangent to null geodesics $\to$ light rays

$A_2 = N_\mu K_\nu A^{\mu\nu}$, $A_3 = K_\mu N_\nu A^{\mu\nu}$, $A_4 = K_\mu K_\nu A$

$\nu A^{\mu\nu}$, $B^{\beta}_{2\perp} = -K_\mu \Omega^{\beta}{}_{\nu} A^{\mu\nu}$, $C^{\alpha}_{1\perp} = -\Omega^{\alpha}{}_{\mu} N_\nu A^{\mu\nu}$, $C^{\alpha}_{2\perp} =$

$\nu A^{\mu\nu}$



$K_\alpha = -\nabla_\alpha S$

$S = \text{const}$

Decompose $\nabla_\alpha K_\beta$

$$K_\alpha = -\nabla_\alpha S$$

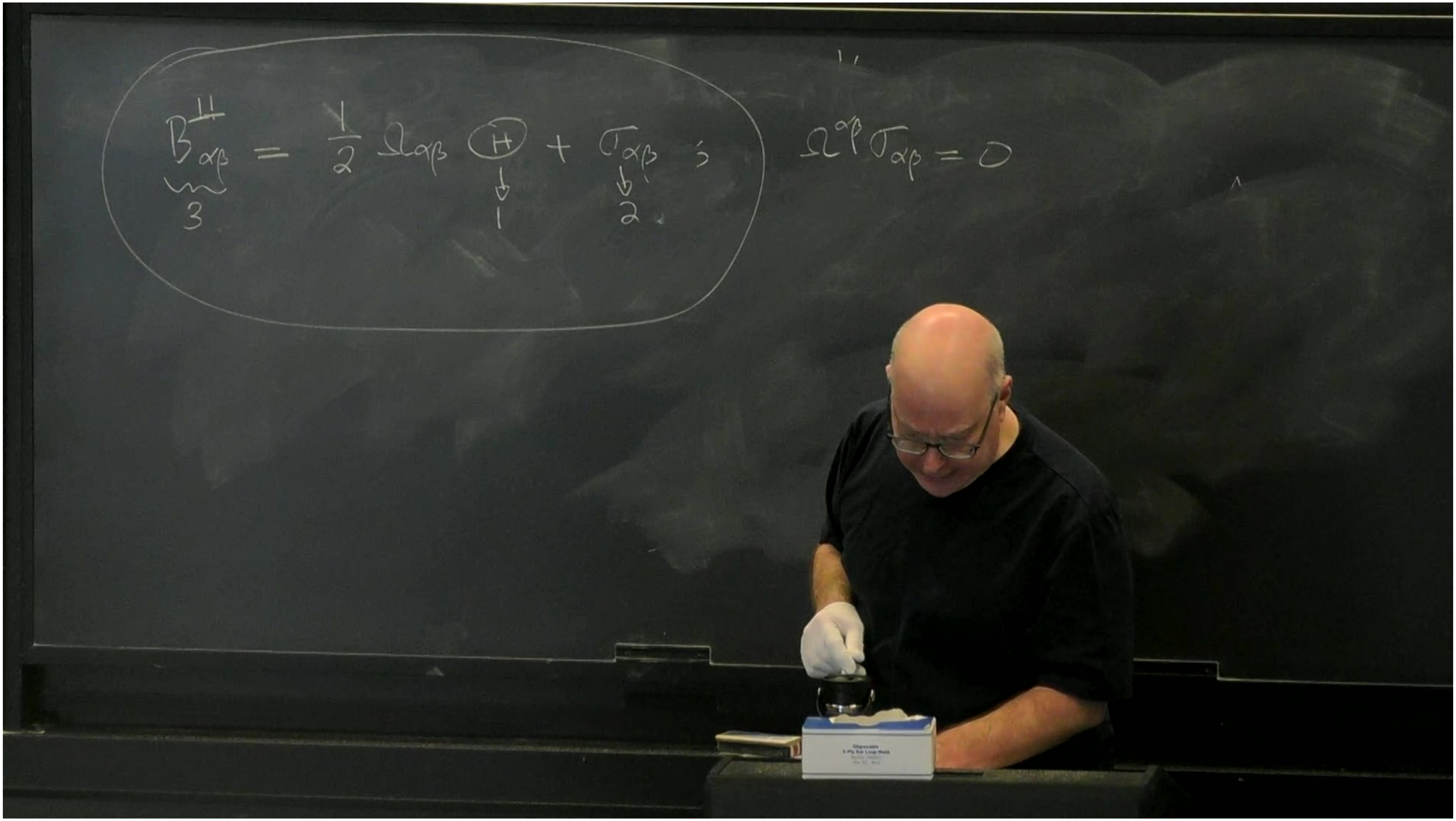
$$\nabla_\beta K_\alpha = -\nabla_{\beta\alpha} S = -\nabla_{\alpha\beta} S = +\nabla_\alpha K_\beta$$

$$\boxed{\nabla_\alpha K_\beta = \underbrace{B\, k_\alpha k_\beta + K_\alpha B^\perp_\beta + B^\perp_\alpha K_\beta}_{\text{useless}} + \underbrace{B^{\perp\perp}_{\alpha\beta}}_{\text{meaningful}}}$$

$$B = N^\mu N^\nu \nabla_\mu k_\nu$$

$$B^\perp_\beta = -N^\mu \Omega^\nu{}_\beta \nabla_\mu k_\nu$$

$$B^{\perp\perp}_{\alpha\beta} = \Omega^\mu{}_\alpha \Omega^\nu{}_\beta \nabla_\mu k_\nu$$


$B^{II}_{\alpha\beta} = \frac{1}{2}\Omega_{\alpha\beta} H + \sigma_{\alpha\beta}$;
3
1
2
$\Omega^{\alpha\beta}\sigma_{\alpha\beta} = 0$