Title: Advanced General Relativity - 240207

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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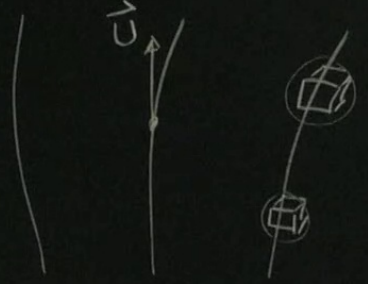
URL: https://pirsa.org/24020000

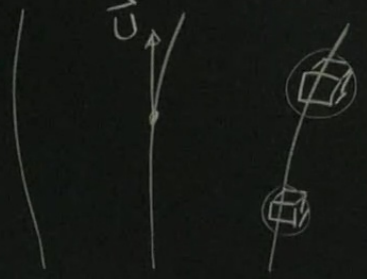
Raychaudhuri's eqn.



$$\frac{D\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta + \nabla_\alpha a^\alpha$$

$$\Theta = \nabla_\alpha U^\alpha = \frac{1}{\delta V}\frac{d}{d\lambda}\delta V = \text{fractional rate of volume change}$$

Raychaudhuri's eqn



$$\frac{D\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta + D_\alpha a^\alpha$$

$$\Theta = D_\alpha U^\alpha = \frac{1}{\delta V}\frac{d}{d\lambda}\delta V = \text{fractional rate of volume change}$$

Focusing theorem: - geodesic ($a^\alpha = 0$)

- hypersurface orthogonal ($\omega_{\alpha\beta} = 0$)

$$\frac{D\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta + \nabla_\alpha a^\alpha$$

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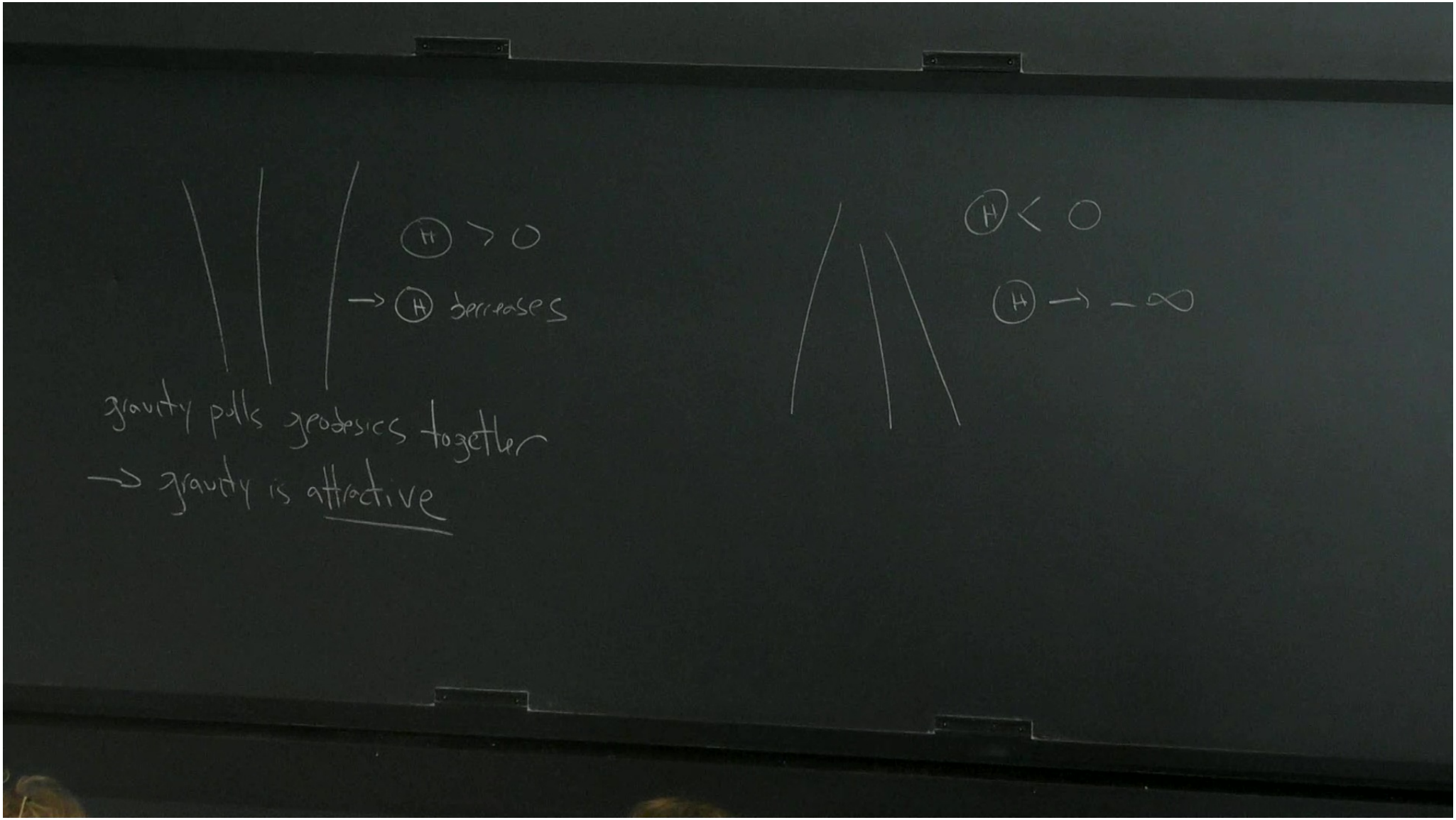
focusing theorem: – geodesic ($a^\alpha = 0$)

– hypersurface orthogonal ($\omega_{\alpha\beta} = 0$)

– Ricci condition: $R_{\alpha\beta}u^\alpha u^\beta \geq 0$

$$\frac{D\Theta}{d\tau} = -\left(\frac{1}{3}\Theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + R_{\alpha\beta}u^\alpha u^\beta\right) \leq 0$$

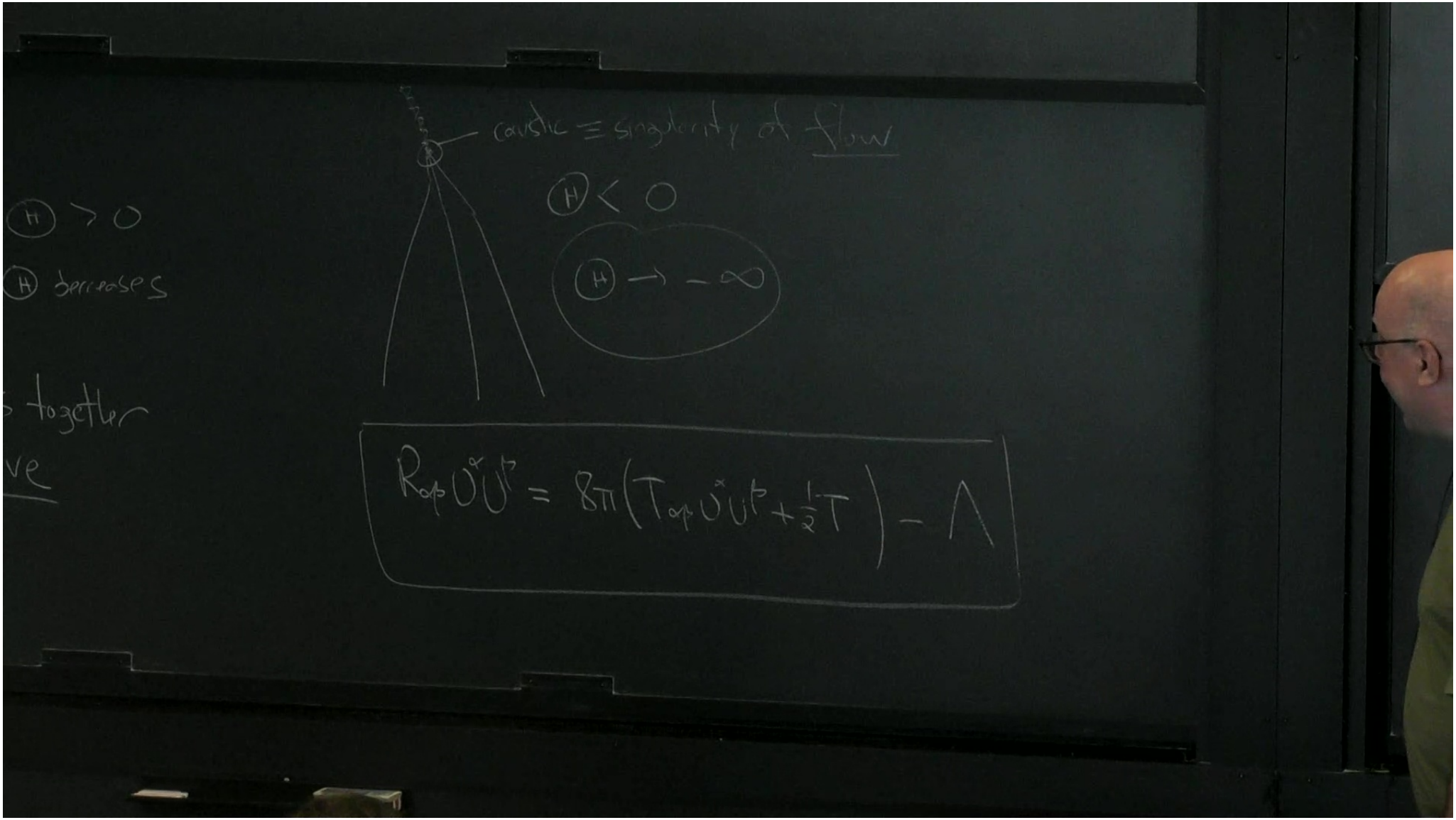

H > 0
→ H decreases
gravity pulls geodesics together
→ gravity is attractive
H < 0
H → −∞

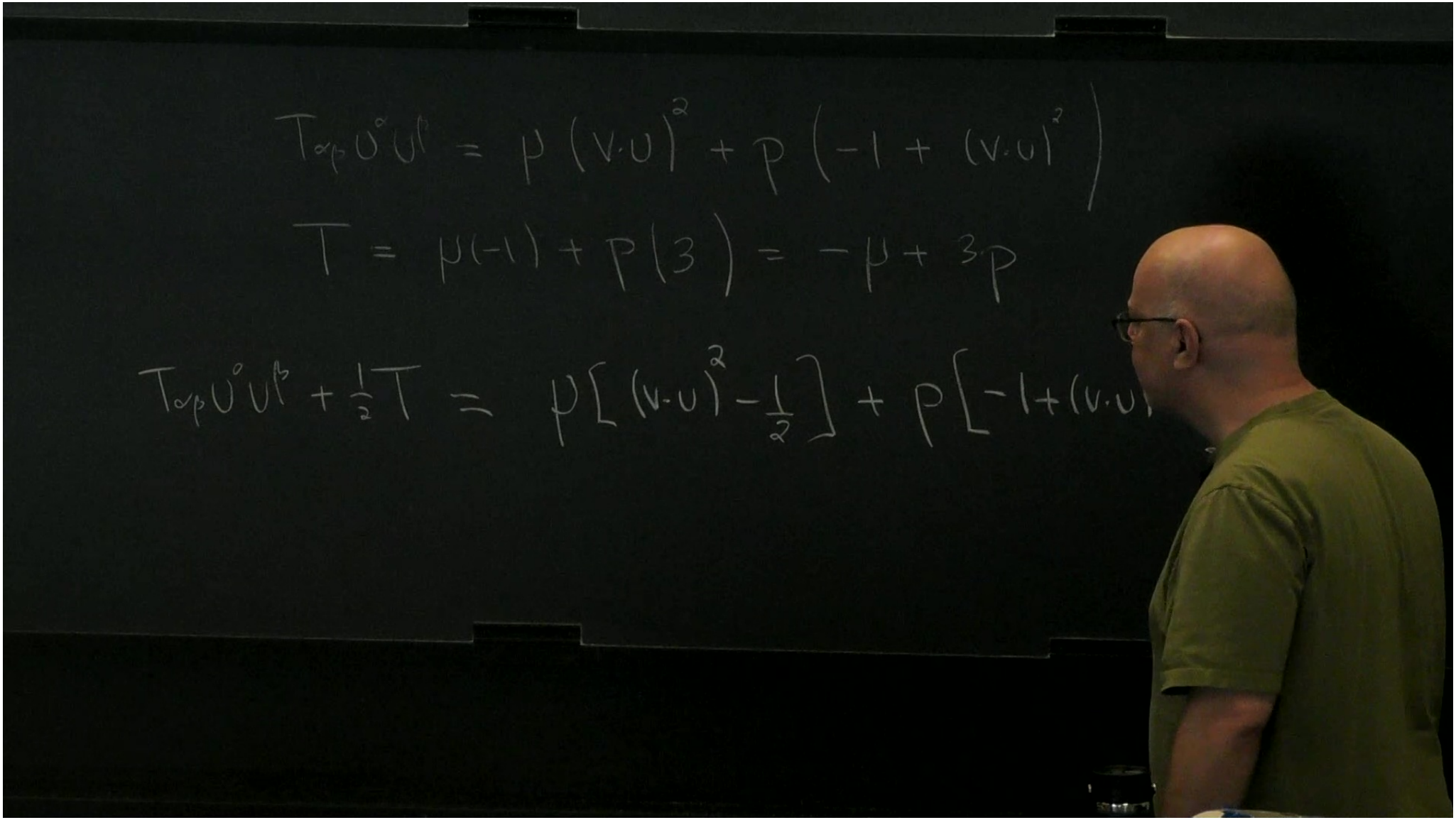
$$\Rightarrow R - \frac{1}{2}R(4) + \Lambda(4) = 8\pi T \Rightarrow +R = +4\Lambda - 8\pi T$$

$$R_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta}\left(4\Lambda - 8\pi T\right) - \Lambda g_{\alpha\beta} + 8\pi T_{\alpha\beta}$$

$$\boxed{R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta}}$$

$$R_{\alpha\beta} U^\alpha U^\beta = 8\pi\left(T_{\alpha\beta} U^\alpha U^\beta - \frac{1}{2} T(-1)\right) + \Lambda(-1)$$


caustic ≡ singularity of flow
H < 0
H → −∞
H > 0
H decreases
together
ve
$R_{\alpha\beta}U^{\alpha}U^{\beta} = 8\pi\left(T_{\alpha\beta}U^{\alpha}U^{\beta} + \frac{1}{2}T\right) - \Lambda$


$T_{\alpha\beta} U^\alpha U^\beta = \mu (v \cdot U)^2 + p \left(-1 + (v \cdot U)^2 \right)$
$T = \mu(-1) + p(3) = -\mu + 3p$
$T_{\alpha\beta} U^\alpha U^\beta + \frac{1}{2} T = \mu \left[(v \cdot U)^2 - \frac{1}{2} \right] + p \left[-1 + (v \cdot U)$

$$\rho U^\alpha U^\beta = \rho (V\cdot U) + p\left(-1 + (V\cdot U)^2\right)$$

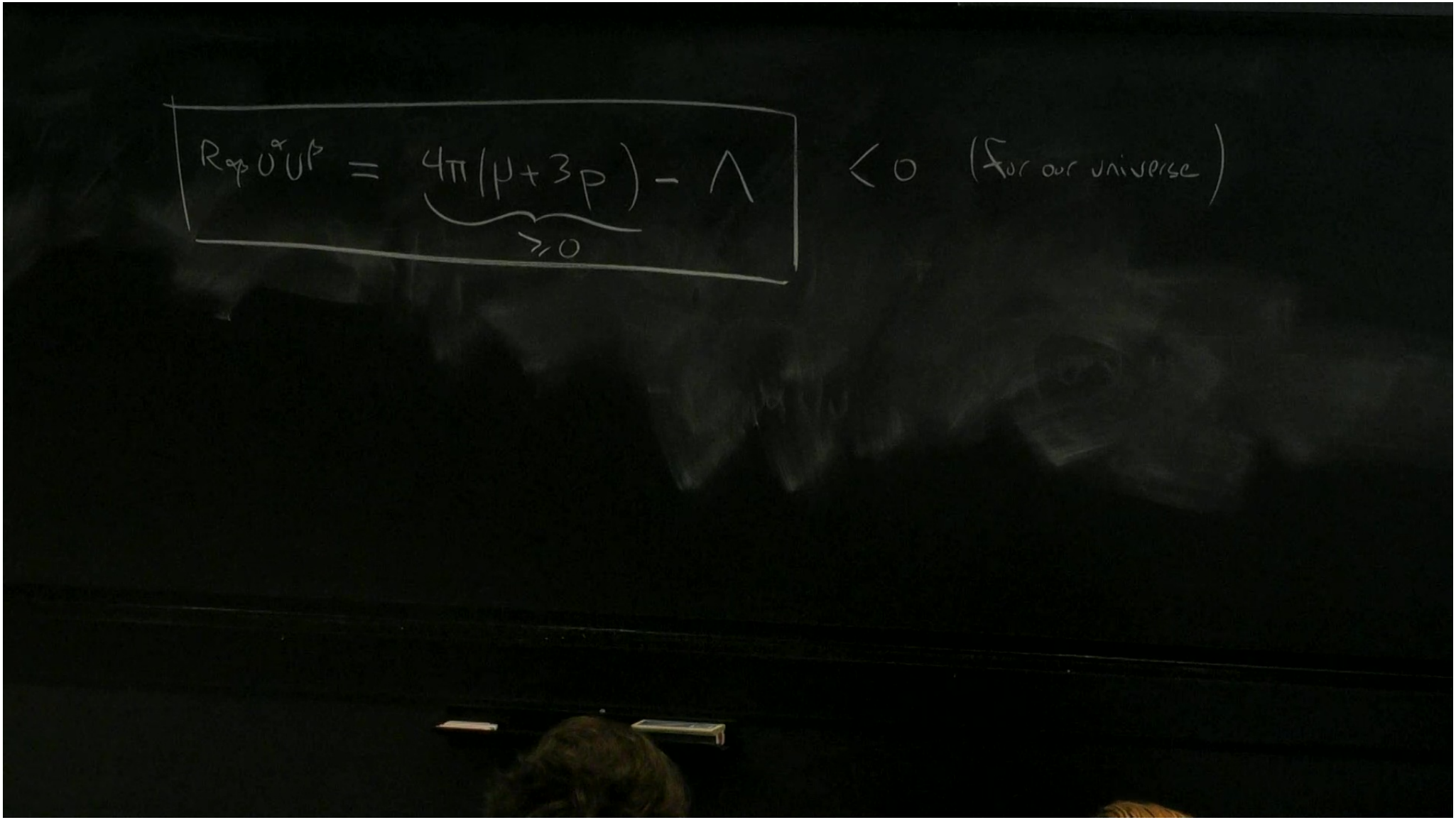
$$= \rho(-1) + p(3) = -\rho + 3p$$

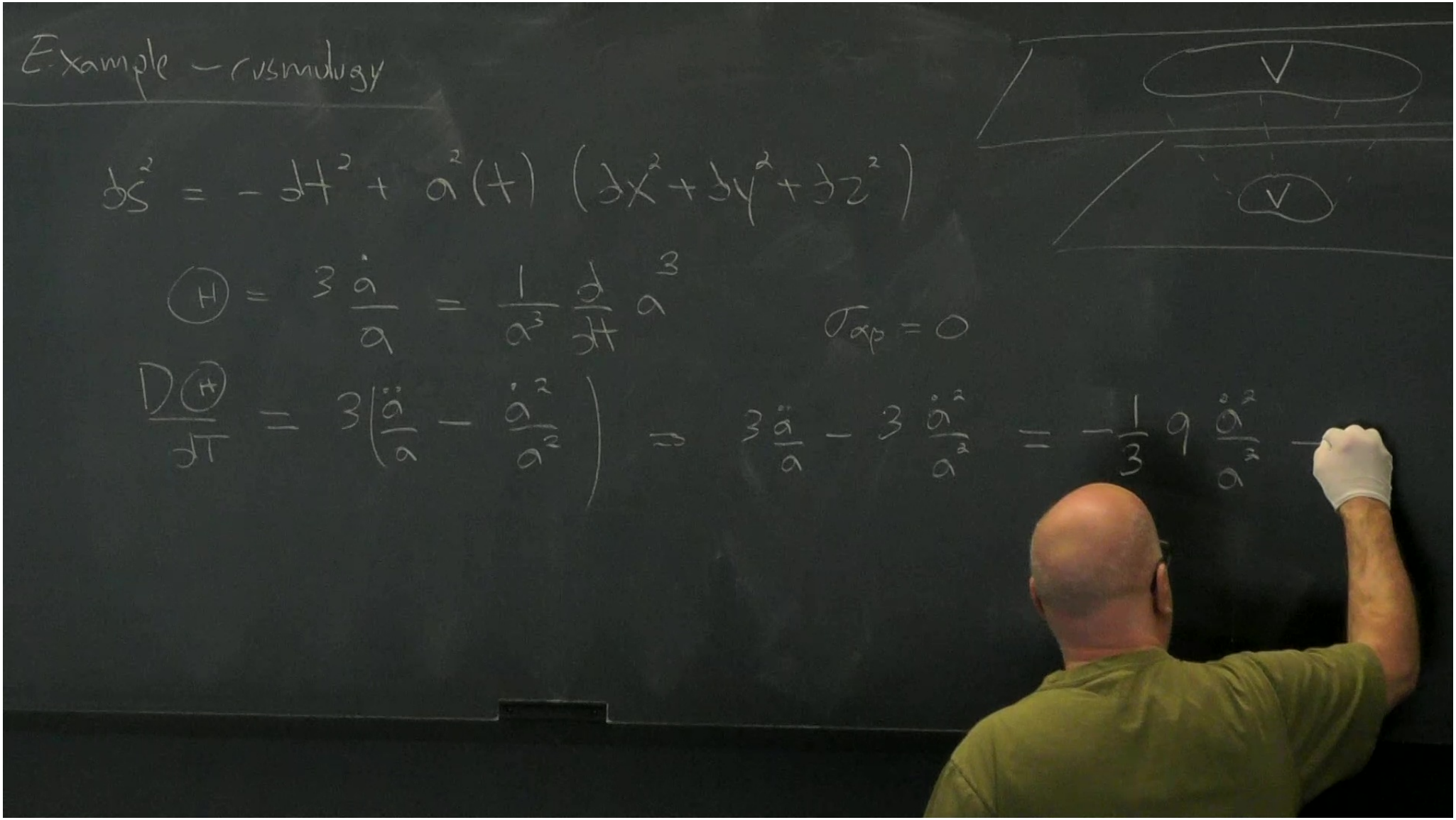
Choose $V^\alpha \equiv U^\alpha$

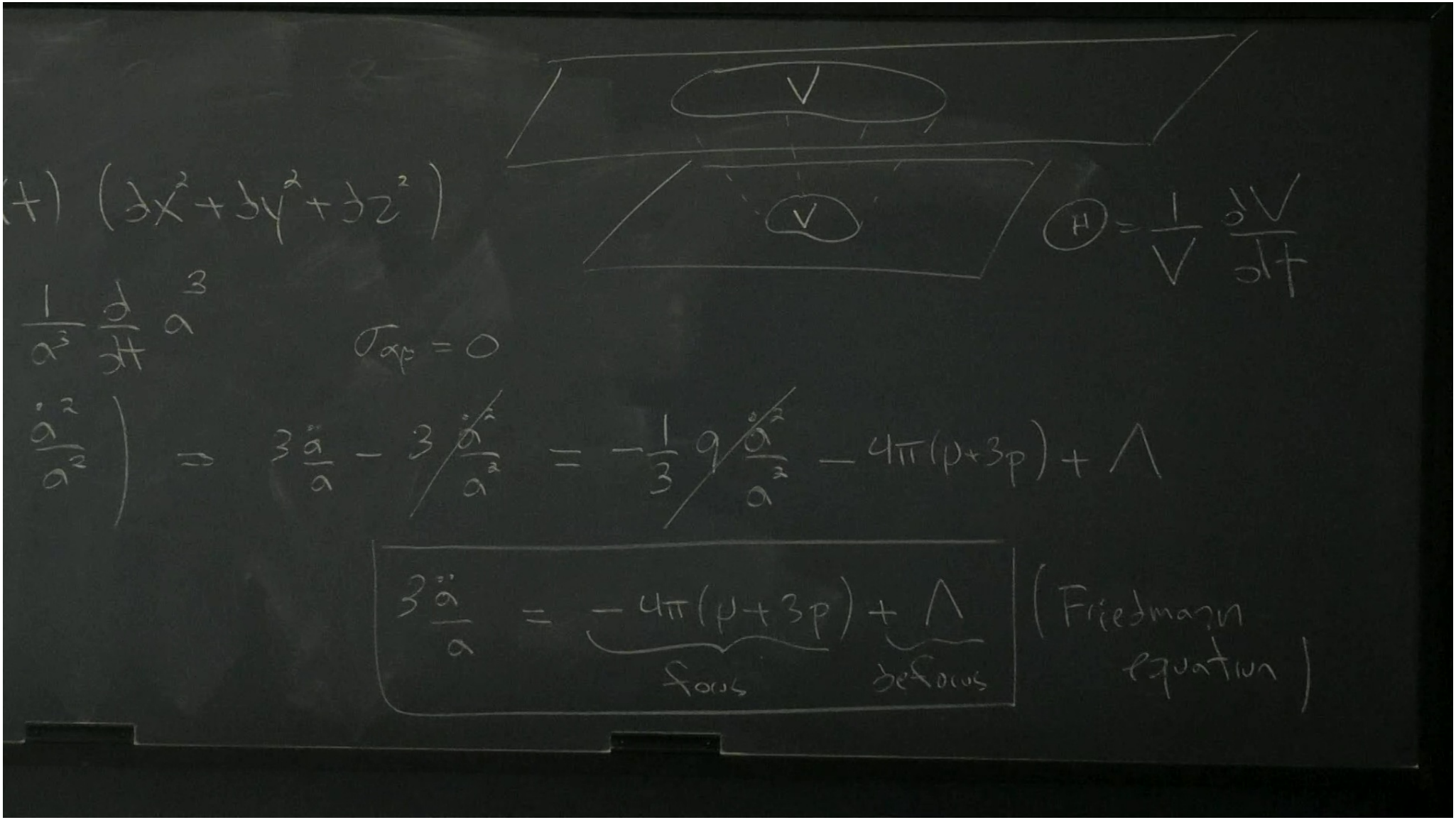
$U\cdot V = -1$

$$-\frac{1}{2}T = \rho\left[(V\cdot U)^2 - \frac{1}{2}\right] + p\left[-1 + (V\cdot U)^2 + \frac{3}{2}\right]$$

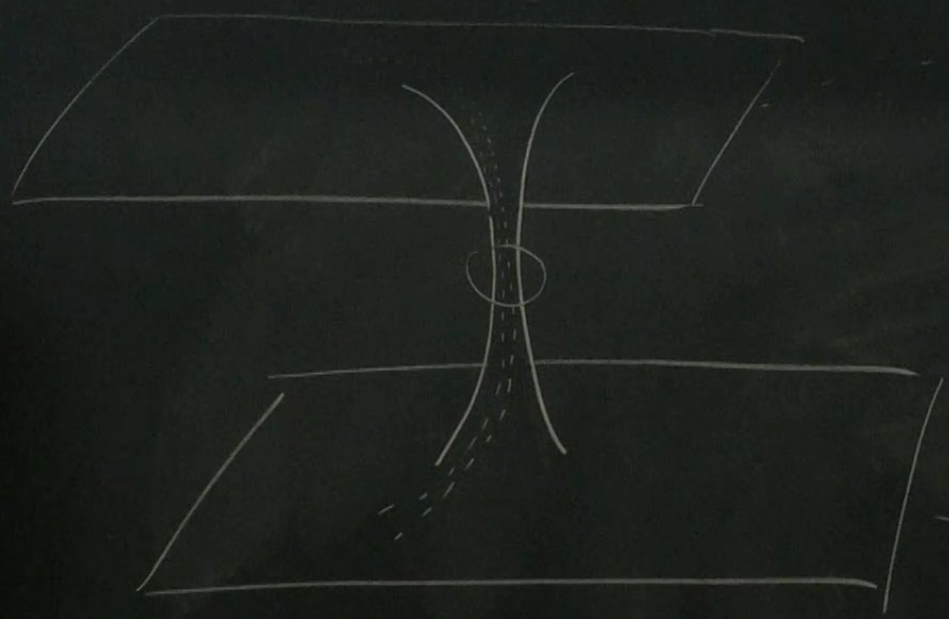
$$= \frac{1}{2}\rho + \frac{3}{2}p = \frac{1}{2}(\rho + 3p)$$


$R_{\alpha\beta} U^\alpha U^\beta = \underbrace{4\pi(\mu + 3p)}_{\geq 0} - \Lambda$
< 0 (for our universe)

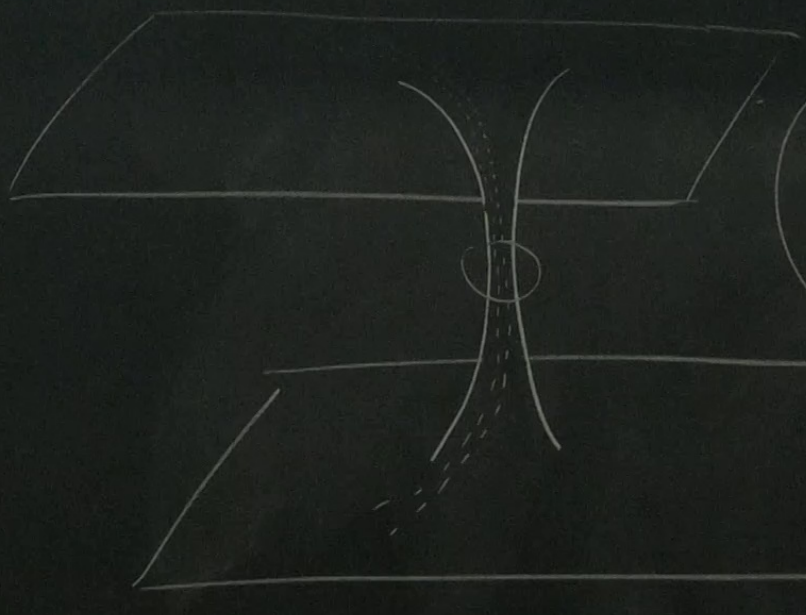

Example – cosmology
$ds^2 = -dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2)$
$H = 3\frac{\dot{a}}{a} = \frac{1}{a^3}\frac{d}{dt}a^3$
$\sigma_{ap} = 0$
$\frac{D\Theta}{dT} = 3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \Rightarrow 3\frac{\ddot{a}}{a} - 3\frac{\dot{a}^2}{a^2} = -\frac{1}{3}9\frac{\dot{a}^2}{a^2}$
V
V


V
V
H = 1/V ∂V/∂t
(+) (∂x² + ∂y² + ∂z²)
1/a³ d/dt a³
σ_αβ = 0
ȧ²/a²) ⇒ 3ä/a − 3ȧ²/a² = −1/3 9ȧ²/a² − 4π(ρ+3p) + Λ
3ä/a = −4π(ρ+3p) + Λ
focus
defocus
(Friedmann equation)

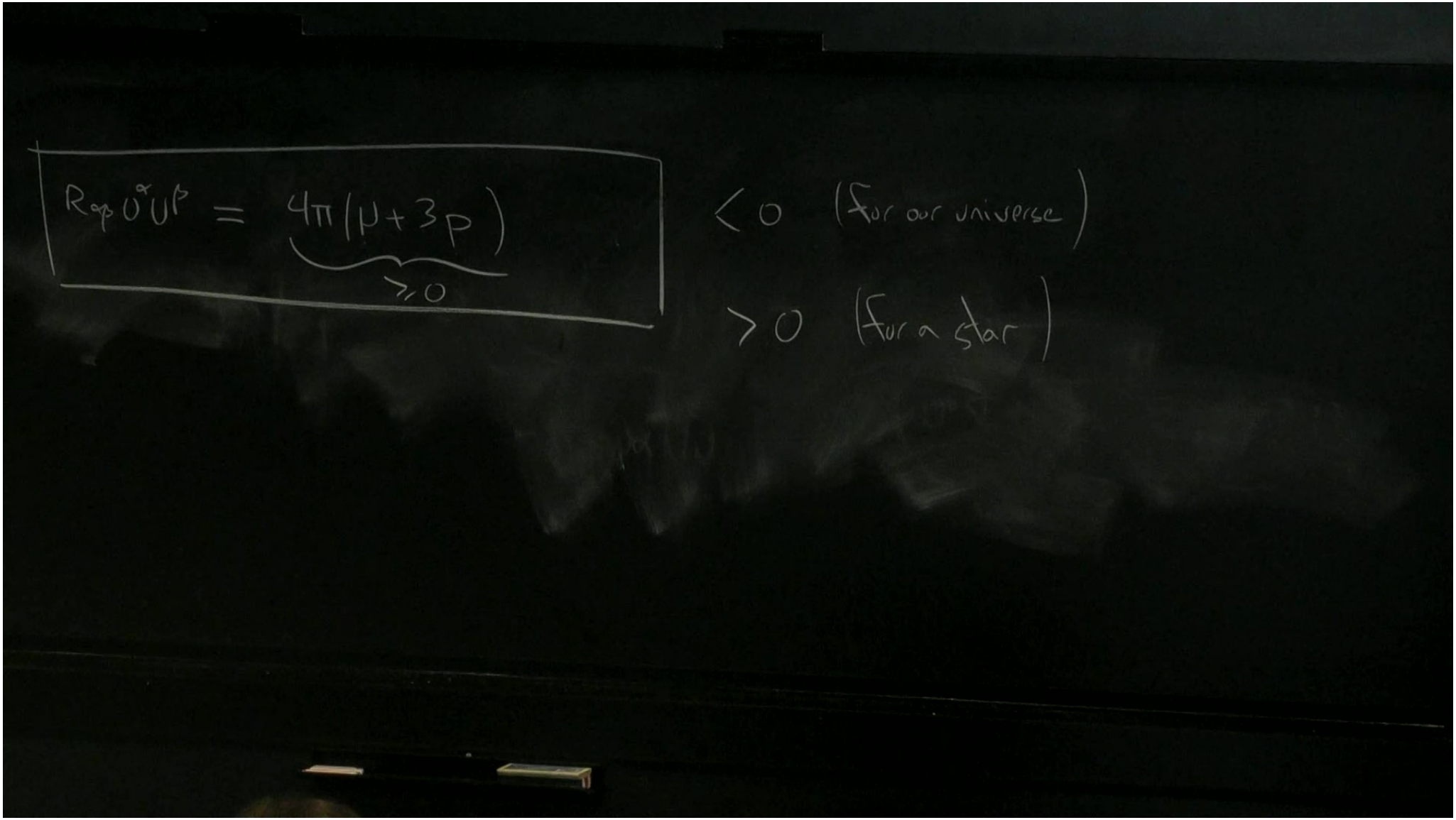
Example – wormhole

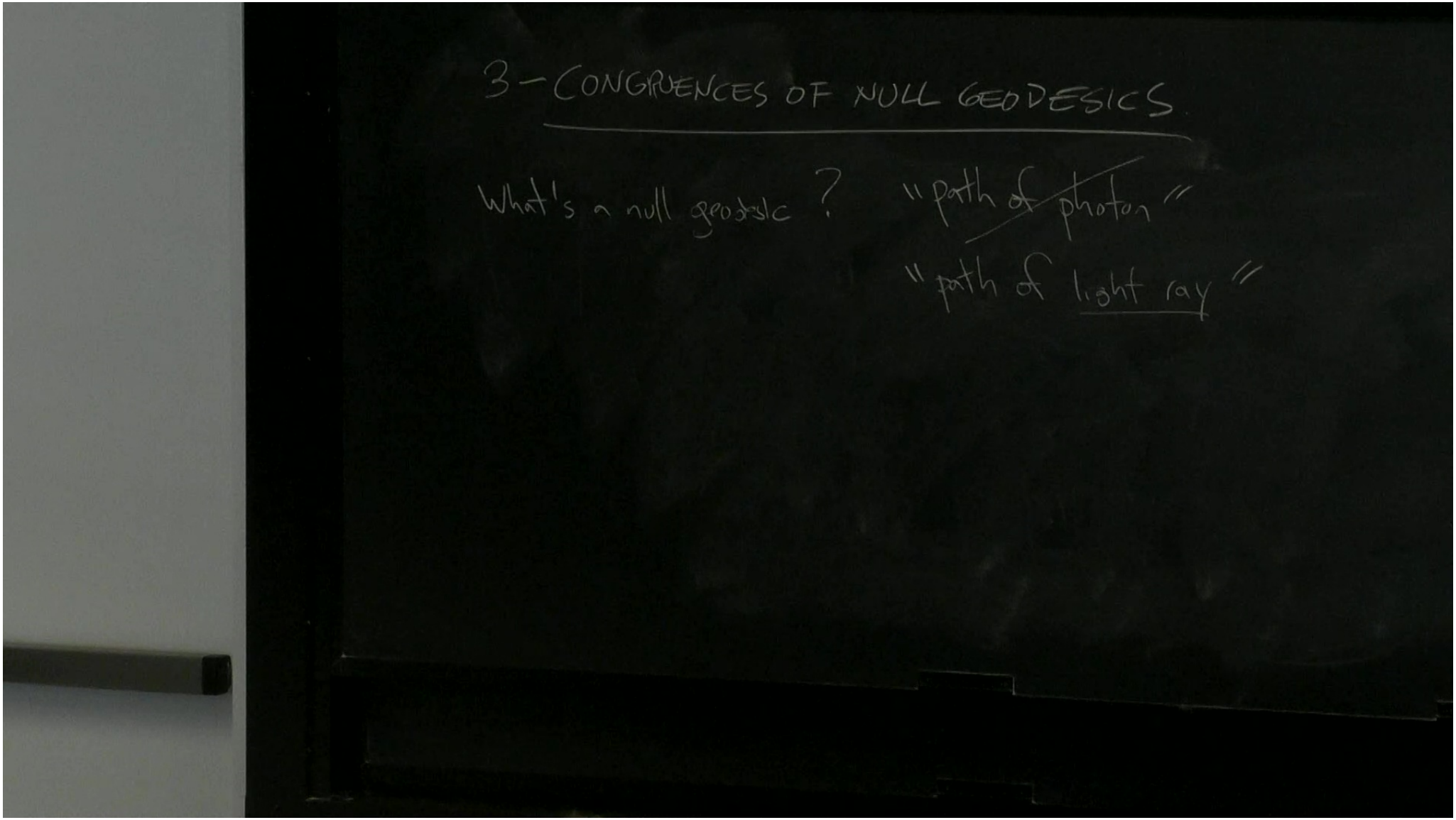


Example – wormhole



$R_{\alpha\beta} U^\alpha U^\beta < 0$


$R_{\alpha\beta} U^\alpha U^\beta = \underbrace{4\pi(\mu + 3p)}_{\geq 0}$
< 0 (for our universe)
> 0 (for a star)


3 - CONGRUENCES OF NULL GEODESICS
What's a null geodesic ?
"path of photon"
"path of light ray"