

Title: Dots and Donâ€™ts of Folding Time

Speakers: Sebastian Mizera

Series: Colloquium

Date: January 31, 2024 - 2:00 PM

URL: <https://pirsa.org/24010096>

Abstract: I will summarize recent progress in uncovering the analytic structure of scattering amplitudes. The overarching theme will be exploiting new intricate ways of analytically continuing time, extending beyond the Wick rotation. I will highlight a broad range of applications: from high-precision calculations in particle physics, through computations of gravitational waves, to formal topics in the scattering of strings.

Zoom link

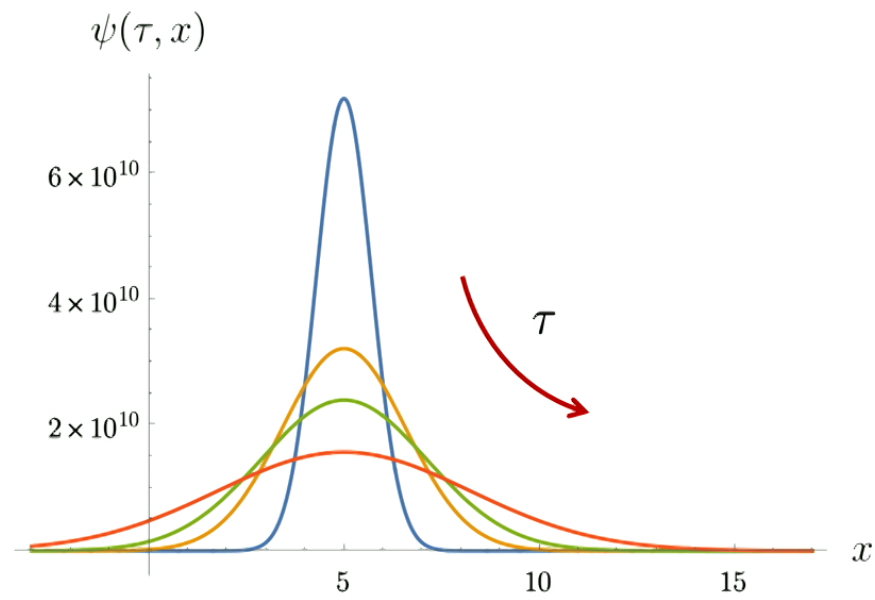


Dos and Don'ts of Folding Time

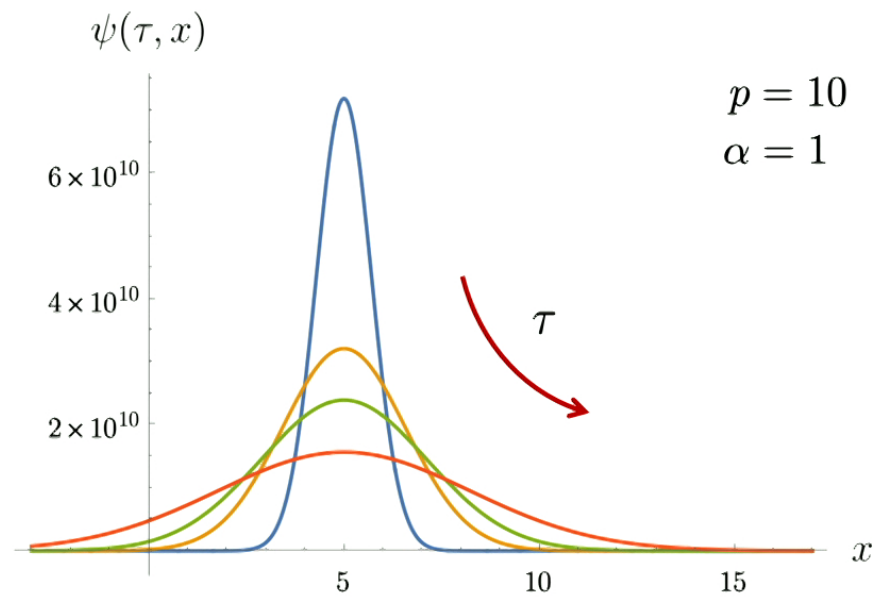
Sebastian Mizera (IAS Princeton)

Perimeter Institute, 31 January 2024

Heat equation: $\frac{\partial \psi}{\partial \tau} = \alpha \frac{\partial^2 \psi}{\partial x^2}$



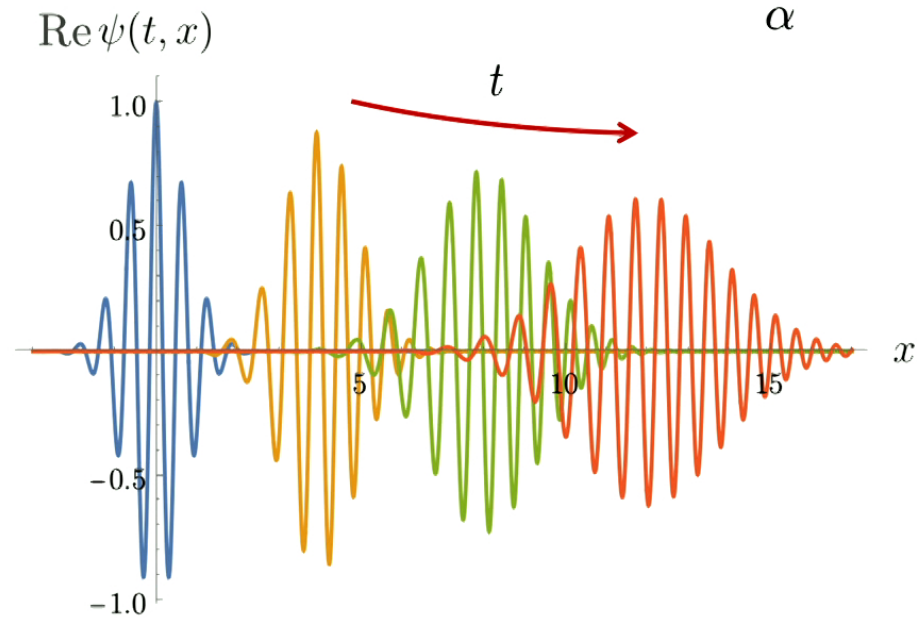
Heat equation: $\frac{\partial \psi}{\partial \tau} = \alpha \frac{\partial^2 \psi}{\partial x^2}$



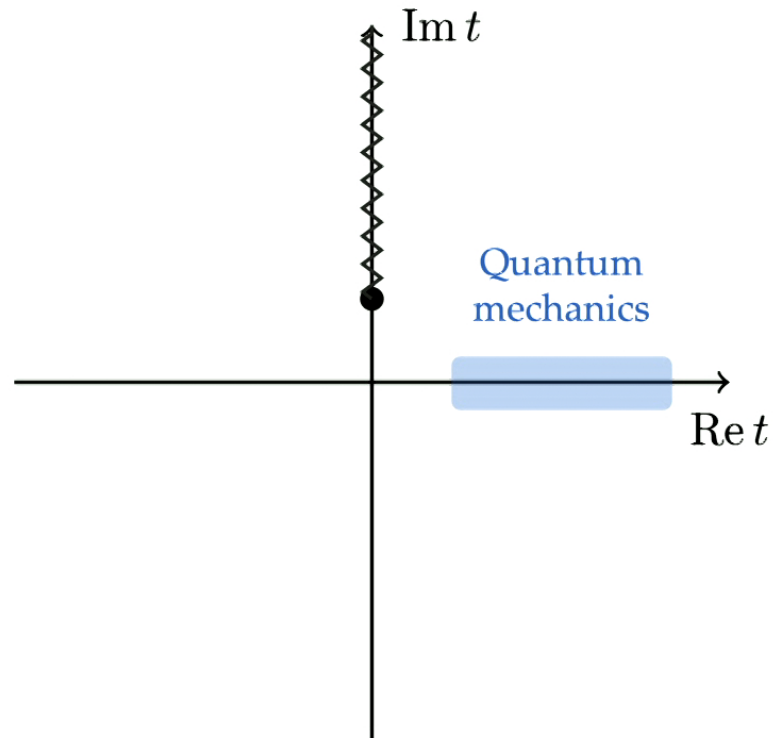
$$\psi(\tau, x) = \frac{1}{\sqrt{1 + 4\alpha\tau}} e^{\frac{p^2\alpha\tau + px - x^2}{1 + 4\alpha\tau}}$$

Free Schrödinger equation:
$$-i\frac{\partial\psi}{\partial t} = \underbrace{\frac{\hbar}{2m}}_{\alpha} \frac{\partial^2\psi}{\partial x^2}$$

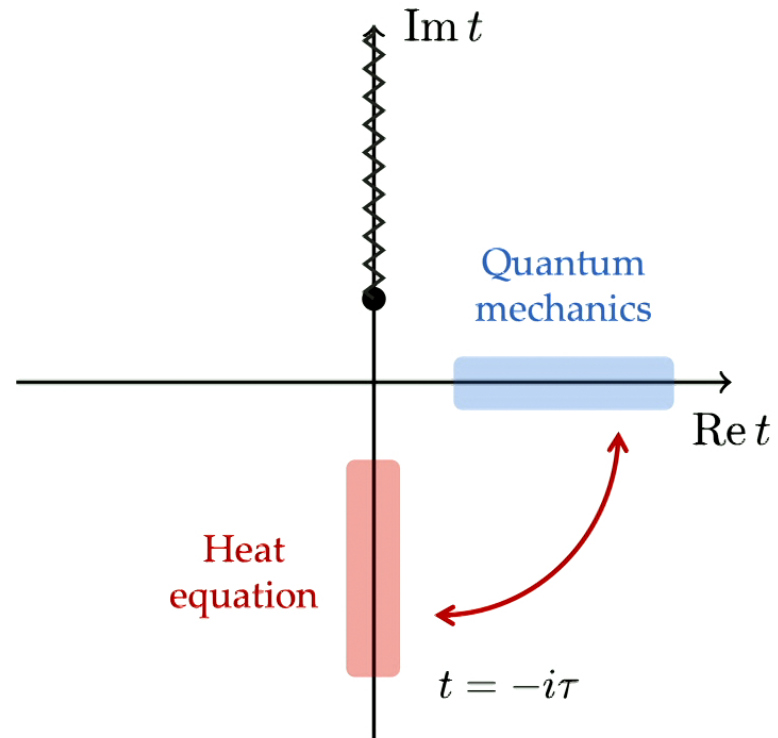
Free Schrödinger equation:
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Example of an analytic continuation: Wick rotation

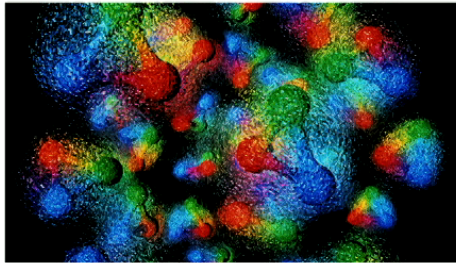


Example of an analytic continuation: Wick rotation

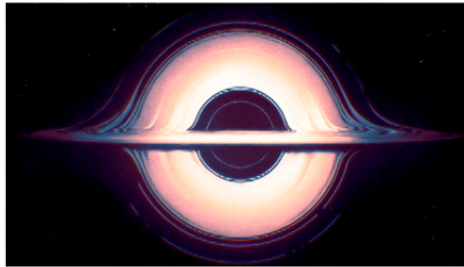


Cornerstone of many fields of physics

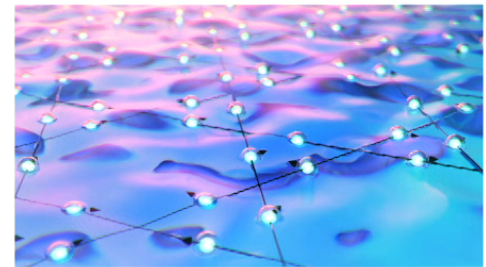
- Lattice field theory



- Black holes



- Condensed matter

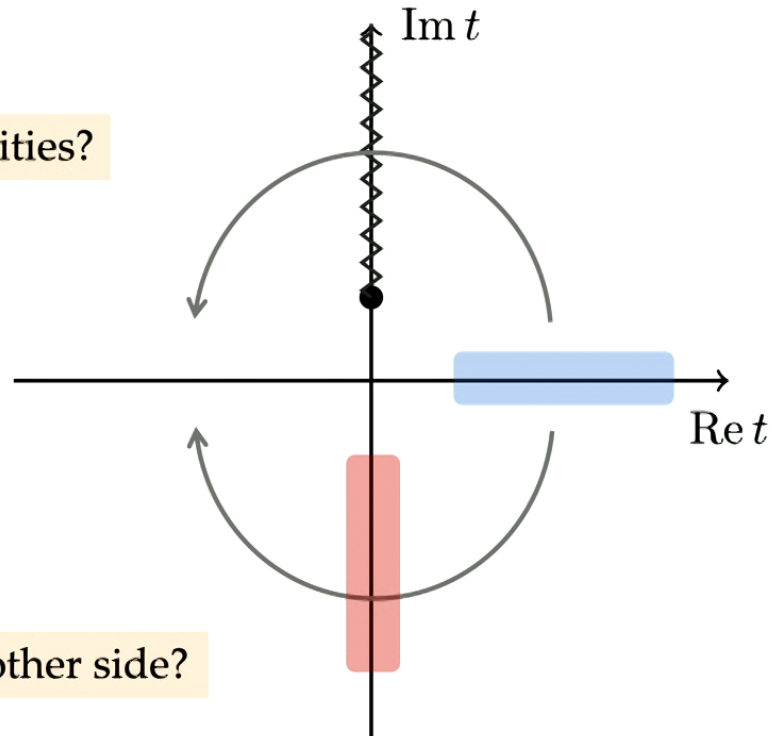


[Graphics: Quanta Magazine]

**This talk is about even more intricate ways of
analytically continuing time**

Requires deep understanding of the analytic structure

Where are the singularities?

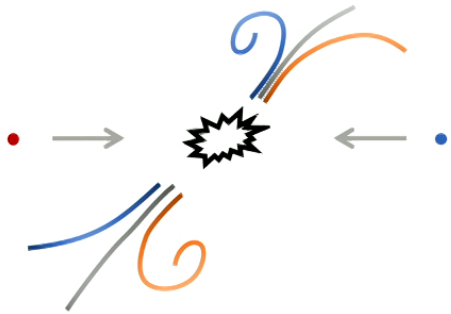


How to avoid them?

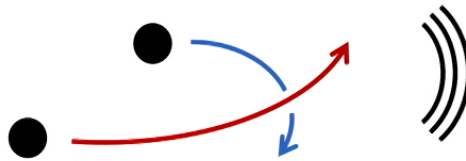
What's on the other side?

Plan for the talk

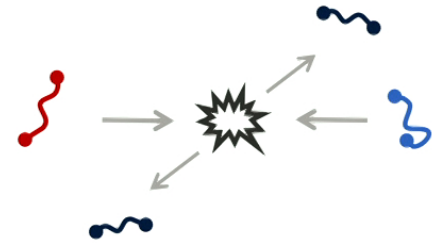
- Particle physics



- Gravitational waves



- String theory



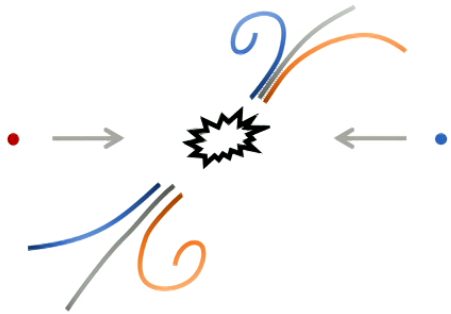
**Foundations of
scattering theory**



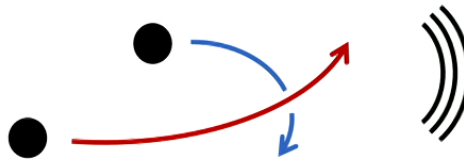
**Practical
applications**

Plan for the talk

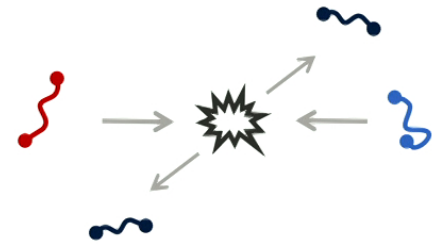
- Particle physics



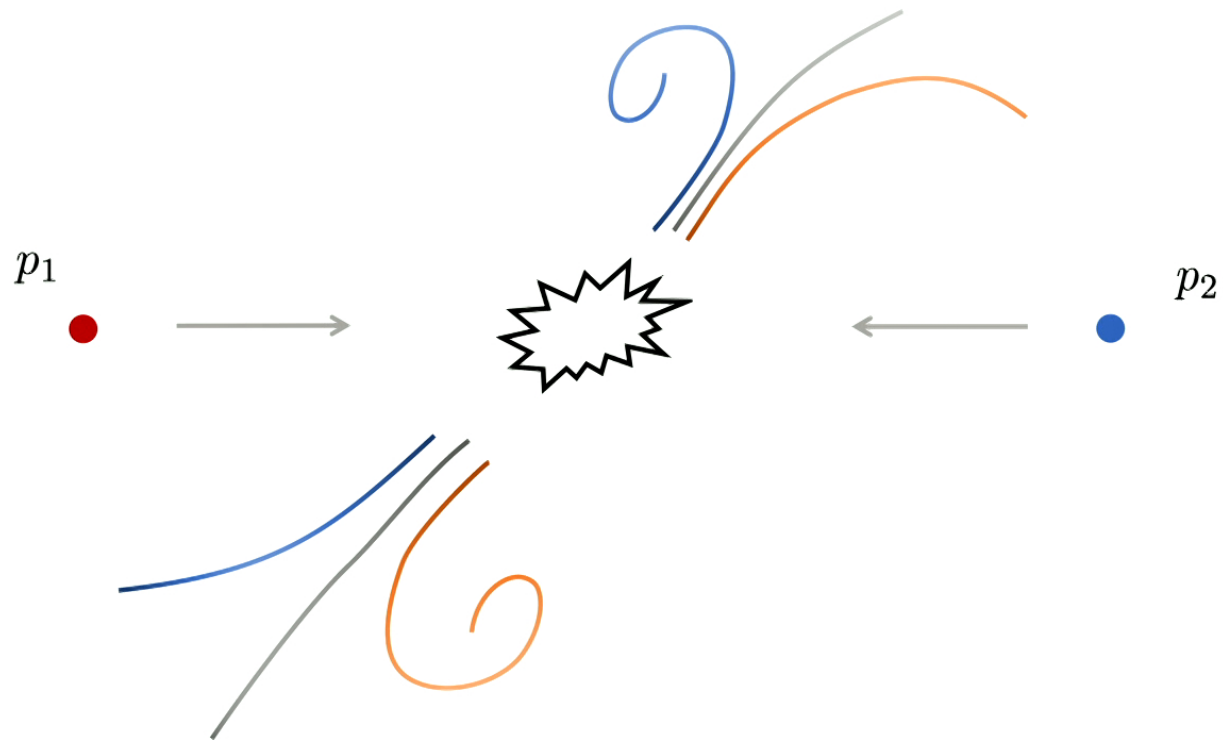
- Gravitational waves



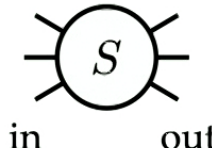
- String theory



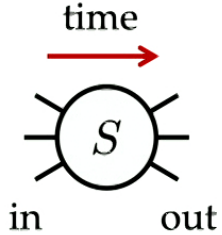
Collider experiments



The S-matrix

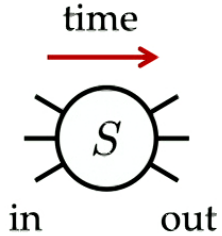
Matrix elements: $\langle \text{out} | S | \text{in} \rangle =$ 

The S-matrix

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Unitarity: $SS^\dagger = \mathbb{1}$

The S-matrix

Matrix elements: $\langle \text{out} | S | \text{in} \rangle =$ 


Unitarity: $SS^\dagger = \mathbb{1}$

Separate out the identity: $S = \mathbb{1} + iT$

Optical theorem

$$\frac{1}{2i}(T - T^\dagger) = \frac{1}{2}T^\dagger T$$

Sandwich between $\langle \text{out} |$ and $| \text{in} \rangle$; insert $\mathbb{1} = \sum_X |X\rangle\langle X|$



Optical theorem

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$$\text{Im} \left(\begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ T \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \text{in} \quad \text{out} \end{array} \right) = \frac{1}{2} \sum_X \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ T \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \text{in} \quad X \quad \text{out} \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ T^\dagger \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array}$$

Optical theorem

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Sandwich between $\langle \text{out} |$ and $|\text{in}\rangle$; insert $\mathbb{1} = \sum_X |X\rangle\langle X|$

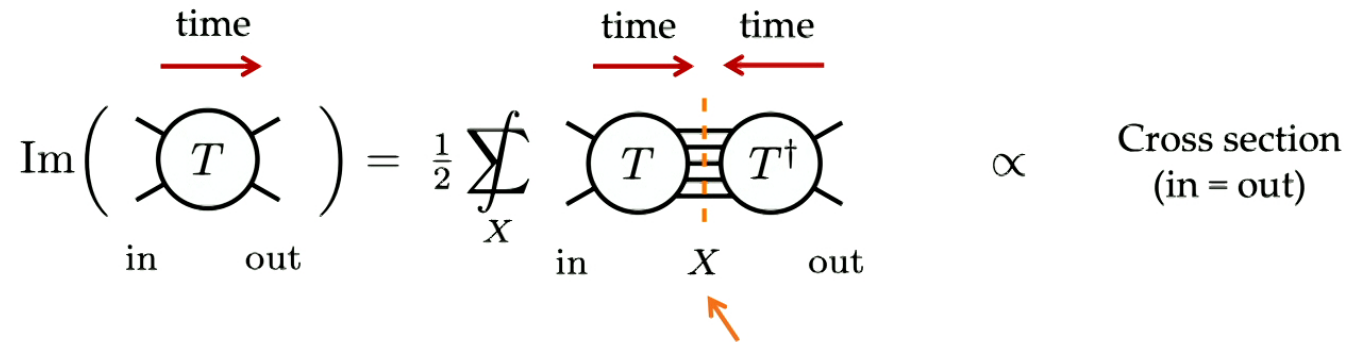
$$\text{Im} \left(\begin{array}{c} \text{---} \text{---} \\ | \quad | \\ \text{---} \text{---} \\ T \\ | \quad | \\ \text{---} \text{---} \\ \text{in} \quad \text{out} \end{array} \right) = \frac{1}{2} \sum_X \begin{array}{c} \text{---} \text{---} \\ | \quad | \\ \text{---} \text{---} \\ T \\ | \quad | \\ \text{---} \text{---} \\ \text{in} \quad X \quad \text{out} \\ | \quad | \\ \text{---} \text{---} \\ T^\dagger \\ | \quad | \\ \text{---} \text{---} \end{array}$$

Unitarity cut: Intermediate on-shell states $s \geq (m_1 + m_2 + \dots)^2$

Optical theorem

$$\frac{1}{2i}(T - T^\dagger) = \frac{1}{2}T^\dagger T$$

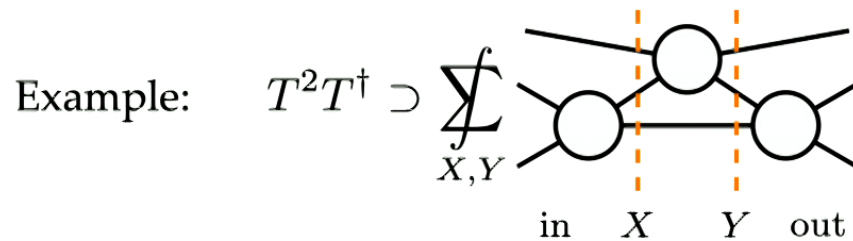
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Unitarity cut: Intermediate on-shell states $s \geq (m_1 + m_2 + \dots)^2$

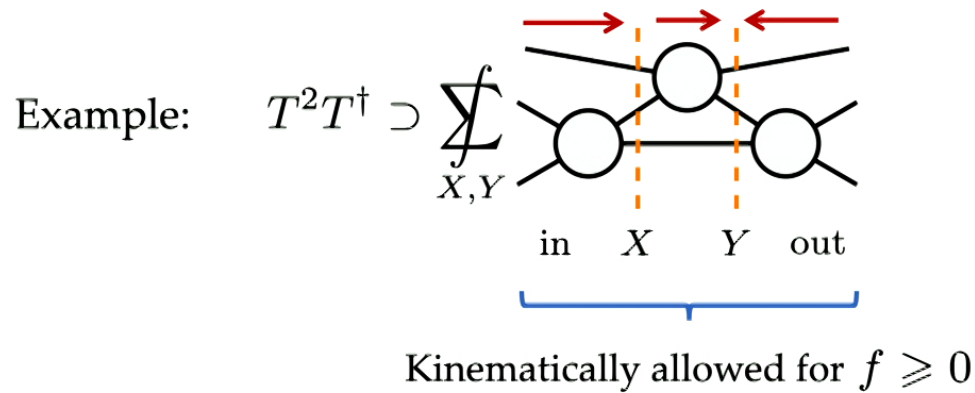
Unitarity implies an infinite number of discontinuities

$$\begin{aligned}
 \frac{1}{2i}(T - T^\dagger) &= \frac{1}{2}TT^\dagger \\
 &= \frac{1}{2}(T^2 - iT^2T^\dagger) \quad \left. \vphantom{\frac{1}{2i}(T - T^\dagger)} \right\} T^\dagger = T - iTT^\dagger \\
 &= \frac{1}{2}(T^2 - iT^3 - T^3T^\dagger) \\
 &= \dots
 \end{aligned}$$



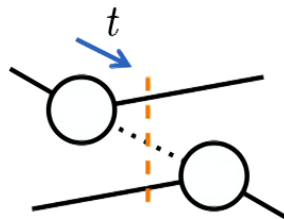
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Singularities encode physics

- Analyticity in momentum transfer $t = (p_2 + p_3)^2 \Leftrightarrow$ **locality**

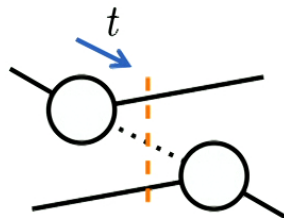


The diagram shows two external lines (solid black) entering from the left and exiting to the right. Two internal lines (solid black) connect two circular vertices. A blue arrow labeled t points to the upper internal line. Two vertical dashed orange lines are drawn between the vertices, and a dotted black line connects them.

$$\sim \frac{1}{t - \mu^2} \quad V(x) \sim \frac{e^{-\mu|x|}}{|x|}$$

Singularities encode physics

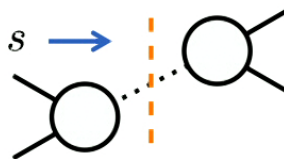
- Analyticity in momentum transfer $t = (p_2 + p_3)^2 \Leftrightarrow$ **locality**



A Feynman diagram showing two vertices connected by a dashed line. The left vertex has two external lines, and the right vertex has two external lines. A blue arrow labeled 't' points to the dashed line. A vertical dashed orange line is drawn between the two vertices.

$$\sim \frac{1}{t - \mu^2} \quad V(x) \sim \frac{e^{-\mu|x|}}{|x|}$$

- Analyticity in energy $s = (p_1 + p_2)^2 \Leftrightarrow$ **causality**

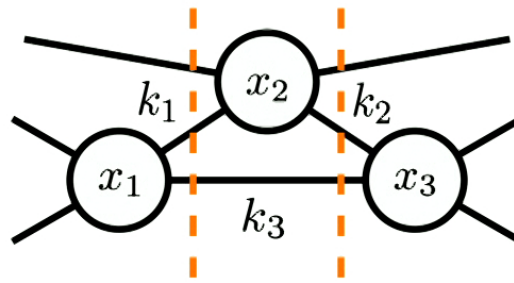


A Feynman diagram showing two vertices connected by a dashed line. The left vertex has two external lines, and the right vertex has two external lines. A blue arrow labeled 's' points to the dashed line. A vertical dashed orange line is drawn between the two vertices.

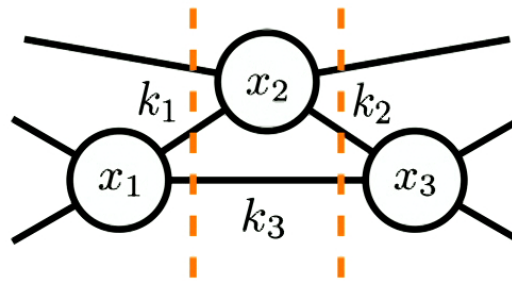
$$\sim \frac{1}{s - \mu^2} \quad \psi(t) \sim e^{-i\mu t} \theta(t)$$

[Reviews: Nussenzveig (Elsevier 1972), SM (Physics Reports 2024)]

Anomalous thresholds – most in complex kinematics (Landau singularities)

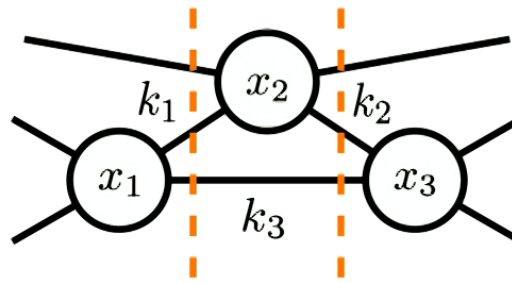


Anomalous thresholds – most in complex kinematics (Landau singularities)



- On-shell particles: $k_i^2 = m_i^2$
- Interacting at space-time points $x_1^\mu, x_2^\mu, x_3^\mu, \dots$

Anomalous thresholds – most in complex kinematics (Landau singularities)



- On-shell particles: $k_i^2 = m_i^2$
- Interacting at space-time points $x_1^\mu, x_2^\mu, x_3^\mu, \dots$
- In principle **all complex**: $k_i^\mu, x_i^\mu \in \mathbb{C}^4$

Why do we care?

Why do we care?

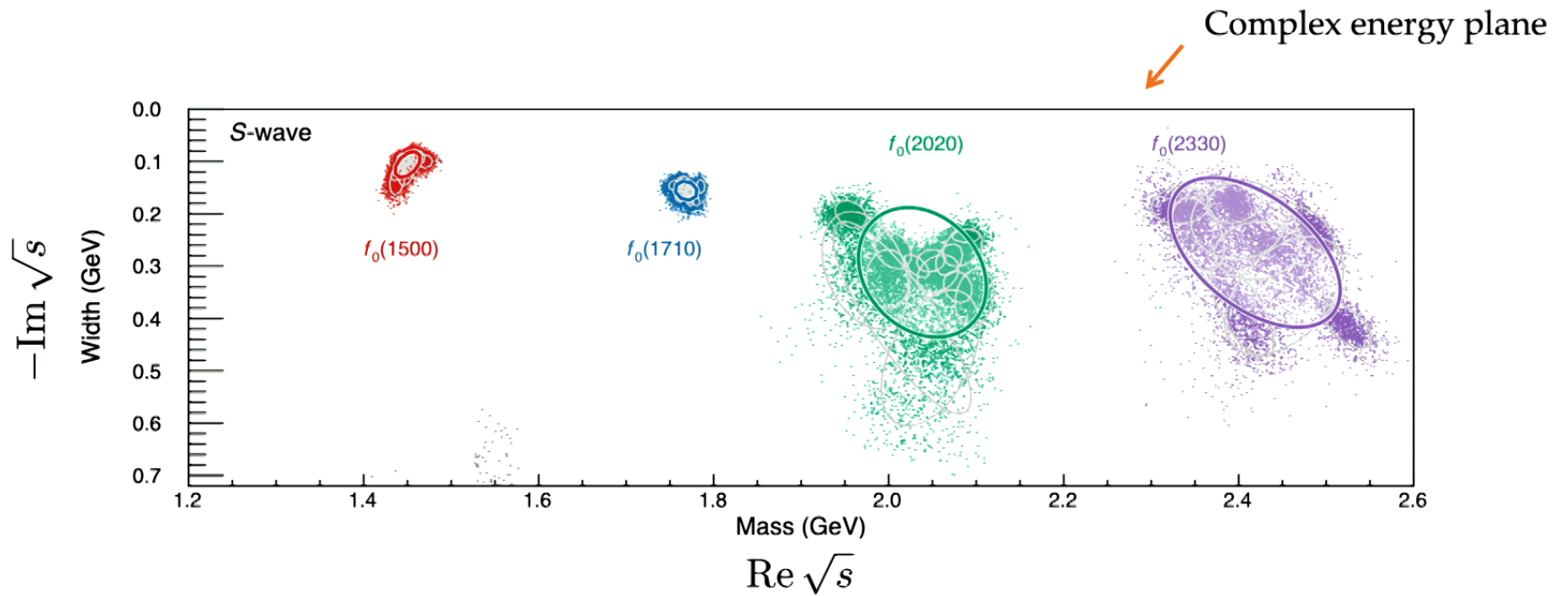
High-precision measurements in QFT

Simple example: Complex singularities in experiments

Example: $J/\psi \rightarrow \gamma\pi^0\pi^0$ and $\gamma K_S^0 K_S^0$ decays

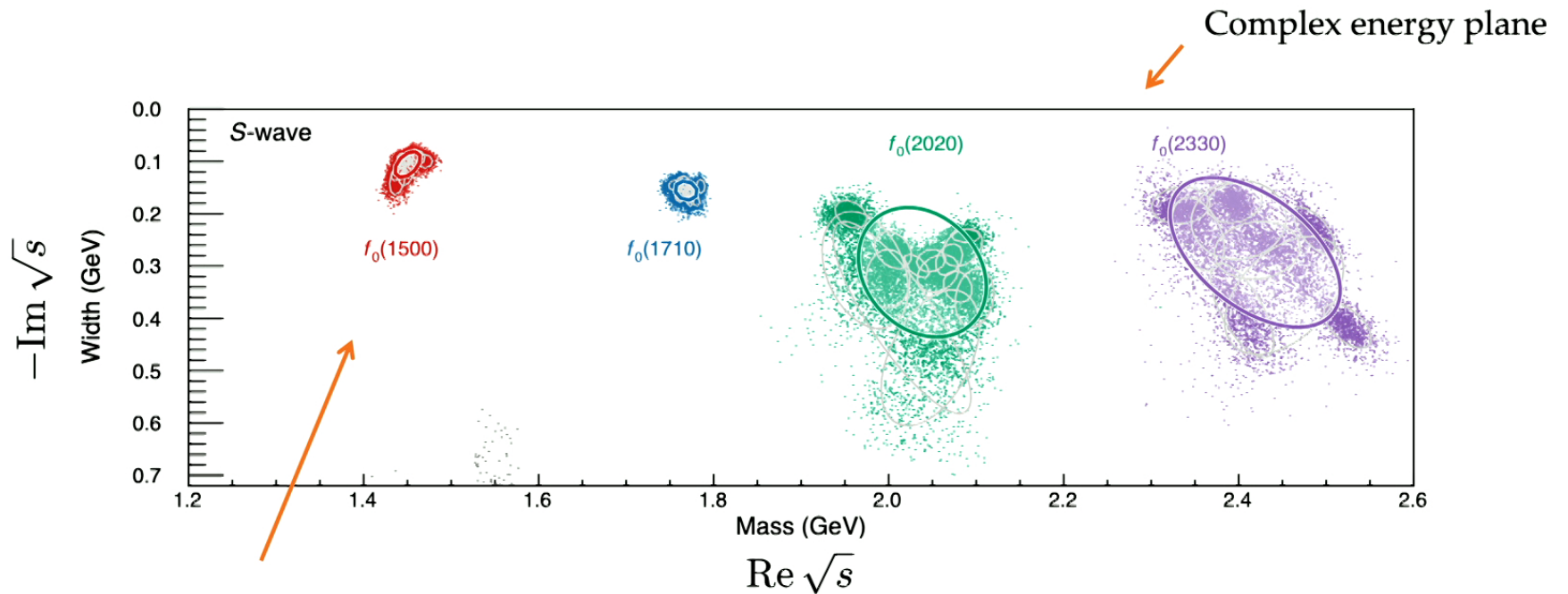
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Simple example: Complex singularities in experiments

Example: $J/\psi \rightarrow \gamma\pi^0\pi^0$ and $\gamma K_S^0 K_S^0$ decays



Complex pole = QCD bound state

[JPAC collaboration (Eur. Phys. J. C 82, 2022)]

[Submitted on 15 Mar 2022]

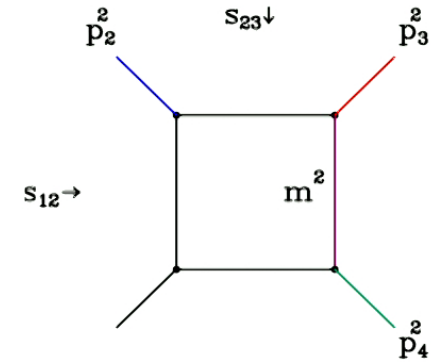
Snowmass white paper: Need for amplitude analysis in the discovery of new hadrons

Miguel Albaladejo, Marco Battaglieri, Lukasz Bibrzycki, Andrea Celentano, Igor V. Danilkin, Sebastian M. Dawid, Michael Doring, Cristiano Fanelli, Cesar Fernandez-Ramirez, Sergi Gonzalez-Solis, Astrid N. Hiller Blin, Andrew W. Jackura, Vincent Mathieu, Mikhail Mikhasenko, Victor I. Mokeev, Emilie Passemar, Robert J. Perry, Alessandro Pilloni, Arkaitz Rodas, Matthew R. Shepherd, Nathaniel Sherrill, Jorge A. Silva-Castro, Tomasz Skwarnicki, Adam P. Szczepaniak, Daniel Winney (for the JPAC Collaboration)

We highlight the need for the development of comprehensive amplitude analysis methods to further our understanding of hadron spectroscopy. Reaction amplitudes constrained by first principles of S -matrix theory and by QCD phenomenology are needed to extract robust interpretations of the data from experiments and from lattice calculations.

Simple example: Complex singularities in perturbative computations

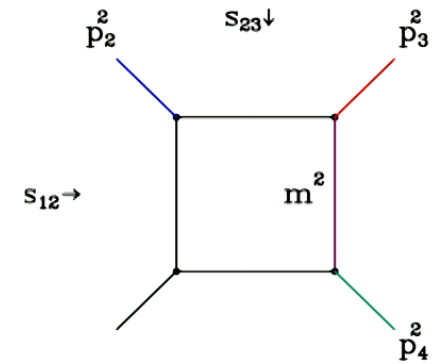
$$\begin{aligned}
 I_4^{\{D=4-2\epsilon\}}(0, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, 0, m^2) &= \frac{1}{(s_{12}s_{23} - m^2s_{12} - p_2^2p_4^2 + m^2p_2^2)} \\
 \times \left[\frac{1}{\epsilon} \ln \left(\frac{(m^2 - p_4^2)p_2^2}{(m^2 - s_{23})s_{12}} \right) + \text{Li}_2 \left(1 + \frac{(m^2 - p_3^2)(m^2 - s_{23})}{p_2^2m^2} \right) - \text{Li}_2 \left(1 + \frac{(m^2 - p_3^2)(m^2 - p_4^2)}{s_{12}m^2} \right) \right. \\
 + 2 \text{Li}_2 \left(1 - \frac{m^2 - s_{23}}{m^2 - p_4^2} \right) - 2 \text{Li}_2 \left(1 - \frac{p_2^2}{s_{12}} \right) + 2 \text{Li}_2 \left(1 - \frac{p_2^2(m^2 - p_4^2)}{s_{12}(m^2 - s_{23})} \right) \\
 \left. + 2 \ln \left(\frac{\mu m}{m^2 - s_{23}} \right) \ln \left(\frac{(m^2 - p_4^2)p_2^2}{(m^2 - s_{23})s_{12}} \right) \right] + \mathcal{O}(\epsilon)
 \end{aligned}$$



[QCDloop repository]

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 \end{aligned}$$



[QCDloop repository]

17 distinct singularities: Need to know them all!

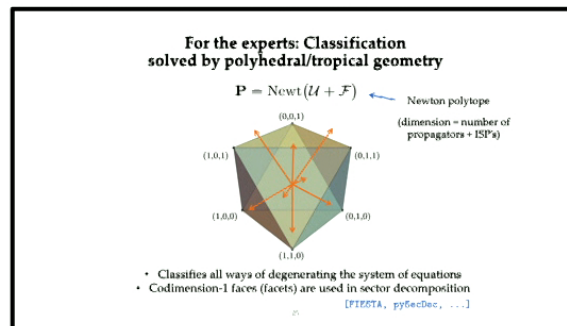
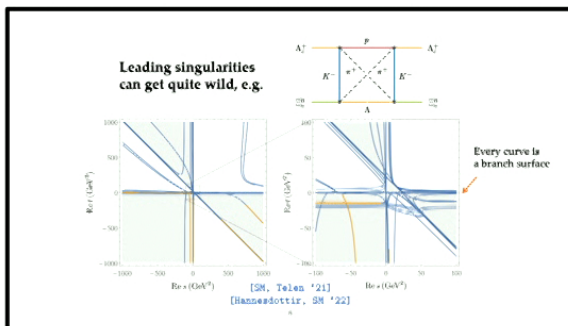
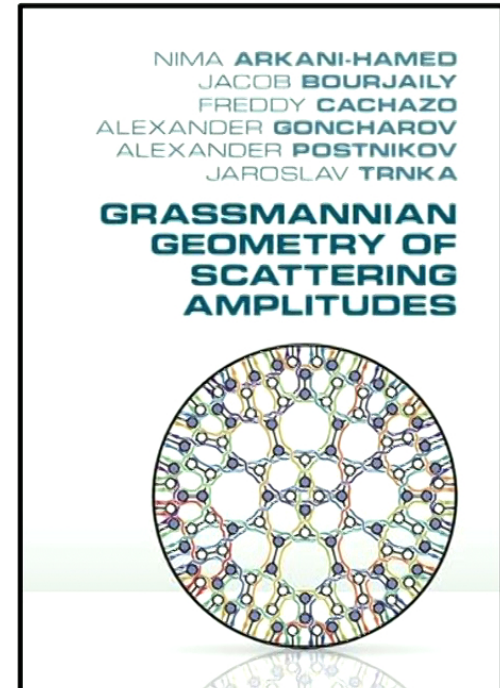
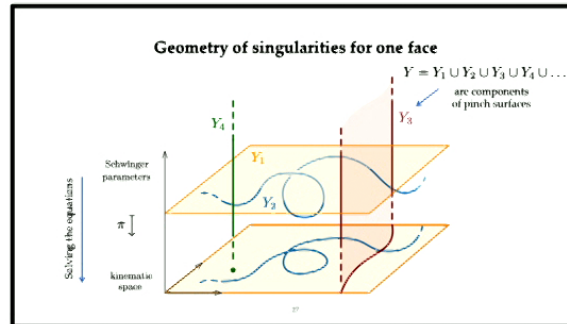
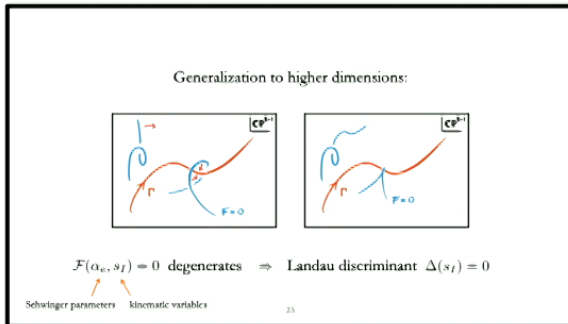
(one of the bottlenecks)

Analyticity enters at all stages of the modern pipeline for Standard Model computations

- Generalized unitarity
- Integration-by-parts reduction
 - Differential equations
 - Symbol alphabet
 - Singularity analysis
 - Elliptic functions
 - Finite-field methods
- Numerical integration
 - ...

[Reviews: Weinzierl (Springer 2022), Badger, Henn, Plefka, Zoia (Springer 2024)]

Start of an adventure into analytic properties of the S-matrix



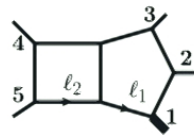
Highlight: 2-loop 6-pt MHV remainder function

17 pages

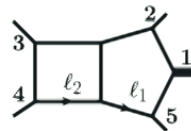
```
(-29u1^4)/3u + (P1^2u6(1 - u11)^(-1), (-1 + u211)/(-1 + u11 + u21), 1)
24 + (P1^2u6(u11)^(-1), (u11 + u211)^(-1), 11)/24 +
(P1^2u6(u11)^(-1), (u11 + u211)^(-1), 11)/24 +
(P1^2u6(1 - u11)^(-1), (-1 + u11)/(-1 + u11 + u21), 11)/24 +
(P1^2u6(u21)^(-1), (u21 + u211)^(-1), 11)/24 +
(P1^2u6(1 - u11)^(-1), (-1 + u11)/(-1 + u11 + u21), 11)/24 +
(P1^2u6(u11)^(-1), (u11 + u211)^(-1), 11)/24 +
(3u6(0, u, u11)^(-1), (u11 + u211)^(-1), 11)/2 +
(3u6(0, u, u21)^(-1), (u21 + u211)^(-1), 11)/2 +
(3u6(0, u, u11)^(-1), (u11 + u211)^(-1), 11)/2 +
(3u6(0, u, u21)^(-1), (u21 + u211)^(-1), 11)/2 +
(3u6(0, u, u11)^(-1), (u11 + u211)^(-1), 11)/2 +
G(0, u11)^(-1), G(0, u21)^(-1), 1/2 + G(0, u11)^(-1), G(0, u11 + u211)^(-1),
1) - G(0, u11)^(-1), G(0, u21)^(-1), 1/2 +
G(0, u11)^(-1), G(0, u11 + u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 +
G(0, u21)^(-1), G(0, u11)^(-1), 1/2 + G(0, u11)^(-1), G(0, u11 + u211)^(-1),
1) - G(0, u21)^(-1), G(0, u11)^(-1), 1/2 +
G(0, u21)^(-1), G(0, u11 + u211)^(-1), 1/2 -
G(0, u21)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u21)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u21)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 +
G(0, (-1 + u211)/(-1 + u11 + u21), (1 - u11)^(-1), 11/4 +
G(0, (-1 + u211)/(-1 + u11 + u21), (1 - u11)^(-1), 0, 11/4 -
G(0, (-1 + u211)/(-1 + u11 + u21), (1 - u11)^(-1), 1, 11/4 +
G(0, (-1 + u211)/(-1 + u11 + u21), (1 - u11)^(-1), (1 - u11)^(-1), 11/
4 - G(0, (-1 + u211)/(-1 + u11 + u21), (-1 + u211)/(-1 + u11 + u21),
(1 - u11)^(-1), 11/4 + G(0, u11)^(-1), G(0, u21)^(-1), 0, (u11 + u211)^(-1),
1) - G(0, u21)^(-1), G(0, u11)^(-1), 1/2 +
G(0, u11)^(-1), G(0, u21)^(-1), 0, u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 -
G(0, u11)^(-1), u211)^(-1), (u11 + u211)^(-1), 1/2 +
G(0, (-1 + u111)/(-1 + u11 + u21), (1 - u111)^(-1), 11/4 +
G(0, (-1 + u111)/(-1 + u11 + u21), (1 - u111)^(-1), 0, 11/4 -
G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1), 1, 11/4 +
G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1), (1 - u111)^(-1), 11/
4 - G(0, (-1 + u111)/(-1 + u11 + u221), (-1 + u111)/(-1 + u11 + u221),
(1 - u111)^(-1), 11/4 + G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1),
u111)^(-1), 0,
(1 - u221)^(-1), 11/4 + G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1),
u111)^(-1), 0,
(1 - u221)^(-1), 11/4 + G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1),
u111)^(-1),
(1 - u221)^(-1), 1, 11/4 + G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1),
u111)^(-1),
(1 - u221)^(-1), 1, 11/4 + G(0, (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1),
u111)^(-1),
(1 - u221)^(-1), 1/4 -
G(0, (-1 + u111)/(-1 + u11 + u221), (-1 + u111)/(-1 + u11 + u221), (1 - u111)^(-1),
u111)^(-1), 1/4 +
G(1 - u111)^(-1), (-1 + u211)/(-1 + u11 + u21), (1 - u111)^(-1), 1, 0, 11/4 -
G(1 - u111)^(-1), (-1 + u211)/(-1 + u11 + u21), (1 - u111)^(-1), 0, 11/
4 + G(1 - u111)^(-1), (-1 + u211)/(-1 + u11 + u21), (1 - u111)^(-1), 1,
.
.
.
(2*Hcal[1, 1, 1, 1, v12, 1, 2]^(-1))/2 + H(0, u11)+Zeta(3) +
H(0, u21)+Zeta(3) + H(0, u31)+Zeta(3) + (8u01, u11)+Zeta(3))/2 +
(8u01, u21)+Zeta(3))/2 + (8u01, u31)+Zeta(3))/2 +
Hcal[1, u1, 2, 2]^(-1)+Zeta(3))/2 - Hcal[1, u12, 2, 1]^(-1)+Zeta(3))/
2 + Hcal[1, u13, 1, 2]^(-1)+Zeta(3))/2 -
Hcal[1, u12, 2, 1]^(-1)+Zeta(3))/4 - Hcal[1, v1, 1, 2]^(-1)+Zeta(3))/
4 + Hcal[1, v12, 1, 2]^(-1)+Zeta(3))/4 - Hcal[1, v13, 1, 2]^(-1)+Zeta(3))/
4 + Hcal[1, v13, 2, 1]^(-1)+Zeta(3))/4
```

[Del Duca, Duhr, Smirnov (JHEP 2010)]

Highlight: State-of-the-art QCD computations (for example, $pp \rightarrow H + \text{jets}$)



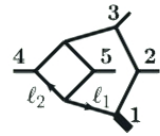
(a) PBmzz



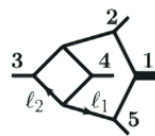
(b) PBzmm



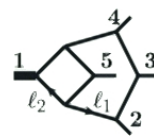
(c) PBzzz



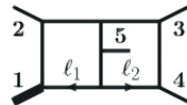
(d) HBmzz



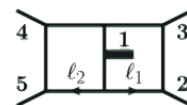
(e) HBzmm



(f) HBzzz



(g) DPMz



(h) DPzz

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia (2023)]

Summary of the 1st part

- Scattering amplitudes have an intricate **analytic structure**
- Understanding it is central for **precision measurements** in QFT

ias.ed.lu/amplitudes2024

Amplitudes 2024

Conference

Institute for Advanced Study, Princeton, NJ, United States
10 - 14 June 2024

Organizers:
Nima Arkani-Hamed, Jacob Bourjaily,
Hofie Hannesdottir, Sebastian Mizera

Image credits: Gaia Fontana

Carl P. Feinberg Program in Cross-Disciplinary Innovation

IAS INSTITUTE FOR
ADVANCED STUDY

ias.ed.lu/amplitudes2024

Amplitudes 2024

Summer School

Institute for Advanced Study, Princeton, NJ, United States
17 - 21 June 2024

Organizers:
Nima Arkani-Hamed, Jacob Bourjaily,
Hofie Hannesdottir, Sebastian Mizera

Lecturers:
Michael Borinsky
Fabrizio Caola
Simon Caron-Huot
Lucía Córdova
Henriette Elvang
Donal O'Connell

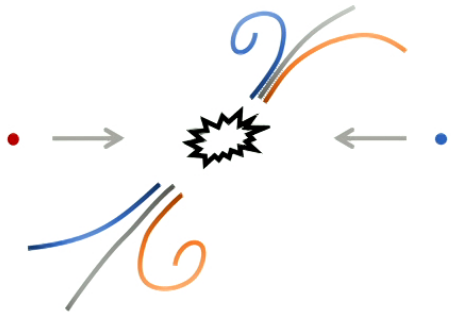
Image credits: Gaia Fontana

Carl P. Feinberg Program in Cross-Disciplinary Innovation

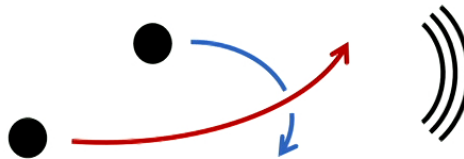
IAS INSTITUTE FOR
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Plan for the talk

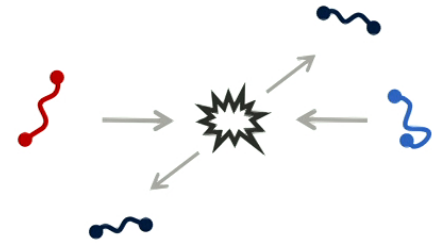
- Particle physics



- Gravitational waves



- String theory

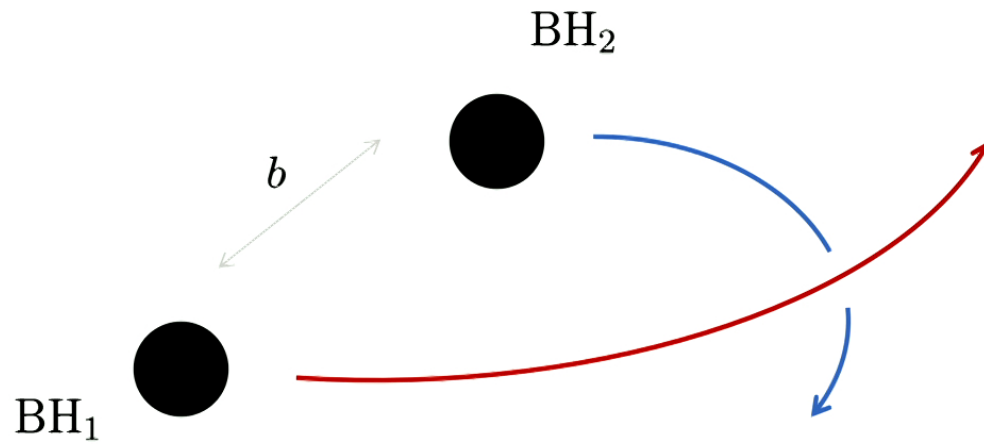


Beyond the S-matrix

Example: Gravitational Bremsstrahlung

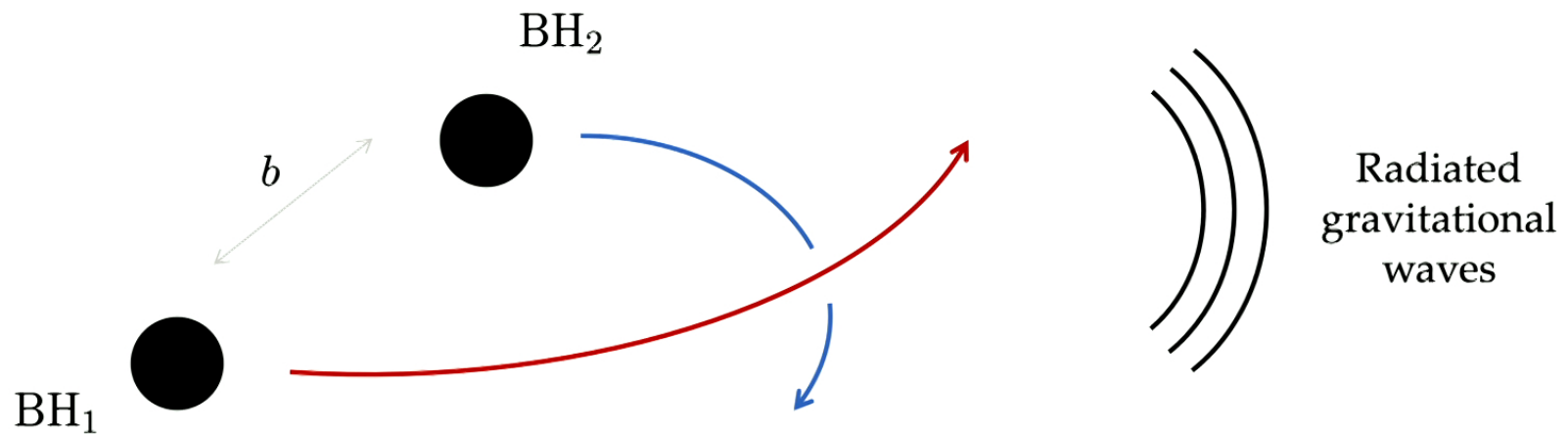
Beyond the S-matrix

Example: Gravitational Bremsstrahlung



Beyond the S-matrix

Example: Gravitational Bremsstrahlung



Leading order in $\frac{GM}{b}$ computed in [Kovacs, Thorne (Astrophys. J. 224, 1978)]

Asymptotic observables beyond the S-matrix

Expectation value of the metric $h^{\mu\nu}$:

$$\langle \text{BH}'_1 \text{BH}'_2 | a_h^{\dagger \text{out}} | \text{BH}_1 \text{BH}_2 \rangle$$

Asymptotic observables beyond the S-matrix

Expectation value of the metric $h^{\mu\nu}$:

$$\langle \text{BH}'_1 \text{BH}'_2 | \underbrace{a_h^{\dagger \text{out}}}_{S^\dagger a_h^{\dagger \text{in}} S} | \text{BH}_1 \text{BH}_2 \rangle$$
$$\uparrow$$
$$\mathbb{1} = \sum_X |X\rangle \langle X|$$

Asymptotic observables beyond the S-matrix

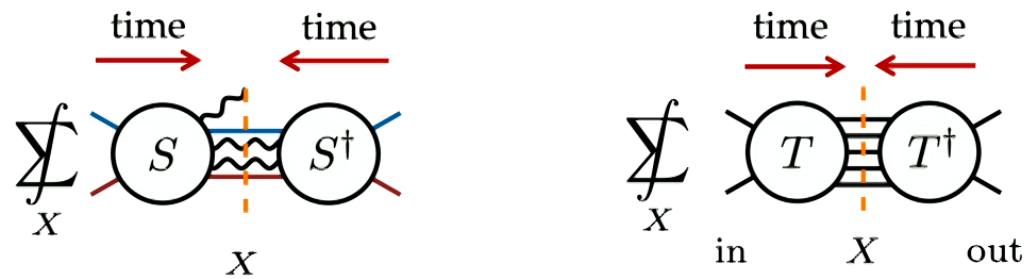
Expectation value of the metric $h^{\mu\nu}$:

$$\langle \text{BH}'_1 \text{BH}'_2 | \underbrace{a_h^{\dagger \text{out}}}_{S^\dagger a_h^{\dagger \text{in}} S} | \text{BH}_1 \text{BH}_2 \rangle = \sum_X \text{[Diagram]}$$

\uparrow
 $\mathbb{1} = \sum_X |X\rangle\langle X|$

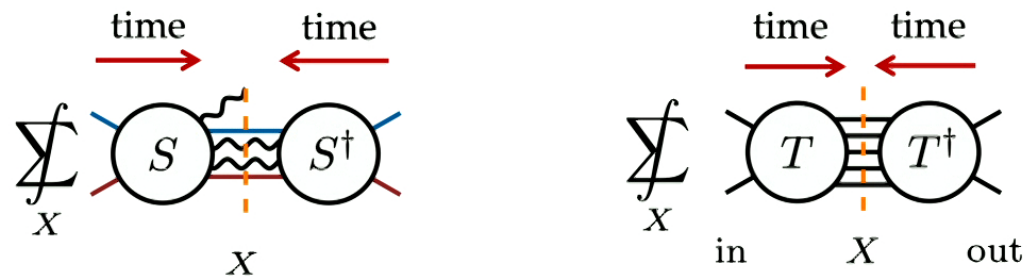
[Kosower, Maybee, O'Connell (JHEP 2019)]

Looks familiar...



Could it be related to the scattering amplitude?

Looks familiar...



Could it be related to the scattering amplitude?

Yes, through crossing!

Interlude: CPT theorem

Interlude: CPT theorem

$$\langle \text{out} | S | \text{in} \rangle = \text{Diagram}$$

Interlude: CPT theorem

$$\begin{aligned}
 \langle \text{out} | S | \text{in} \rangle &= \text{Diagram 1} \\
 &\parallel \\
 \langle \overline{\text{in}} | S | \overline{\text{out}} \rangle &= \text{Diagram 2}
 \end{aligned}$$

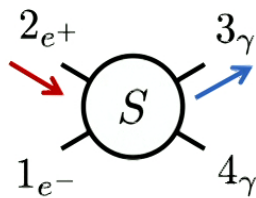
Diagram 1: A circle labeled S with a red arrow labeled "time" pointing to the right above it. On the left side, two lines enter labeled e^+ (top) and e^- (bottom), with the word "in" centered below them. On the right side, three lines exit labeled γ (top), γ (middle), and γ (bottom), with the word "out" centered below them.

Diagram 2: A circle labeled S with a red arrow labeled "time" pointing to the right above it. On the left side, three lines enter labeled γ (top), γ (middle), and γ (bottom), with the word " $\overline{\text{out}}$ " centered below them. On the right side, two lines enter labeled e^- (top) and e^+ (bottom), with the word " $\overline{\text{in}}$ " centered below them.

CP = exchange particles and anti-particles

T = exchange incoming and outgoing states

Crossing: Analog of CPT for individual particles (consider $2 \rightarrow 2$ case first)

$$\langle 34 | S | 12 \rangle = \text{Diagram}$$


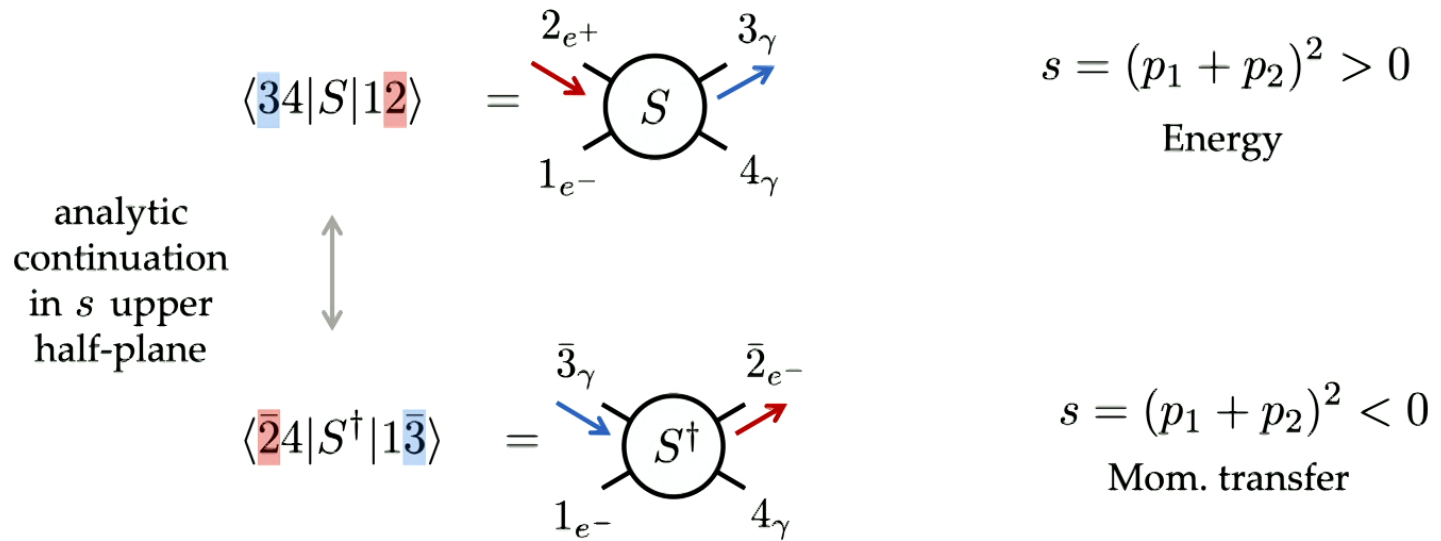
Crossing: Analog of CPT for individual particles

(consider $2 \rightarrow 2$ case first)

$$\begin{aligned}
 \langle \bar{3}_4 | S | 1 \bar{2} \rangle &= \text{Diagram with } S \text{ vertex} \\
 &\quad \text{Inputs: } 1_{e^-} \text{ (bottom-left), } 2_{e^+} \text{ (top-left, red arrow)} \\
 &\quad \text{Outputs: } 3_\gamma \text{ (top-right, blue arrow), } 4_\gamma \text{ (bottom-right)} \\
 &\quad \updownarrow \\
 \langle \bar{2}_4 | S^\dagger | 1 \bar{3} \rangle &= \text{Diagram with } S^\dagger \text{ vertex} \\
 &\quad \text{Inputs: } 1_{e^-} \text{ (bottom-left), } \bar{3}_\gamma \text{ (top-left, blue arrow)} \\
 &\quad \text{Outputs: } 2_{e^-} \text{ (top-right, red arrow), } 4_\gamma \text{ (bottom-right)}
 \end{aligned}$$

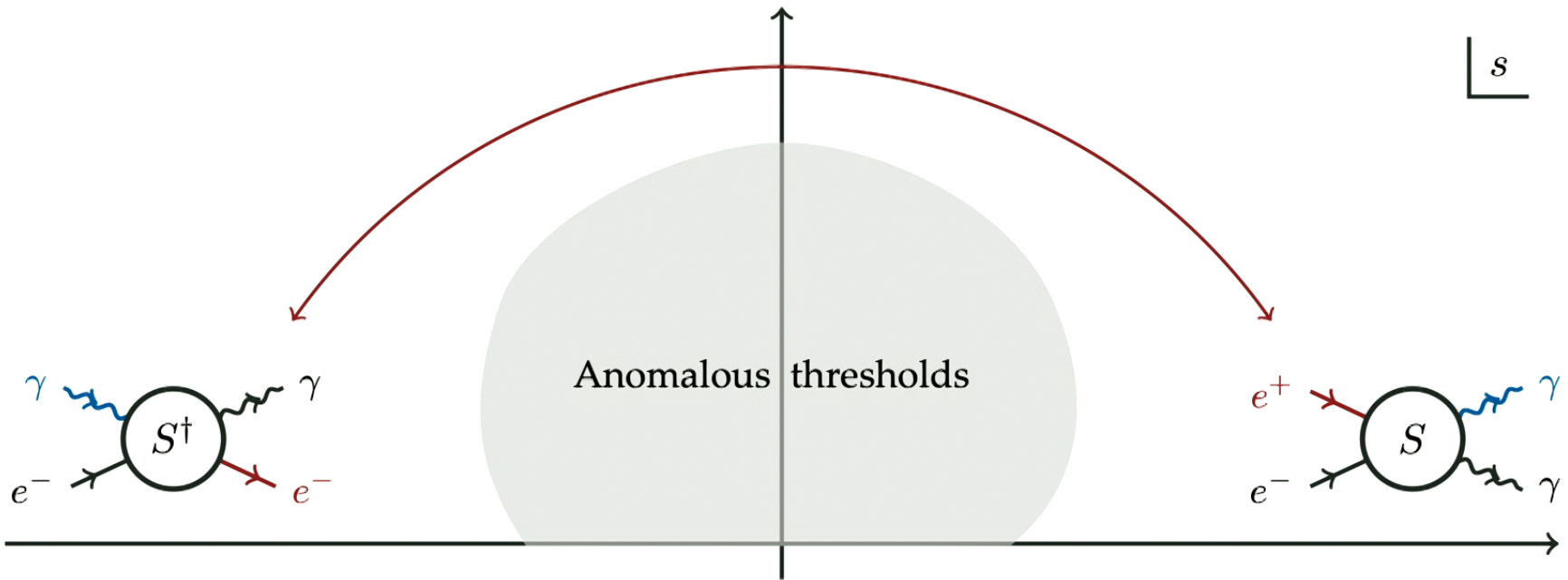
Crossing: Analog of CPT for individual particles

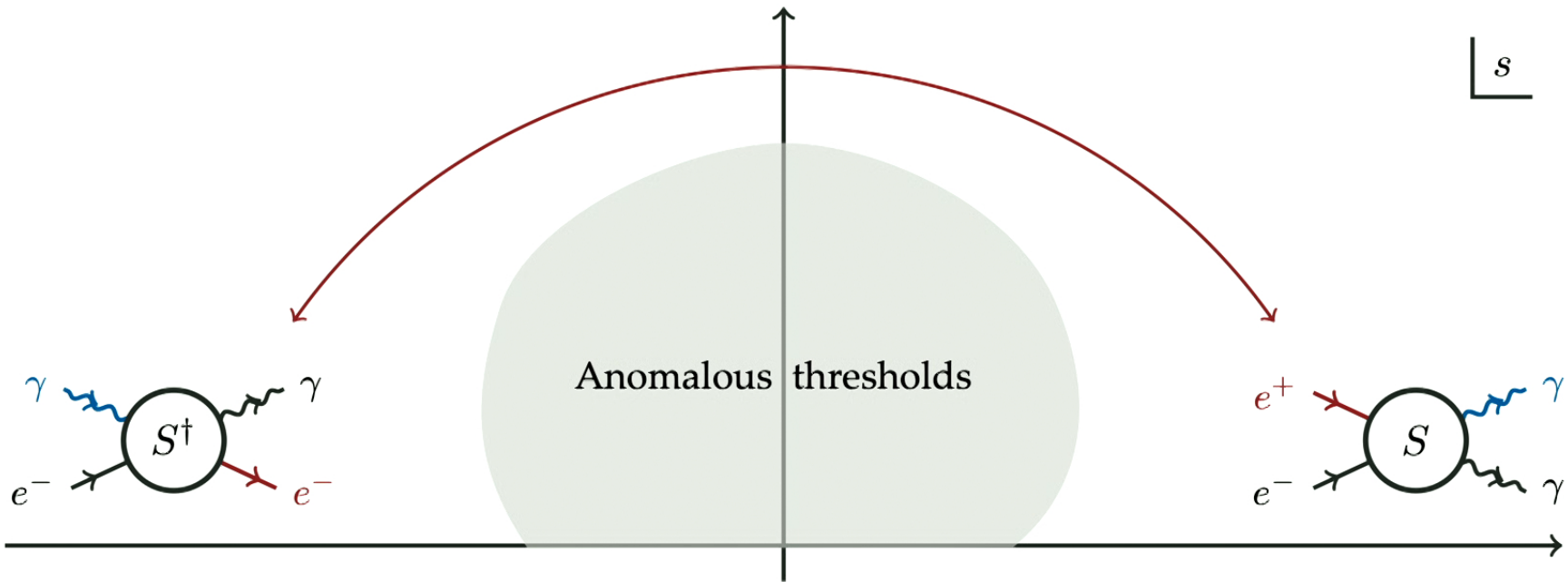
(consider $2 \rightarrow 2$ case first)



[Gell-Mann, Goldberger, Thirring (Phys. Rev. 1954)]

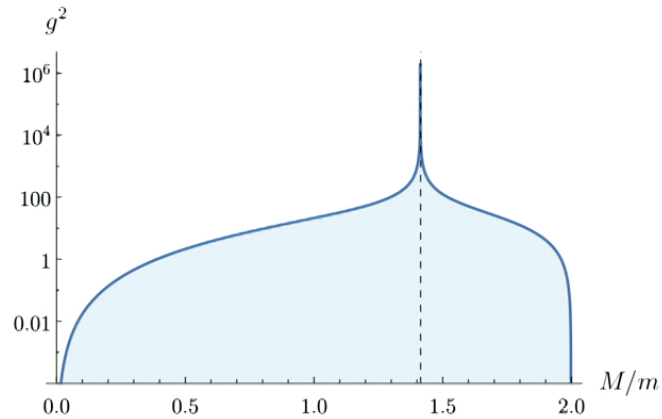
[Bros, Epstein, Glaser (Helv. Phys. Acta 1964)]





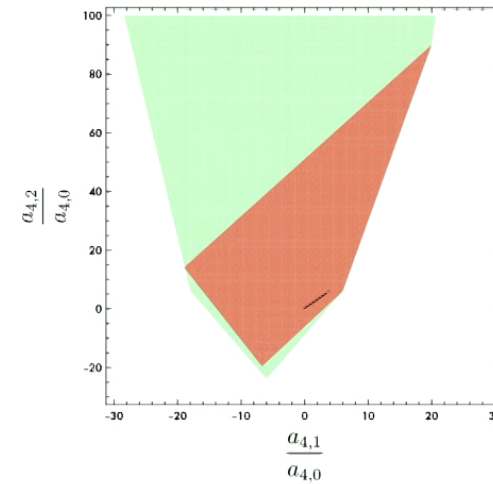
Turns out to be incredibly useful for $2 \rightarrow 2$ scattering

- S-matrix bootstrap



[Paulos, Penedones, Toledo, van Rees, Vieira (JHEP 2017), ...]

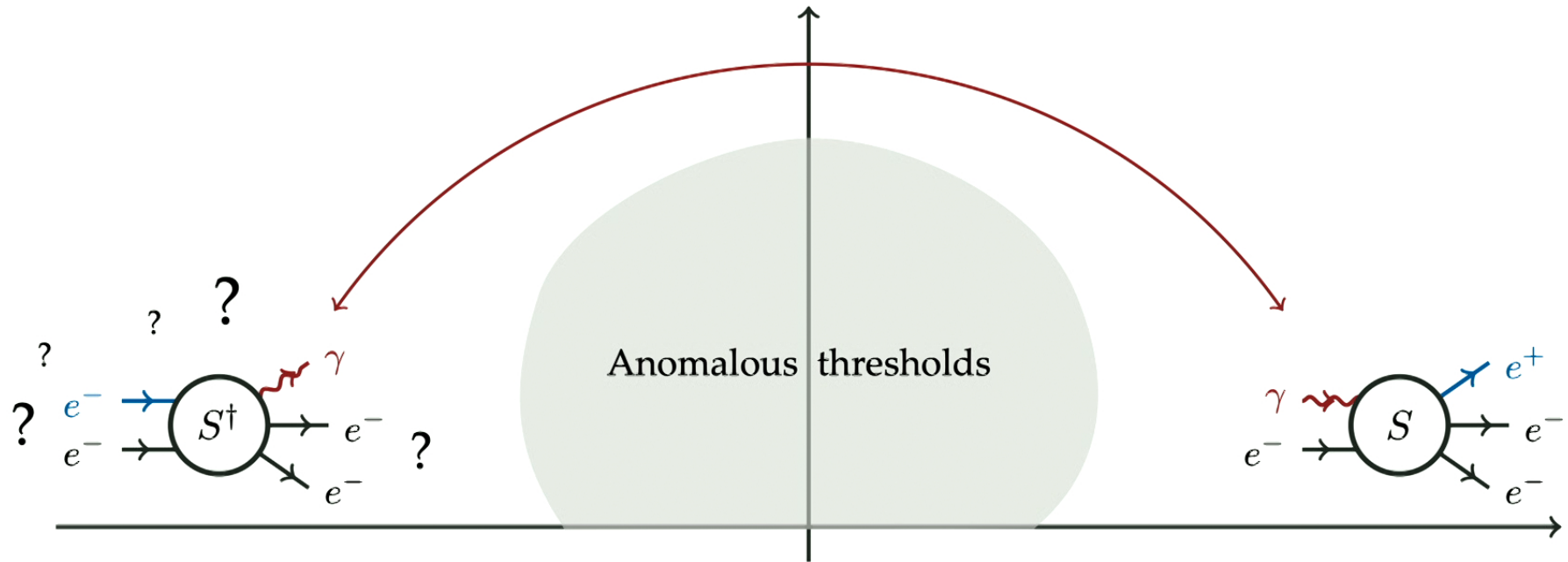
- Bounds on effective field theories



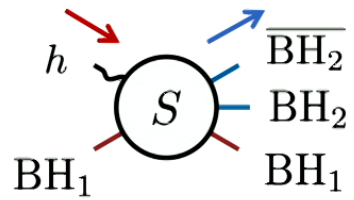
[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi (JHEP 2006), ...]

[Plot from Bern, Kosmopoulos, Zhiboedov (J. Phys. A 2021)]

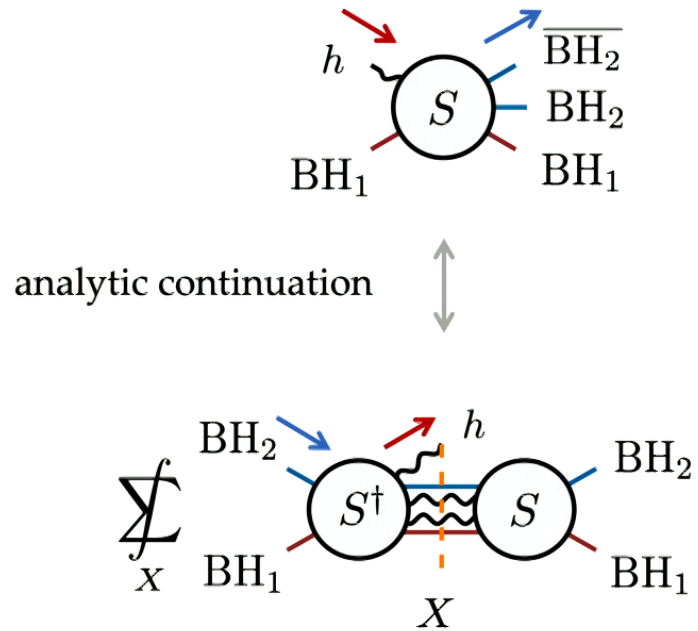
Open question since 1960's



Surprises in crossing beyond $2 \rightarrow 2$ scattering



Surprises in crossing beyond $2 \rightarrow 2$ scattering



Start of an adventure into analytic continuation between observables

CROSSING SYMMETRY IN 2 TO 2 SCATTERING

Proven for the non-perturbative amplitude at fixed momentum transfer $t < 0$ in theories with mass gap

Paths of analytic continuation in unshaded region

u-channel physical region approached from opposite side

s-channel physical region

Possible singularities in shaded region

[Feyn, Epstein, Glaser 1964, 1965]

SIMPLE EXAMPLE AT TREE LEVEL

$$\mathcal{M}_{s_{45} \rightarrow 21} = \frac{g^2}{(s_{45} - m_{45}^2 + i\epsilon)(s_{13} - m_{13}^2)}$$

Try rotating s_{13} in the lower half plane at fixed s_{45}

$$[\mathcal{M}_{s_{45} \rightarrow 21}]_{\mathcal{M}'} = \frac{g^2}{(s_{45} - m_{45}^2 + i\epsilon)(s_{13} - m_{13}^2 - i\epsilon)}$$

$$= \frac{g^2}{(s_{45} - m_{45}^2 - i\epsilon)(s_{13} - m_{13}^2 - i\epsilon)} - 2\pi i \delta(s_{45} - m_{45}^2) \frac{g^2}{(s_{13} - m_{13}^2 - i\epsilon)}$$

\mathcal{M}'

\mathcal{M}

\mathcal{M}'

EXAMPLE 6 PT ASYMPTOTIC MEASUREMENTS

Scattering amplitudes

Inclusive amplitudes

Out-of-time-ordered correlators

CONTINUING AROUND ANOMALOUS THRESHOLD

Local analyticity can be subtle: might need to continue past anomalous threshold on second sheet

Expected from axiomatic field theory

CROSSING CHECK FOR PENTAGON

[Chicherin, Henn, Mitev 2017]

BOCHNER'S TUBE THEOREM

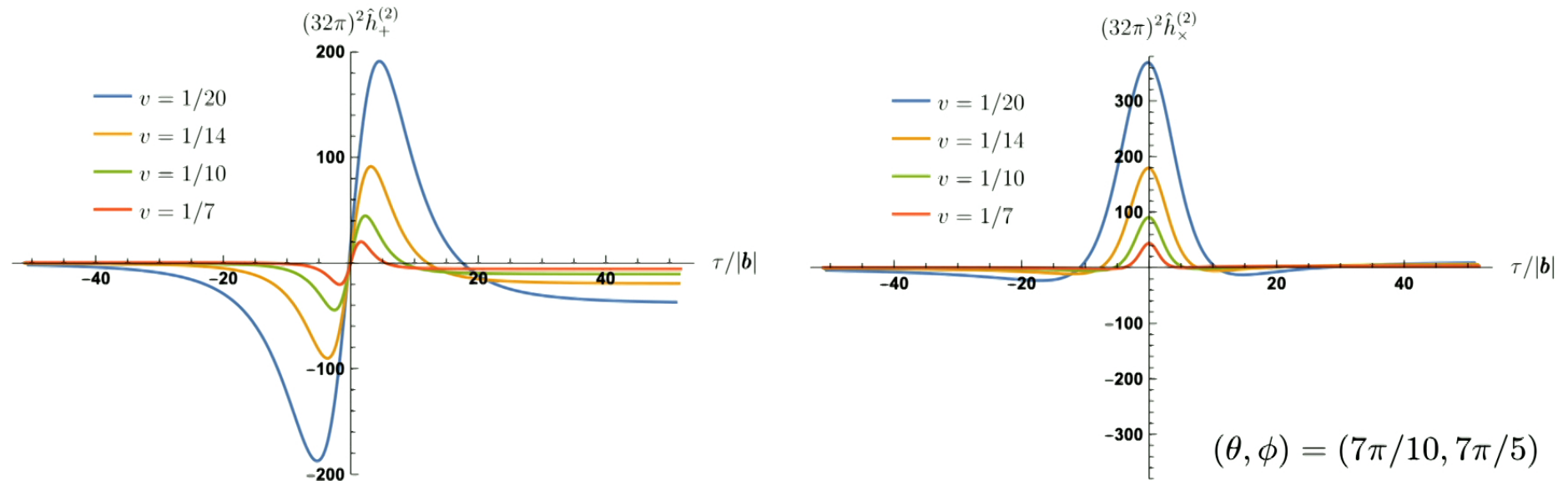
(I.) Local analyticity

(II.) Global analyticity

(III.) Connect physical regions on sheet

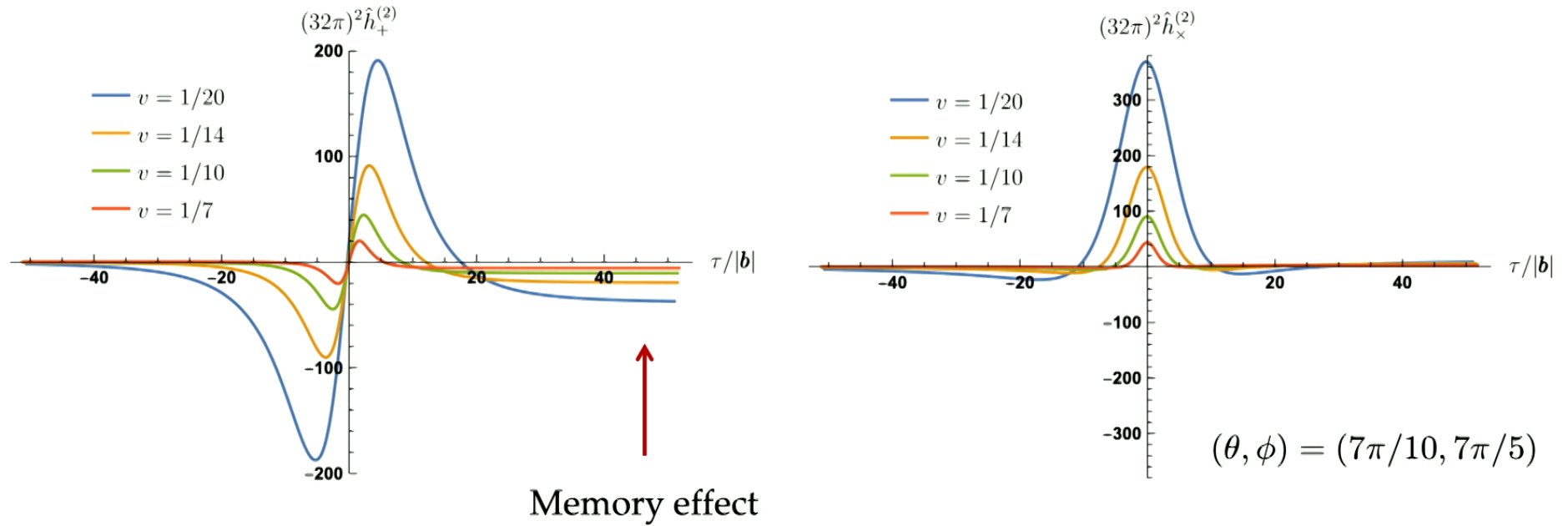
Highlight: First computation Gravitational Bremsstrahlung at NLO

Highlight: First computation Gravitational Bremsstrahlung at NLO



[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini (JHEP 2023), Herderschee, Roiban, Teng (JHEP 2023), Elkhidir, O'Connell, Sergola, Vazquez-Holm (JHEP 2023), Caron-Huot, Giroux, Hannesdottir, SM (JHEP 2023a)]

Highlight: First computation Gravitational Bremsstrahlung at NLO

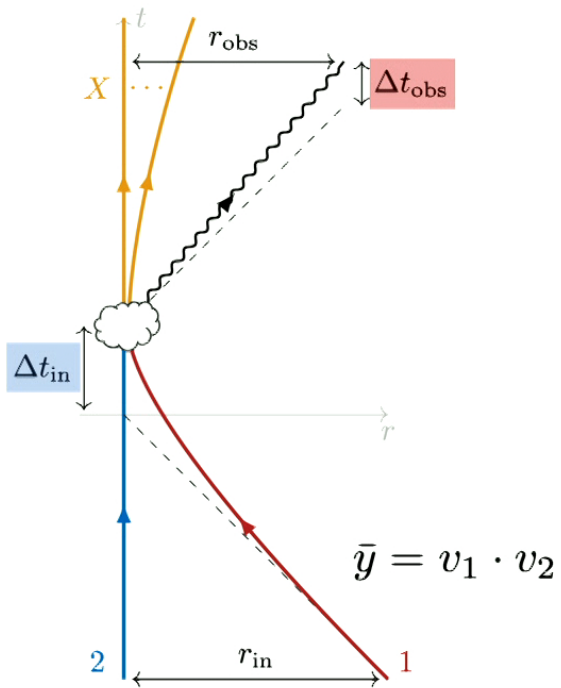


[Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini (JHEP 2023), Herderschee, Roiban, Teng (JHEP 2023), Elkhidir, O'Connell, Sergola, Vazquez-Holm (JHEP 2023), Caron-Huot, Giroux, Hannesdottir, SM (JHEP 2023a)]

Physical intuition for the result: Infrared divergence

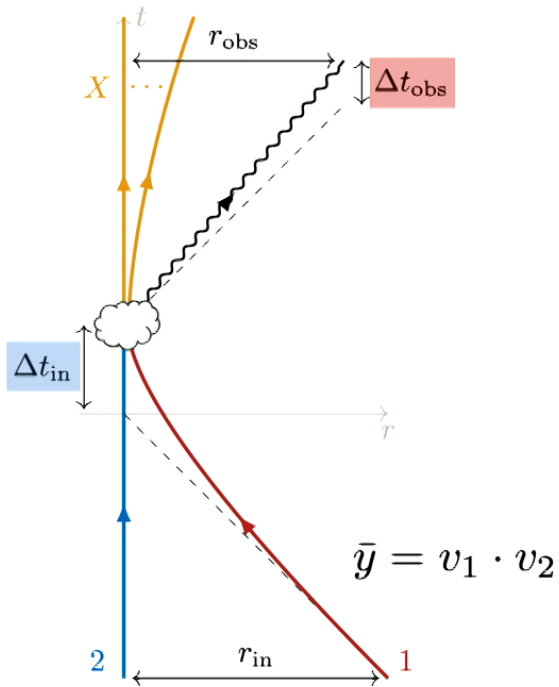
Physical intuition for the result: Infrared divergence

$$\sum_X \text{diagram} \sim (\text{tree-level}) \exp \left[-iG_N \hat{k} \cdot (\bar{p}_1 + \bar{p}_2) \left(2 + \frac{2 - 3\bar{y}^{-2}}{(1 - \bar{y}^{-2})^{\frac{3}{2}}} \right) \log \frac{\Lambda}{\mu_{\text{IR}}} \right]$$



Physical intuition for the result: Infrared divergence

$$\sum_X \text{tree-level} \left[S \text{---} S^\dagger \right] \sim \text{(tree-level)} \exp \left[-iG_N \hat{k} \cdot (\bar{p}_1 + \bar{p}_2) \left(2 + \frac{2 - 3\bar{y}^{-2}}{(1 - \bar{y}^{-2})^{\frac{3}{2}}} \right) \log \frac{\Lambda}{\mu_{\text{IR}}} \right]$$

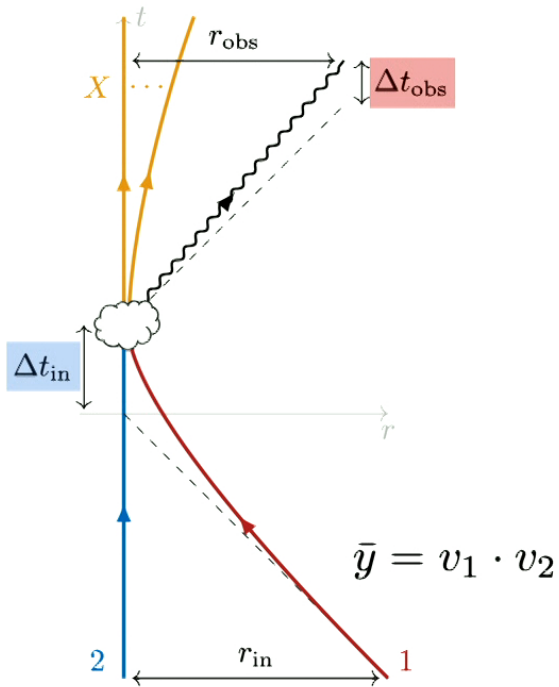


Two sources of time delays:

$$\Delta t_{\text{in}} = -G_N \hat{k} \cdot (\bar{p}_1 + \bar{p}_2) \times \frac{2 - 3\bar{y}^{-2}}{(1 - \bar{y}^{-2})^{\frac{3}{2}}} \log \frac{r_{\text{in}}}{b}$$

Physical intuition for the result: Infrared divergence

$$\sum_X \text{tree-level} \left(S \text{---} S^\dagger \right) \sim (\text{tree-level}) \exp \left[-iG_N \hat{k} \cdot (\bar{p}_1 + \bar{p}_2) \left(2 + \frac{2 - 3\bar{y}^{-2}}{(1 - \bar{y}^{-2})^{\frac{3}{2}}} \right) \log \frac{\Lambda}{\mu_{\text{IR}}} \right]$$



Two sources of time delays:

$$\Delta t_{\text{in}} = -G_N \hat{k} \cdot (\bar{p}_1 + \bar{p}_2) \times \frac{2 - 3\bar{y}^{-2}}{(1 - \bar{y}^{-2})^{\frac{3}{2}}} \log \frac{r_{\text{in}}}{b}$$

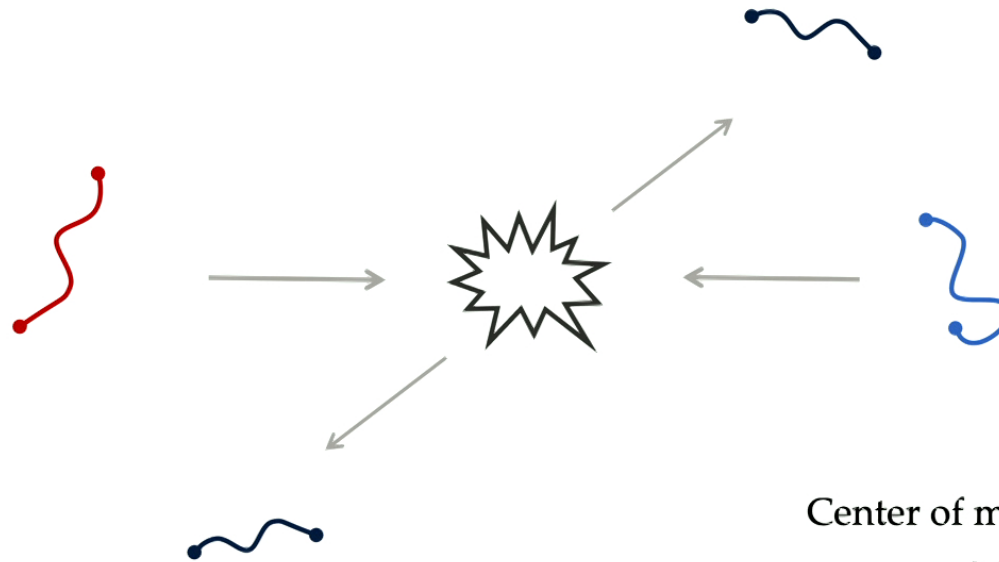
$$\Delta t_{\text{obs}} = -2G_N \hat{k} \cdot (\bar{p}_1 + \bar{p}_2) \log \frac{r_{\text{obs}}}{b}$$

[Caron-Huot, Giroux, Hannesdottir, SM (JHEP 2023a)]
[cf. Sahoo, Sen (JHEP 2019)]

Summary of the 2nd part

- Crossing connects different **observables**
- Used to obtain new results in **gravitational-wave physics** and beyond

String theory as a practical window for quantum gravity



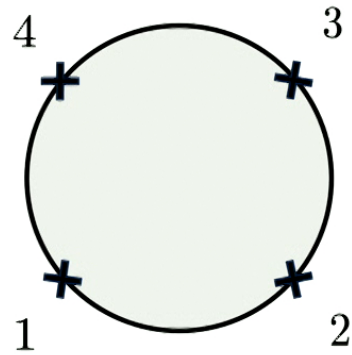
Center of mass energy

$$s \sim M_{\text{Planck}}^2$$

[Gross, Mende (Nucl. Phys. B 1988)]

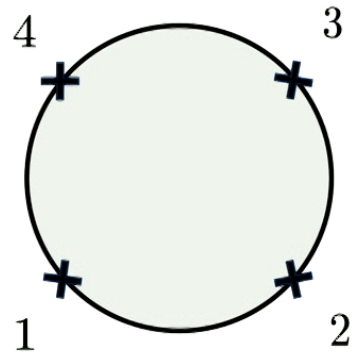
Euclidean vs. Lorentzian worldsheets

Euclidean vs. Lorentzian worldsheets



**Euclidean worldsheet
(Riemann surface)**

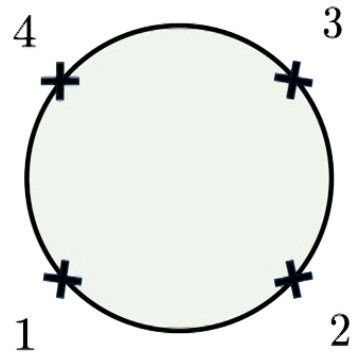
Euclidean vs. Lorentzian worldsheets



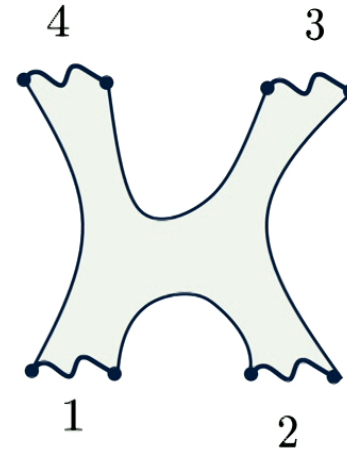
Euclidean worldsheet (Riemann surface)

- Textbook formulation
- Easy to use CFT technology

Euclidean vs. Lorentzian worldsheets



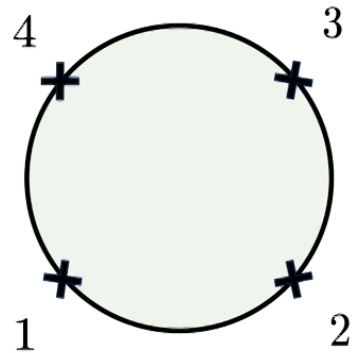
**Euclidean worldsheet
(Riemann surface)**



Lorentzian worldsheet

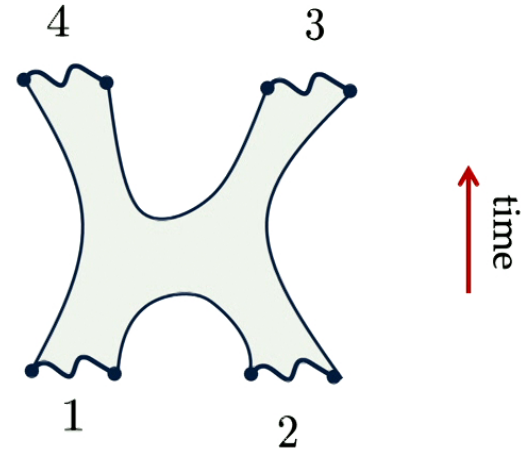
- Textbook formulation
- Easy to use CFT technology

Euclidean vs. Lorentzian worldsheets



**Euclidean worldsheet
(Riemann surface)**

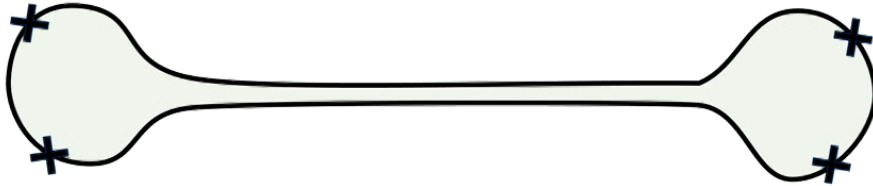
- Textbook formulation
- Easy to use CFT technology



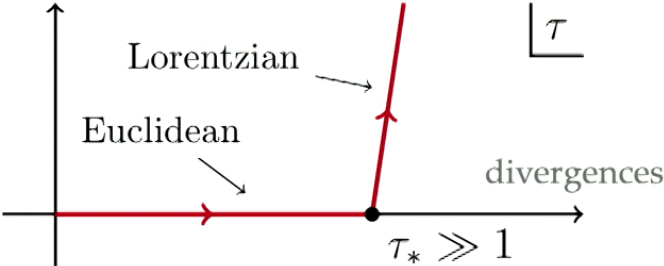
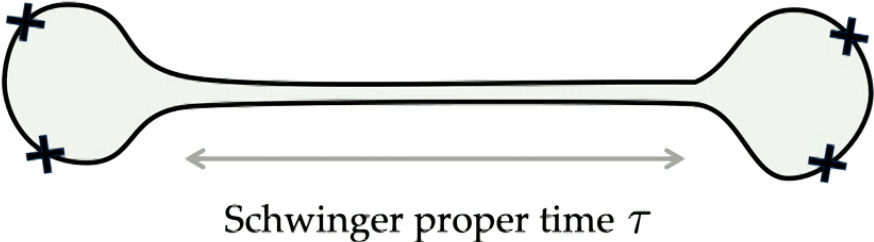
Lorentzian worldsheet

- Not well understood
- Essential for practical computations!

Wick rotation of the worldsheet

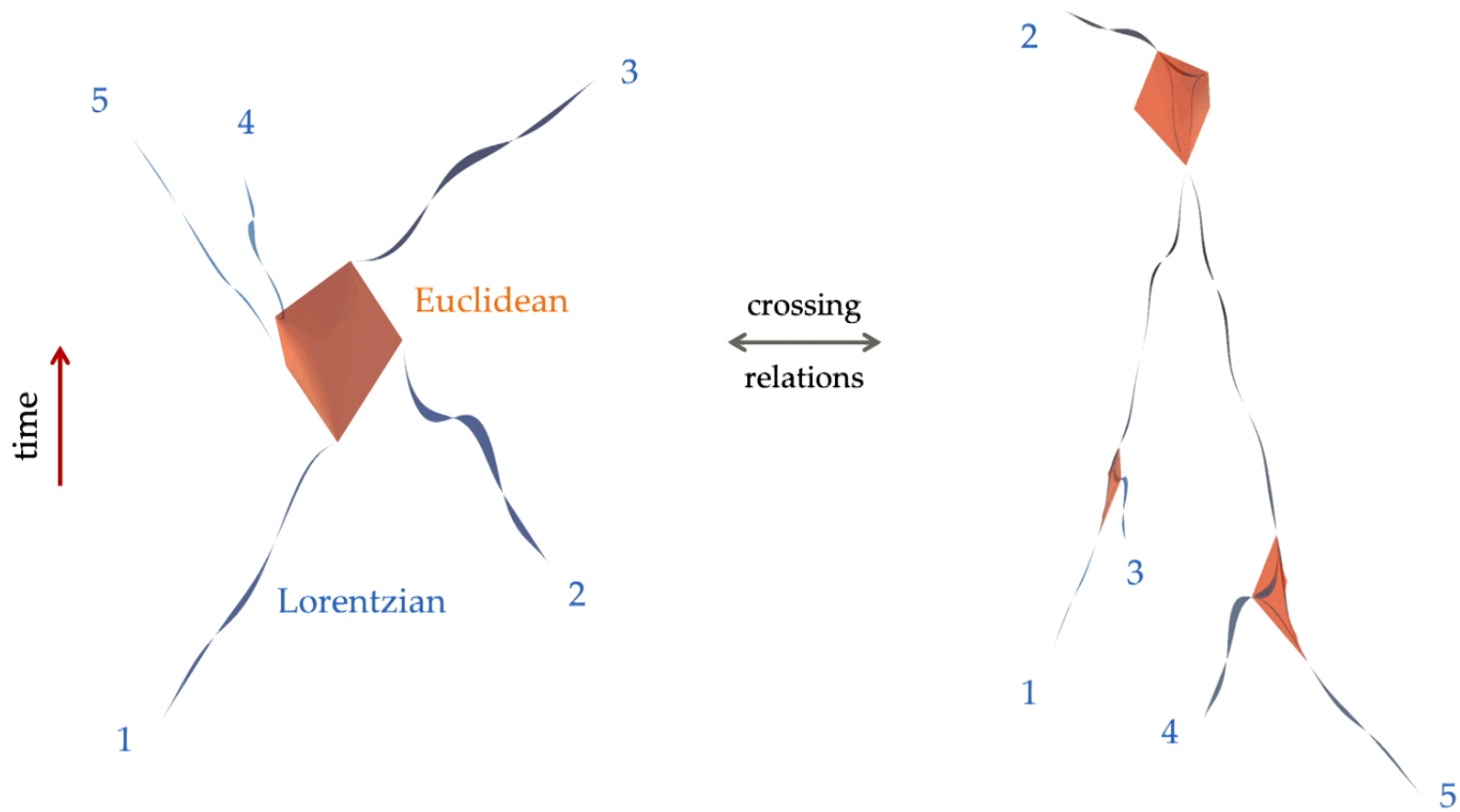


Wick rotation of the worldsheet



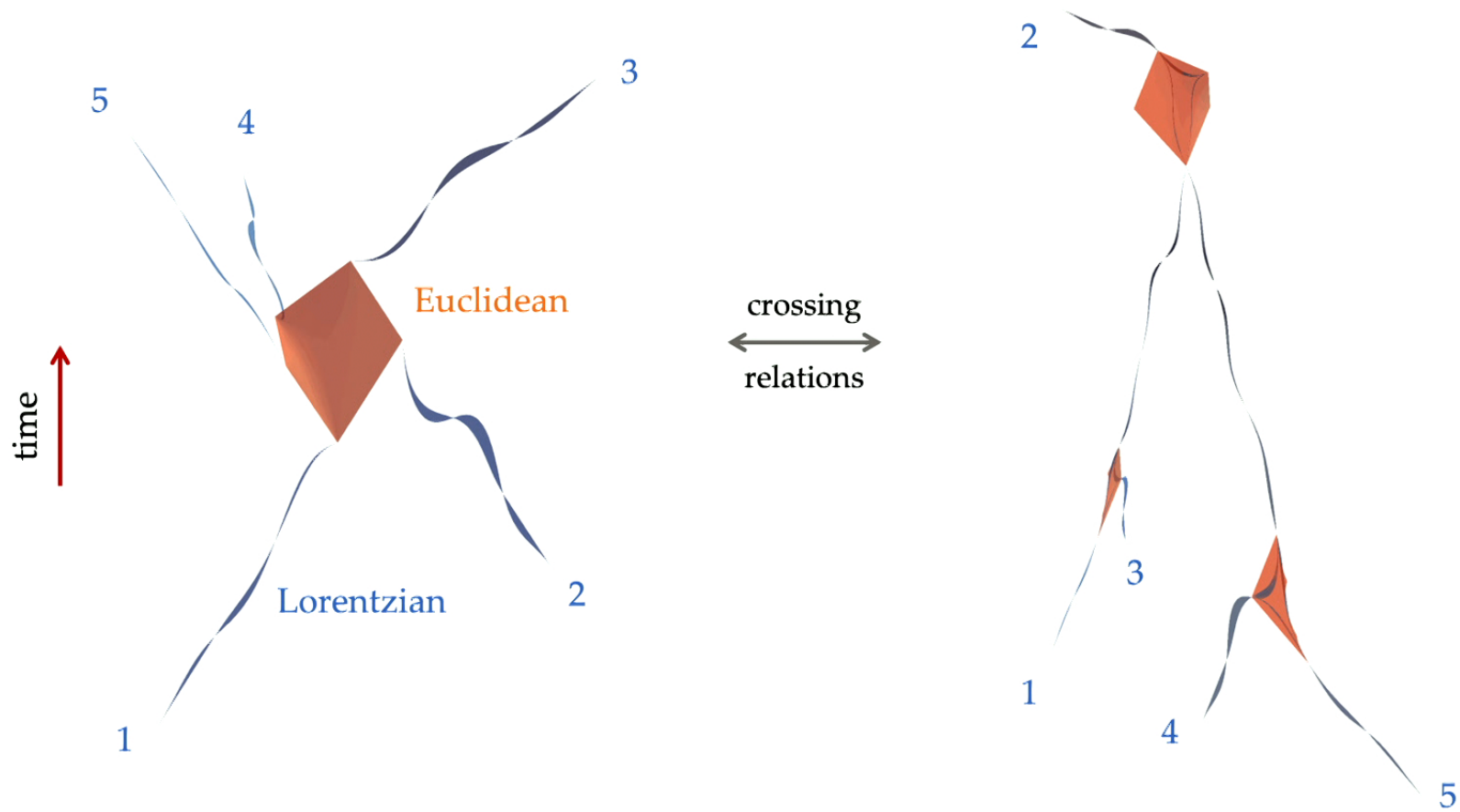
[Witten (JHEP 2015)]

Dancing worldsheets




Quantum corrections to the string amplitude

Dancing worldsheet




Quantum corrections to the string amplitude

$$\text{NIntegrate} \left[\left(\frac{\theta_1[z_2 - z_1, \tau] \theta_1[z_4 - z_3, \tau]}{\theta_1[z_3 - z_1, \tau] \theta_1[z_4 - z_2, \tau]} \right)^{-s} \left(\frac{\theta_1[z_3 - z_2, \tau] \theta_1[z_4 - z_1, \tau]}{\theta_1[z_3 - z_1, \tau] \theta_1[z_4 - z_2, \tau]} \right)^{-t} \right. \\ \left. /. \{s \rightarrow 3/2, t \rightarrow -1/2, z_4 \rightarrow 1, \tau \rightarrow \text{I Im}\tau\}, \right. \\ \left. \{\text{Im}\tau, 0, \infty\}, \{z_1, 0, 1\}, \{z_2, z_1, 1\}, \{z_3, z_2, 1\} \right]$$


Textbook formula for type-I four-point amplitude [Green, Schwarz (Nucl. Phys. B 1982)]


Quantum corrections to the string amplitude

$$\text{NIntegrate}\left[\left(\frac{\theta_1[z_2 - z_1, \tau] \theta_1[z_4 - z_3, \tau]}{\theta_1[z_3 - z_1, \tau] \theta_1[z_4 - z_2, \tau]}\right)^{-s} \left(\frac{\theta_1[z_3 - z_2, \tau] \theta_1[z_4 - z_1, \tau]}{\theta_1[z_3 - z_1, \tau] \theta_1[z_4 - z_2, \tau]}\right)^{-t}\right. \\ \left./, \{s \rightarrow 3/2, t \rightarrow -1/2, z_4 \rightarrow 1, \tau \rightarrow \text{I Im}\tau\},\right. \\ \left.\{\text{Im}\tau, 0, \infty\}, \{z_1, 0, 1\}, \{z_2, z_1, 1\}, \{z_3, z_2, 1\}\right]$$


Textbook formula for type-I four-point amplitude [Green, Schwarz (Nucl. Phys. B 1982)]

⋯ **NIntegrate**: Catastrophic loss of precision in the global error estimate due to insufficient WorkingPrecision or divergent integral.

Quantum corrections to the string amplitude

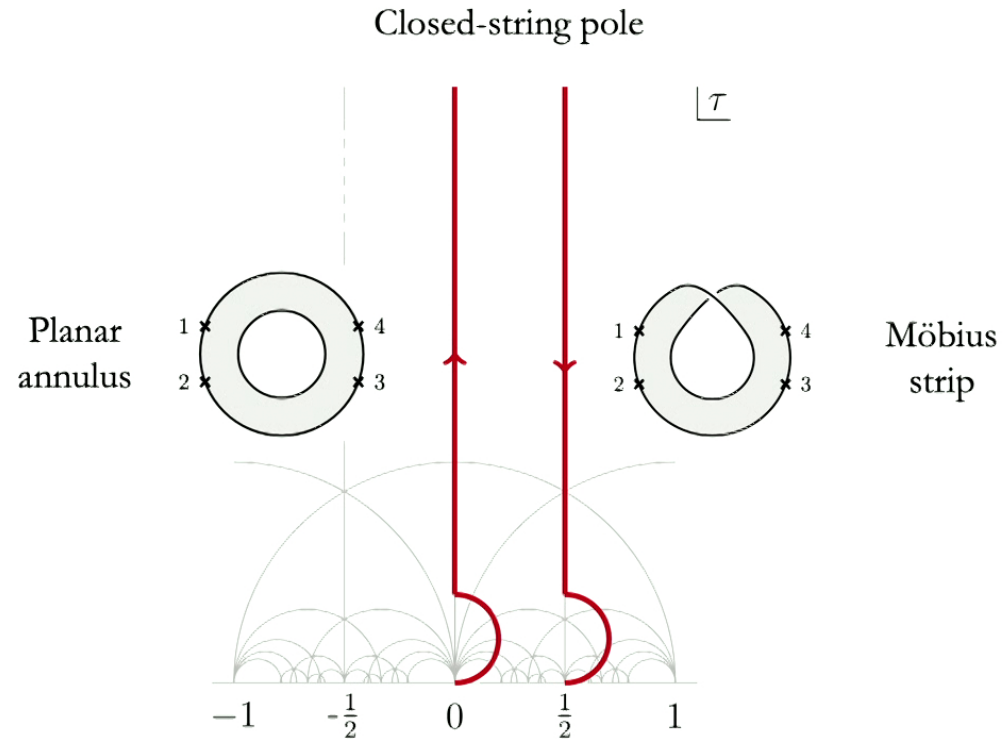
$$\text{NIntegrate}\left[\left(\frac{\theta_1[z_2 - z_1, \tau] \theta_1[z_4 - z_3, \tau]}{\theta_1[z_3 - z_1, \tau] \theta_1[z_4 - z_2, \tau]}\right)^{-s} \left(\frac{\theta_1[z_3 - z_2, \tau] \theta_1[z_4 - z_1, \tau]}{\theta_1[z_3 - z_1, \tau] \theta_1[z_4 - z_2, \tau]}\right)^{-t}\right. \\ \left./, \{s \rightarrow 3/2, t \rightarrow -1/2, z_4 \rightarrow 1, \tau \rightarrow \text{I Im}\tau\},\right. \\ \left.\{\text{Im}\tau, 0, \infty\}, \{z_1, 0, 1\}, \{z_2, z_1, 1\}, \{z_3, z_2, 1\}\right]$$


Textbook formula for type-I four-point amplitude [Green, Schwarz (Nucl. Phys. B 1982)]

⋯ **NIntegrate**: Catastrophic loss of precision in the global error estimate due to insufficient WorkingPrecision or divergent integral.

**Textbook formulas
can't be used in practice!**

Implementing the Lorentzian time evolution



[Eberhardt, SM (SciPost 2023b)]

Start of an adventure into the physics of string scattering

In the limit, we enclose all the circles

We call it the Rademacher contour
[Rademacher '43]

Shortcut computation using worldsheet methods

For the purposes of this talk, we only compute a toy model:

$$I = \int_0^1 \frac{d\tau}{\eta(\tau)^{24}}$$

Dedekind eta function
 $\eta\left(\frac{\tau+d}{\tau+d}\right) = (\tau+d)^{12} \eta(\tau)^{24}$

After modular transformation $\tau \rightarrow -1/\tau$:

$$I = - \int_{-\infty}^0 \frac{d\tau}{\tau^{11} \eta(\tau)^{24}}$$

The full computation is technically much more involved, but conceptually similar; the final result is

$$A^p = \Delta A^p + \sum_{m=1}^{\infty} \sum_{\substack{l, a, b \\ (a, b)=1}} \sum_{\substack{n_L, n_D, n_R, n_1, n_2 \\ n_L + n_D + n_R + n_1 + n_2 = m-1}} A_{l, a, b}^{n_L, n_D, n_R, n_1, n_2}$$

Cusp contribution (gauge)

Every term can be interpreted as summing over c windings with punctures distributed on the folds:

$(n_L, n_D, n_R, n_1, n_2) = (0, 1, 7, 1)$

First, just the imaginary part of the planar annulus

Most of the known results

Total cross section

Contribution from masses \sqrt{m} and $\sqrt{m'}$ flowing through the unitarity cut

- $(n_1, n_2) = (0, 0)$
- $(n_1, n_2) = (1, 0)$
- $(n_1, n_2) = (2, 0)$
- $(n_1, n_2) = (3, 0)$
- $(n_1, n_2) = (1, 1)$
- $(n_1, n_2) = (4, 0)$
- $(n_1, n_2) = (5, 0)$
- $(n_1, n_2) = (2, 1)$
- $(n_1, n_2) = (6, 0)$

New thresholds opening up very slowly

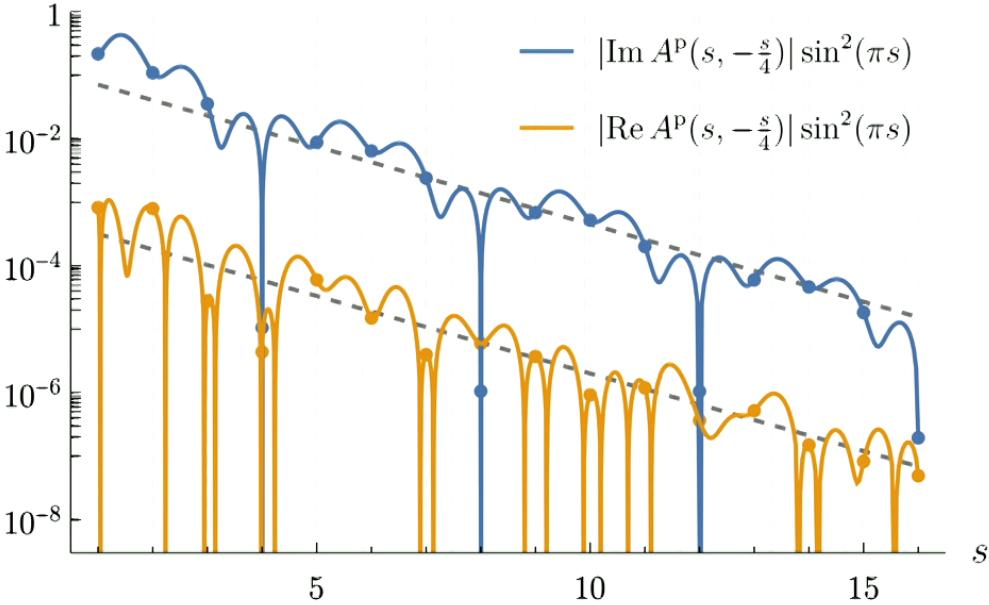
Low-spin dominance

(cf. [Arkani-Hamed, Huang, Huang '20], [Bern, Kamphuis, Zhiboedov '21] at tree level)

Partial wave coefficients $f_j(s) \propto \int_{-1}^1 dz (1-z^2)^j G_j^{(m)}(z) A_{l, a, b}^{n_L, n_D, n_R, n_1, n_2}$

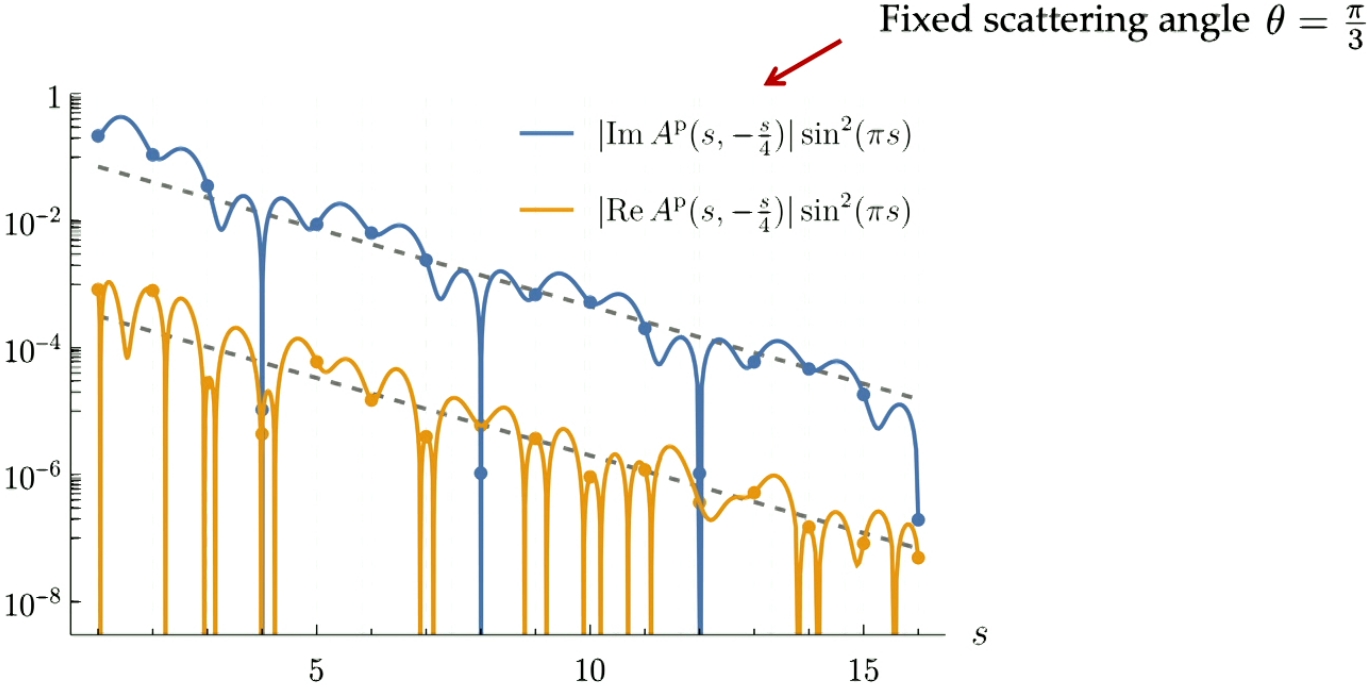
Almost all contributions from spins $j+1 \leq s$

Highlight: First evaluation of a quantum string amplitude at finite energy



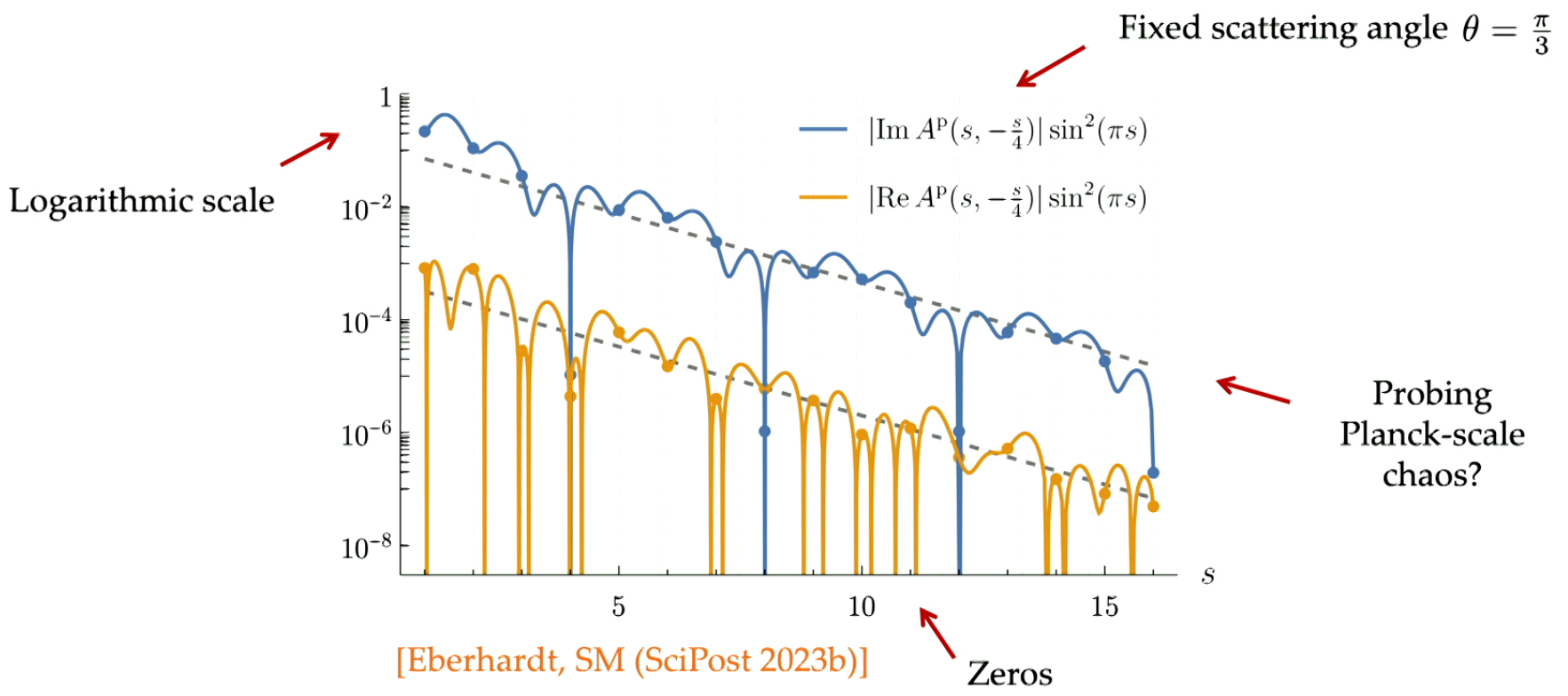
[Eberhardt, SM (SciPost 2023b)]

Highlight: First evaluation of a quantum string amplitude at finite energy



[Eberhardt, SM (SciPost 2023b)]

Highlight: First evaluation of a quantum string amplitude at finite energy

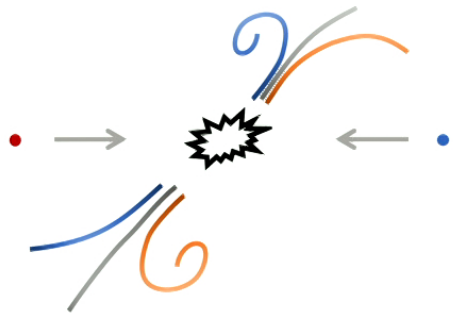


Summary of the 3rd part

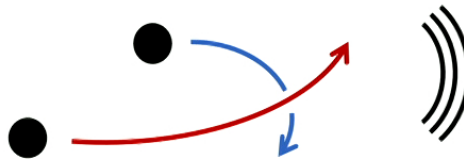
- Essential to understand **Lorentzian worldsheets**
- Used to extract **physics** of string scattering

Summary of the talk

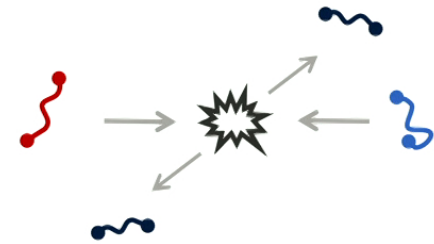
- Particle physics



- Gravitational waves



- String theory



Thank you

