Title: Extended Phase Space: Constraints, Charges and Conditional Expectations

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Abstract:

I'll give a review of extended phase space in gauge theories and/or diffeomorphism-invariant theories, beginning with its original conception up to our most recent work. The primary focus in a classical theory is on the proper representation of symmetries and gauge symmetries on phase space. Correspondingly, this is understood in the quantum theory through the crossed product construction and conditional expectation. I'll discuss several examples.

Zoom link

Extended Phase Space: Constraints, Charges and Conditional Expectation

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Perimeter

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Comments

- there is of course much interest in both quantum gravity and quantum information, and their relationships
- in this talk, I'll review work of the past years on extended phase space (classical) and the crossed product construction (quantum)
- main thesis is that a properly extended phase space is a key to quantization, as it is what allows for the proper representation of symmetries (whether they be gauge or global)
  - earlier we suggested that there is a close relationship between extended phase space and quantum operator algebras
    - **there were of course some unexplored details…**
  - our December paper studies quantization from the extended phase space perspective

References

- **related:**
  - Isolated Surfaces and Symmetries of Gravity
    - **2104.07643 (with Luca Ciambelli)**
  - Embeddings and Integrable Charges for Extended Corner Symmetry
    - **2111.13181 (with Luca Ciambelli, Pin-Chun Pai)**
  - Extended Phase Spaces in General Gauge Theories
    - **2303.06786 (with Marc Klinger, Pin-Chun Pai)**
  - Crossed Products, Extended Phase Spaces and the Resolution of Entanglement Singularities
    - **2306.09314 (with Marc Klinger)**
  - Null Raychaudhuri: Canonical Structure and the Dressing Time
    - **2309.03932 (with Luca Ciambelli, Laurent Freidel)**
  - Crossed Products, Conditional Expectations and Constraint Quantization
    - **2312.16678 (with Marc Klinger)**

Embeddings

- **the original implementation of extended phase space was to include embeddings of submanifolds (such as the boundary of a spatial region) into the phase space**

Donnelly-Freidel '16
Ciambelli-RGL '21

- **an embedding is described by a map $\phi_k : S_k \to M$**
  - described in local coordinates by $\phi_k : \sigma^\alpha \mapsto y^\mu(\sigma)$
  - one identifies a maximal closed subalgebra of diff(M)

$$\Big(Diff(S) \ltimes GL(k,\mathbb{R})\Big) \ltimes \mathbb{R}^k$$

background independent

  - **now referred to as *universal corner symmetry* (UCS) $\mathcal{A}_k$**

- **these are the diffeomorphisms that can become physical at a corner** (k=2)
  - i.e., support non-zero gauge charges
  - in context of covariant phase space formalism, by including embeddings, one finds that these are generated by integrable charges
  - various subalgebras of $\mathcal{A}_k$ are relevant in different situations

Corners

- **codimension-2 surfaces (corners) are also important in many contexts**
  - in any gauge theory (including diff-invariant theories), Noether's second theorem reports that the current is of the form

$$J_{\underline{\lambda}} = M_{\underline{\lambda}} + dq_{\underline{\lambda}}$$

0-form sym -> J is a (d–1)-form

  - in the case of a gauge symmetry, $\underline{\lambda}$ is an element of the Lie algebra
  - in the case of diffeomorphisms, it is a vector field
  - $M_{\underline{\lambda}}$ is the constraint, which vanishes in some sense (classically, on-shell)
  - $q_{\underline{\lambda}}$ is the charge density or aspect, whose integration over a codim-2 surface gives the (generally non-vanishing) gauge charge

Lagrangian Theories

- let's be very formal with embeddings. Given a spacetime M, and a region $\phi_0 : R \hookrightarrow M$

$$S_R[\Phi, \phi] = \int_R \phi_0^*(L[\Phi, y])$$

- variation gives

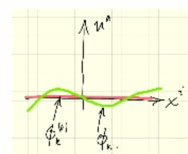
$$\delta S_R[\Phi, \phi] \hat{=} \int_{\partial R} \phi_1^*(\theta[\Phi, \delta\Phi, y]) + \int_R \phi_0^*(\mathcal{L}_{\underline{\chi}} L[\Phi, y])$$

$$\hat{=} \int_{\partial R} \phi_1^*(\theta[\Phi, \delta\Phi, y]) + i_{\underline{\chi}} L)$$

- where the first term is the usual one, coming from varying fields, and the second comes from varying the embedding, thought of as

$$y^\mu \to y^\mu + \chi^\mu$$

- given a hypersurface $\varphi_1 : \Sigma \to M$, we are led to define



$$\Theta_\Sigma^{ext.} \hat{=} \int_\Sigma \varphi_1^*(\theta[\Phi, \delta\Phi, y]) + i_{\underline{\chi}} L)$$

$$\Omega_\Sigma^{ext.} = \int_\Sigma \varphi_1^*(\delta\theta[\Phi, \delta\Phi, y]) + \int_{\partial\Sigma} \phi_2^*(i_{\underline{\chi}}\theta + \frac{1}{2} i_{\underline{\chi}} i_{\underline{\chi}} L)$$

- **then one finds** $I_{\underline{\hat{\eta}}}\Omega_{\Sigma}^{ext.} \triangleq -\delta \int_{\partial\Sigma} \varphi_2^*(q_{\underline{\eta}}) \equiv -\delta H_{\underline{\eta}}$ [Ciambelli, RGL, P.-C. Pai '21]
- **that is for each diffeomorphism, there is an integrable Hamiltonian function on phase space generating it**
  - this extension works for ALL diffeomorphisms, including all elements of UCS
  - the algebra of charges closes,

$$I_{\underline{\hat{\xi}}} I_{\underline{\hat{\eta}}}\Omega_{\Sigma}^{ext.} = -I_{\underline{\hat{\xi}}}\delta H_{\underline{\eta}} = H_{[\underline{\xi},\underline{\eta}]} = \left\{ H_{\underline{\xi}}, H_{\underline{\eta}} \right\}$$

- **note that** $\underline{\chi}$ **is a variation**
  - nilpotency of δ implies $\delta\underline{\chi} = -\frac{1}{2}[\underline{\chi}, \underline{\chi}]$ 'ghost-like'
  - in this respect, the extended phase space brings back degrees of freedom that were 'pure gauge' (in the absence of an embedding)
  - correspondingly, we can expect that the extended phase space has an interesting geometric structure

Comments

- here we have focussed on codimension-2, as it relevant for a discussion of the representation of gauge charges and their algebra
- (classically), this is an on-shell analysis, with constraints $M_{\underline{\xi}}$ vanishing
- off-shell, one expects that the constraints also represent the symmetry
  - generally one finds that the phase space needs to be extended in codimension-1 as well in order for this to work out
  - so generally, the algebra of Noether currents gives a representation when the phase space is extended in both codimension-1 (constraints) and codimension-2 (gauge charges)
    - **we gave an example recently: the symplectic structure of gravity on a (generic) null hypersurface** Ciambelli, Freidel, RGL '23
      - **showed explicitly that constraint algebra closes when spin-2 dof are supplemented by spin-0 dof**

Gauge Theories

- **in fact, this analysis can be extended to any gauge theory**
- **much of the previous discussion about diff-invariant theories flows from the fact that their Lagrangian densities are not quite diff-invariant**
  - there is a close analogue of this in gauge theories: Chern-Simons
  - one can frame such theories in a very similar way to the previous discussion: in this case, there is an extension of the phase space corresponding to gauge transformations (as well as diffs, if *pure* CS)
    - **a 'ghost' field $\underline{\lambda}$: a one-form in field space, valued in the Lie algebra**
- **here the formalism of (Atiyah) Lie algebroids is very useful, as it neatly combines diffeomorphisms and gauge transformations**

Lie Algebroids

- given a principal G-bundle $\pi : P \to M$, a connection on P can be thought of as a <u>1-form on P</u> valued in the Lie algebra $\mathfrak{g}$
  - connection consists of gauge field (pullback to T^*M) plus a vertical form
    - we usually throw away the vertical part, but essentially ~ghost
- the Atiyah Lie algebroid is TP/G
  - this is a vector bundle on M of rank $\dim M + \dim \mathfrak{g}$
  - locally, a fibre looks like $T_xM \oplus \mathfrak{g}$ and the algebroid has a bracket such that j, ρ are *morphisms* (relating $[\cdot,\cdot]_A$ to $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{TM}$)

$$0 \longrightarrow L \xrightarrow{j} A = TP/G \xrightarrow{\rho} TM \longrightarrow 0$$

- this means a section of the algebroid can be thought of as a vector field on M together with an element of the Lie algebra
- in the context of extended phase space, can associate $(\underline{\chi}, \underline{\lambda})$ with a section $\underline{\mathfrak{X}}$

Lie Algebroids

- **a connection on A determines a global splitting** $A = V \oplus H$

$$\underline{\mu} \in L, \quad j(\underline{\mu}) \in V; \qquad \underline{\mathfrak{X}} \in H, \quad \rho(\underline{\mathfrak{X}}) = \underline{X} \in TM$$

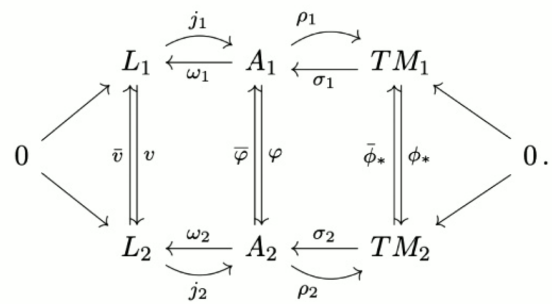
- **the exterior derivative on forms on** M **extends to an exterior derivative** $\hat{d}$ **on** A
  - on a representation (associated bundle), splits into covariant derivative (horizontal) and BRST (vertical) — algebroid bicomplex
  - in fact, the cohomology of $\hat{d}$ is BRST cohomology [Jia, Klinger, RGL'23]

- **so gauge theories can be recast completely in algebroid terms, with the degrees of freedom corresponding to a connection (gauge fields, ghosts) and a choice of associated bundles (charged matter fields)**
  - in such applications, we're interested not just in a particular algebroid with connection (A, ω), but in the entire space of such
  - organizing it this way is useful: the configuration space fields are acted upon in a precise way by diffs and gauge — essentially the algebroid acts on these fields

Examples

- *TM* is a Lie algebroid, with trivial structure group
- an ordinary gauge theory is associated to a Lie algebroid with a compact Lie group as structure group
- gravity, in first order formalism, can be thought of as associated with a Lie algebroid of the (orthonormal) frame bundle, along with a solder form (identified with an associated rank-d bundle)
  - similar theories associated with other G-structures descending from the frame bundle (e.g., Lorentz-Weyl, Carroll, …)

- consider a map between two algebroids over two (generally different) manifolds



- in the figure, ϕ is a diffeomorphism, *v* is a morphism of Lie algebras

- the condition for φ to be a morphism is that the curvatures of the connections match, $\Omega_1 = \varphi^* \Omega_2$
  - this suggests that we identify the set of Lie algebroid morphisms with diffeomorphisms of M and gauge transformations of L

Configuration Algebroid

- **this suggests an interpretation where Lie algebroid morphisms correspond to the 'gauge group' on configuration space**
  - so we have a group(oid) $\mathbb{G}$ acting on a configuration space $\mathbb{F}$ — Klinger, RGL, Pai '23-03; Jia, Klinger, RGL '23-06
  - corresponds formally to a principal bundle $\mathbb{P}$ over phase space, and a corresponding Lie algebroid — we call this the 'configuration algebroid'

$$0 \longrightarrow \mathbb{L} = \mathbb{P} \times_{\mathrm{Ad}} A \xrightarrow{\mathbf{j}} \mathbb{A} = T\mathbb{P}/\mathbb{G} \xrightarrow{\mathbf{r}} T\mathbb{F} \longrightarrow 0$$

- **whereas the space of gauge orbits is a quotient $\mathbb{F}/\mathbb{G}$, we will regard the product $\mathbb{P} \sim \mathbb{G} \times \mathbb{F}$ as an extended configuration space**
  - in fact, this is the necessary structure to define extended phase space in general

- locally, we regard a section of $\mathbb{A}$ as

variational derivs

$$\underline{\mathfrak{X}} = \mathbb{V} \oplus \boldsymbol{\mu}, \qquad \mathbb{V} \in \mathrm{T}\mathbb{F} \text{ and } \boldsymbol{\mu} \in \mathbb{L}.$$

gauge+diffs

- this is a little more complicated than previous, but

$$\mathbb{r}(\mathbb{V} \oplus \boldsymbol{\mu}) = \mathbb{V} + \mathbb{V}_{\boldsymbol{\mu}} \qquad \mathbb{j}(\boldsymbol{\mu}) = -\mathbb{V}_{\boldsymbol{\mu}} \oplus \boldsymbol{\mu}.$$

- locally the configuration algebroid exterior derivative $\hat{\delta}$ can be written in terms of the usual δ and a Maurer-Cartan form ϖ
  - the M-C form is invariant $\hat{\mathrm{L}}_{-\mathbb{j}(\boldsymbol{\mu})}\boldsymbol{\varpi} = 0, \qquad \hat{\mathrm{I}}_{\boldsymbol{\mu}}\boldsymbol{\varpi} = \hat{\mathrm{I}}_{\mathbb{j}(\boldsymbol{\mu})}\boldsymbol{\varpi} = \boldsymbol{\mu}.$
  - the M-C form is the key ingredient to describe extended phase space
  - classically, we would begin with a Lagrangian density $\mathfrak{L}$ (now regarded as a top horizontal form on an algebroid A)
  - studying $\hat{\delta}\mathfrak{L}$ (rather than just $\delta\mathfrak{L}$) yields equations of motion and an extended symplectic potential

$$\theta^{ext.} = \theta + \hat{i}_{\boldsymbol{\varpi}}\mathfrak{L} \in \Omega^1(\mathbb{A}; \Omega^{d-1}(A))$$

- **this result $\theta^{ext.} = \theta + \hat{i}_{\varpi}\mathfrak{L}$ is reminiscent of the extended symplectic potential discussed before**
  - the M-C form encodes $\varpi \leftrightarrow (\underline{\chi}, \underline{\lambda})$
- **one finds integrable charges with Noether current densities for gauge/diffs**

$$\mathrm{J}_{\boldsymbol{\mu}} := \hat{\mathrm{I}}_{-\mathbb{j}(\boldsymbol{\mu})}\theta^{ext.} = \mathrm{I}_{\mathbb{V}_{\boldsymbol{\mu}}}\theta - \hat{i}_{\boldsymbol{\mu}}\mathfrak{L}.$$

- **at the level of the phase space, we will regard this as an extension by the group**
  - we have set up a formalism where much of the structure will end up being 'pure gauge'
  - this depends strongly on context though
    - **the theory already has any required degrees of freedom built in, which either go away or become physical**

Extended Phase Space

ExtPhSp vs. Crossed Products

- we've seen so far that a group(oid) action on a phase space can be formulated via the configuration algebroid construction, yielding a notion of extended phase space, and integrable charges
- think about quantization in rough terms: [Klinger, RGL '23-06]

phase space	⟶	operator algebra
symplectomorphism	⟶	automorphism
equivariant (Hamiltonian)	⟶	inner automorphism

- all about representing (gauge) symmetries:
- extended phase space yields Hamiltonian actions (integrable charges)
- in operator algebra theory, the crossed product construction yields inner automorphisms
    - familiar example: modular automorphism, crossed product : type III to type II …

Quantization

- we suggested that extended phase space already contains within it 'observer' degrees of freedom
- part of a quantization procedure is to set constraints to zero, in some sense
  - recall Noether current $\mathrm{J}_{\underline{\mathfrak{x}}} = \mathbb{M}_{\underline{\mathfrak{x}}} + \hat{d}\mathbb{Q}_{\underline{\mathfrak{x}}},$
  - want to achieve "$\mathbb{M}_{\underline{\mathfrak{x}}} = 0$" without affecting $\mathbb{Q}_{\underline{\mathfrak{x}}}$
- have been studying various quantization scenarios from the perspective of extended phase space / crossed products
  - refined algebraic quantization
  - BRST quantization [Klinger, RGL '23-12]
  - functional integral quantization
- from our perspective they are now all on a quite similar footing

Extended Phase Space

- **here I'll focus on path integrals**
  - interpret gauge theory path integrals as *conditional expectations*
    - **recall that a conditional expectation is a probability distribution conditioned on some event**
    - **here there is a conditionality relating to gauge invariance, a choice of measure**
  - essential ingredients here are 'dressing' to invariant operators and gauge fixing
    - **the conditional expectation achieves both at the path integral level**

Constraints and M-C Forms

- **write the extended symplectic potential as**

$$\theta^{ext}_{\Sigma} \equiv \varphi^*_{(1)}\left(\Pi_i \wedge_A \delta\Phi^i + \hat{i}_{\boldsymbol{\varpi}}\mathfrak{L}\right)$$

- denote the group(oid) action on phase space as $a : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ with

$$a(\varphi, \tilde{\Phi}, \tilde{\Pi}) \equiv a_{\varphi}(\tilde{\Phi}, \tilde{\Pi}) = (\Phi, \Pi).$$

- this gives

$$\delta(\Phi, \Pi) = (\delta\tilde{\Phi} + \delta_{\boldsymbol{\varpi}}\tilde{\Phi}, \delta\tilde{\Pi} + \delta_{\boldsymbol{\varpi}}\tilde{\Pi}) \qquad \delta_{\underline{\mathfrak{x}}}\Phi = \hat{i}_{\underline{\mathfrak{x}}}\delta_{\boldsymbol{\varpi}}\Phi$$

- and so

$$\theta^{ext}_{\Sigma} = \varphi^*_{(1)}\left(\tilde{\Pi}_i \wedge_A \delta\tilde{\Phi}^i + \underbrace{\tilde{\Pi}_i \wedge_A \delta_{\boldsymbol{\varpi}}\tilde{\Phi}^i - \hat{i}_{\boldsymbol{\varpi}}\tilde{\mathfrak{L}}}_{\theta^{ext}_{\mathbb{G}}}\right)$$

- contraction gives Noether

$$\hat{\mathrm{I}}_{-\mathbb{j}(\underline{\mathfrak{x}})}\theta^{ext}_{\mathbb{G}} = \varphi^*_{(1)}\left(\tilde{\Pi}_i \wedge_A \delta_{\underline{\mathfrak{x}}}\tilde{\Phi}^i - \hat{i}_{\underline{\mathfrak{x}}}\tilde{\mathfrak{L}}\right) \equiv \mathrm{J}_{\underline{\mathfrak{x}}}$$

- suggests that we can think of this as $\theta^{ext}_{\mathbb{G}} \equiv \mathbb{M}_M \boldsymbol{\varpi}^M_{(1)} + \hat{d}\left(\mathbb{Q}_M \boldsymbol{\varpi}^M_{(2)}\right)$;

Extended Phase Space

- we've rewritten the symplectic potential in terms of the tilded variables plus 'codimension-1 and -2 M-C forms'

$$\Theta_\Sigma^{ext} = \int_\Sigma \widetilde{\theta}_F^{ext} + \int_{\partial\Sigma} \mathbb{Q}_M \boldsymbol{\varpi}^M_{(2)} + \int_\Sigma \mathbb{M}_M \boldsymbol{\varpi}^M_{(1)}.$$

contraction gives an element of Lie algebra

- in fact, precisely this form can be found in our recent work on gravity on a null hypersurface [Ciambelli, Freidel, RGL '23]
  - there the a action was the diff mapping coordinate null time to 'dressing time', and the constraint term is the Raychaudhuri constraint
  - indeed spin-0 fields can be interpreted as codimension-1 extended fields
- this becomes a (canonical) change of variables on phase space if we impose a condition (gauge fixing)
- this can be implemented at the path integral level, and it closely related to the Faddeev-Popov procedure (here for phase space path integrals)

Conditional Expectation

- generally, a path integral can be regarded as a map $\varphi : \mathcal{M}_{\mathbb{X}} \to \mathbb{C}$ that assigns to each element of the Poisson algebra a numerical value, interpreted as its expectation value. That is, it is a weight,

$$\varphi(f) \equiv \int \nu_{\varphi}(x) f(x(t))$$

- the choice of measure (integrating over a path in phase space with certain boundary conditions) corresponds to a choice of φ
- in the case that there are local symmetries, we have to be more careful, and we usually think of this through the F-P procedure
- but certainly we need to choose the measure carefully, and this is where the conditioning idea comes in

Extended Poisson Algebra

[Klinger, RGL '23-12]

- group action on phase space: $a : G \times X \to X, \quad a(g,x) = a_g(x)$
- consider a projection $\pi : X \to X/G, \quad x \mapsto [x]$

identify
$G \times X \sim X_{ext.}$

  - maps phase space to space of orbits
  - might specify by a function $\mathcal{F} : X \to \mathfrak{g}$ whose kernel intersects each orbit once
  - or can think of it as a section (of $X_{ext.}$) $z : X \to G$ such that $a_{z(x)}(x) = [x]$
  - the projection maps a function on X to an invariant function, $f \mapsto f \circ \pi$

$$a_g^*(f \circ \pi)(x) = f \circ \pi \circ a_g(x) = f \circ \pi(x)$$

- regard Poisson algebra of $X_{ext.}$ as maps into the Poisson algebra of X

$$\mathcal{M}_{X_{ext.}} := \{\mathfrak{F} : G \to \mathcal{M}_X\}$$

  - define a (symplecto)morphism $i : \mathcal{M}_X \to \mathcal{M}_{X_{ext.}}, \qquad i(f)(g,x) = f \circ a_g(x)$
  - already resembles a crossed product, as we have an algebra along with a group acting on it…

Extended Phase Space

Conditional Probability Distribution

- write a (F-P) determinant as

$$\Delta_{\mathcal{F}}(x)^{-1} \equiv \int_G \mu(g)\ \delta(\mathcal{F} \circ a_g(x)).$$

left invariance of Haar measure
= invariance of determinant

- then introduce a conditional probability measure

$$\Phi(g \mid x) = \Delta_{\mathcal{F}} \circ a_g(x)\ \delta(\mathcal{F} \circ a_g(x))$$

- this is a measure on G, conditioned on an element of phase space
  - **automatically satisfies the properties**

$$\int_G \mu(g)\ \Phi(g \mid x) = 1,\ \forall x \in X$$
$$\int_G \mu(g)\ \Phi(g \mid x) i(f)(x,g) = f([x]).$$

  - **notice the latter can be interpreted as a "dressed version" of the function**

Extended Phase Space

Conditional Expectation

- can use this to construct a conditional expectation

$$T : \mathcal{M}^{pq}_{X_{ext}} \to \mathcal{M}^{pq}_{X}, \qquad \mathfrak{F}(x,g) \mapsto \int_G \mu(g)\ \Phi(g \mid x)\mathfrak{F}(x,g) = \mathfrak{F}(x, z(x)).$$

- the usual notion of gauge fixing is then given by composition

$$T \circ i(f)(x,g) = f([x]).$$

- so given a weight $\varphi : \mathcal{M}_X \to \mathbb{C}$ on $\mathcal{M}_X$ we get a 'dual weight'

$$\tilde{\varphi} = \varphi \circ T : \mathcal{M}_{X_{ext.}} \to \mathbb{C}$$

- compacting notation, we then find

$$\tilde{\varphi} \circ i(f) = \int_{X_{ext}} \nu^{ext}_{\varphi}(x,g)\ i(f) = \int_{X_{ext}} \nu_{\varphi}(x) \wedge \mu(g)\ \Phi(g \mid x)\ f \circ a_g(x) = \int_{X/G} \nu^{X/G}_{\varphi}([x])\ f([x]).$$

- this is more general than F-P: we don't need to only evaluate the path integral on invariant functions, the path integral projects to an invariant function directly
  - **the details of which depend on a choice of conditioning (gauge fixing)**
- can relate directly to crossed product construction from operator algebra pov

Examples

[Klinger, RGL '23-12]

- **we considered three examples in the paper**
  - gauge the $SO(D) \subset SU(D)$ subgroup in the D-dim'l symmetric oscillator
    - **a QM problem: no corners**
  - 4d Yang-Mills or QED:
    - **phase space path integral constructed as above**
    - **constraint/gauge fixing appear in path integral side-by-side**
  - gravity, formulated on a null hypersurface — as in [Ciambelli,Freidel,RGL '23]
    - **each detail can be re-interpreted in the above terms**
      - **here the gauge symmetries are boosts and reparamaterizations of the null fibre (generalize in the future)**
    - **stopped short of quantization**

Comments/Questions/Hand-wringing

- **this works very well in the case where the group is locally compact**
  - need a Haar measure to implement group averaging and the path integral
- **alas, the diffeomorphism group doesn't have that property and so it's not yet clear how gravity will work**
  - should we introduce an ad hoc regulator for this?
    - **can that be done in a 'diff-invariant' fashion?**
  - is this comment related to non-renormalizability?
  - one feature of the extended phase space formalism that i've emphasized throughout the talk is that we in fact don't have a *group*, we have instead a *groupoid*.
    - **does it make things worse? or better?**
    - **this is something we are pursuing now — Lie groupoids have Haar measures…**
  - is this related to relational ideas?