Title: Positive traces on deformations of Kleinian singularities

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Abstract: For a noncommutative algebra A and an antilinear automorphism ρ of A there is a notion of positive trace. On the physics side, positive traces are related to quantizations of superconformal field theories. On the mathematical side, positive traces are connected to spherical unitary representations of complex Lie groups. We will discuss the classification of positive traces in the case when the algebra A is a deformation or a q-deformation of a Kleinian singularity of type A.

Zoom link

Positive traces on deformations of Kleinian singularities

Daniil Klyuev

Plan:

1. Positive traces and motivation. (Some slides provided by Pavel Etingof from his talk.)
2. Kleinian singularities of type A, their deformations and q-deformations. Formulation of a problem.
3. Description of positive traces for deformations and q-deformations.
4. The case of bimodules, alternative way of classifying positive traces for q-deformations and future plans.

Positive traces

Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with an antilinear automorphism ρ.

Definition

A linear map $T\colon \mathcal{A} \to \mathbb{C}$ is a ρ^2-twisted trace if $T(ab) = T(b\rho^2(a))$. The trace is positive if $T(a\rho(a)) > 0$ for nonzero $a \in A$.

- When T is positive $(a, b) = T(a\rho(b))$ is positive definite and

$$(ab, c) = T(ab\rho(c)) = T(b\rho(c)\rho^2(a)) = (b, c\rho(a)).$$

- For central reduction of $U(\mathfrak{g})$ this corresponds to spherical unitary representations of G: $\mathcal{A}$ is a complexification of the corresponding $(\mathfrak{g}, K)$-module.
- Positive traces appear in the study of Coulomb branches for 3-dimensional $N = 4$ and 4-dimensional $N = 2$ superconformal field theories.

Spherical unitary representations

- Let G be a semisimple Lie group, K its maximal compact subgroup.
- Representation V of G is called spherical if there exists nonzero $v \in V$ such that $Kv = \{v\}$.
- There is one-to-one correspondence between unitary representations of G and $(\mathfrak{g}, K)$-modules with positive definite invariant Hermitian form: take K-finite vectors in one direction, take completion in the other.
- When the representation if spherical, the complexification of the corresponding $(\mathfrak{g}, K)$-module is a central reduction of $U(\mathfrak{g})$ with the natural action of $\mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Hermitian form gives a positive trace, $T(a) = (a, 1)$.

Short star-products

Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded algebra ($A_n A_m \subset A_{n+m}$ for all n, m).

Definition

A ($\mathbb{Z}/2$-equivariant) star-product on A is an associative multiplication $* : A \otimes A \to A$ such that for $a \in A_n$ and $b \in A_m$,

$$a * b = \sum_{k=0}^{\lfloor \frac{n+m}{2} \rfloor} C_k(a, b),$$

where $C_k : A_n \otimes A_m \to A_{n+m-2k}$ are bilinear maps such that $C_0(a, b) = ab$

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Definition (Beem, Peelaers, Rastelli, 2016)

A star-product $*$ on A is short if for any $m, n \in \mathbb{Z}_{\geq 0}$ and any $a \in A_n$, $b \in A_m$ one has

$$C_k(a, b) = 0, k > \min(n, m).$$

In other words, $a * b$ has no terms in A_d for $d < |n - m|$.

Filtered deformations

- Let A be a commutative graded algebra. Fix a filtered associative algebra $\mathcal{A} = \bigcup_{d \geq 0} F_d\mathcal{A}$ with $F_n\mathcal{A}F_m\mathcal{A} \subset F_{n+m}\mathcal{A}$ such that its associated graded algebra is identified with the graded algebra A; namely,

$$\mathrm{gr}\mathcal{A} = \bigoplus_{d \geq 0} F_d\mathcal{A}/F_{d-1}\mathcal{A}$$

and $F_d\mathcal{A}/F_{d-1}\mathcal{A} \cong A_d$ for $d \geq 0$ (compatibly with multiplication) with $F_{-1}\mathcal{A} := 0$.
- Such $\mathcal{A}$ is called filtered deformation (quantization) of A.
- For example, Weyl algebra $\mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ is a filtered deformation of $\mathbb{C}[x, y]$.

Bijection between star-products and quantization maps

A linear map $\phi\colon A \to \mathcal{A}$ is called quantization map if $\operatorname{gr}\phi = \mathrm{id}$.
This means for each $a \in A_d \cong F_d\mathcal{A}/F_{d-1}\mathcal{A}$ we choose its lift to $F_d\mathcal{A}$. In this case ϕ is an isomorphism of vector spaces.
Given a quantization map $\phi : A \to \mathcal{A}$ we can define the star-product on A by

$$a * b := \phi^{-1}(\phi(a)\phi(b)).$$

Conversely, given a star-product on A, we can set $\mathcal{A} := (A, *)$ with the filtration induced by the grading, and $\phi := \mathrm{Id}$.

Lemma

This defines a pair of mutually inverse bijections between star-products on A and ($\mathbb{Z}/2$-equivariant) quantizations of A equipped with a quantization map.

Correspondence between star-products and positive traces

- A short star-product is positive if $C_k(a, \rho(a)) > 0$ for all nonzero a of degree k.
- It follows from the results in Etingof-Stryker that there is a bijection between positive short star-products and pairs $(\mathcal{A}, T)$, where $\mathcal{A}$ is a filtered deformation of A and T is a positive trace.
- Namely, given A and $*$ we set $\mathcal{A} = (A, *)$ and T equal to constant term map. For $a \in A_k$ we have $T(a\rho(a)) = CT(a * \rho(a)) = C_k(a, \rho(a)) > 0$.
- Given $(\mathcal{A}, T)$ we define $\mathcal{A}_d = F_{d-1}^{\perp}A \cap F_d A$. We have $\mathcal{A}_d \cong F_d A / F_{d-1} A \cong A_d$. This gives the required map ϕ.

Star-products and positive traces, ctd.

- Instead of infinite-dimensional space of all possible ϕ we get a finite-dimensional convex cone of positive traces. In some cases a positive trace is unique, so we recover a star-product from a deformation.
- This is one of the motivations for considering positive traces.

Kleinian singularities

- Let H be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then $\mathbb{C}^2/H = \operatorname{Spec}\mathbb{C}[x,y]^H$ is called a Klenian singularity.
- Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to conjugation are classified by simply-laced Dynkin diagrams.
- For example, a group corresponding to A_{n-1} is $\mathbb{Z}/n\mathbb{Z}$ generated by $\begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix}$.
- Kleinian singularity of type A_{n-1} corresponds to $\mathbb{C}[x,y]^{C_n} \cong \mathbb{C}[u,v,z]/(uv - z^n)$ with $u = x^n$, $v = y^n$, $z = xy$.
- Kleinian singularities are examples of Coulomb/Higgs branches.

Deformations of Kleinian singularities

Definition

Let P be a polynomial of degree n. Consider an algebra $\mathcal{A}$ with generators u, v, z and relations $[z, u] = -u$, $[z, v] = v$, $vu = P(z - \frac{1}{2})$, $uv = P(z + \frac{1}{2})$.

The algebra $\operatorname{gr}\mathcal{A}$ is isomorphic to an A_{n-1} Kleinian singularity $\mathbb{C}[x, y]^{C_n}$.

1. If $n = 1$, $P(x) = x$, $\mathcal{A}$ is generated by u, v with relation $[u, v] = 1$ and $z = uv - \frac{1}{2} = vu + \frac{1}{2}$.
2. If $n = 2$, $P(x) = x^2 + C$, this is central reduction of $U(\mathfrak{sl}_2)$ with $e = v$, $f = -u$, $h = 2z$. The Casimir element is $ef + fe + \frac{h^2}{2} = \frac{h^2}{2} + h + 2fe$.

q-deformations of Kleinian singularities

Definition

Let P be a Laurent polynomial. Consider an algebra $\mathcal{A}$ with generators u, v, Z and relations $ZuZ^{-1} = q^2 u$, $ZvZ^{-1} = q^{-2} v$, $uv = P(q^{-1}Z)$, $vu = P(qZ)$.

Taking a limit $q \to 1$ in a certain way we get an algebra from the previous slide.

1. If $P(x) = 1$, $\mathcal{A}$ is generated by $u^{\pm 1}, Z^{\pm 1}$ with relation $Zu = q^2 uZ$, a q-Weyl algebra.
2. If $P(x) = \frac{x + x^{-1}}{(q - q^{-1})^2}$ this is central reduction of $U_q(\mathfrak{sl}_2)$: $u = E$, $v = F$, $Z = K$, the Casimir element is $\Lambda = (q - q^{-1})^2 FE + qK + q^{-1}K^{-1}$.

Involution ρ

- Recall that $\mathcal{A}$ is generated by u, v, z with $[z, u] = -u$, $[z, v] = v$, $vu = P(z - \frac{1}{2})$, $uv = P(z + \frac{1}{2})$. For even n we define
$$\rho(u) = v, \quad \rho(v) = u, \quad \rho(z) = -z.$$
- ρ is well-defined when $P(x) = \overline{P}(-x)$, hence P is real on $i\mathbb{R}$, in particular the roots of P are symmetric with respect to the map $\alpha \mapsto -\overline{\alpha}$.
- The problem: classify traces T on $\mathcal{A}$ such that $T(a\rho(a)) > 0$ for all nonzero $a \in \mathcal{A}$.
- More generally, $\rho = \rho_\pm$ is given by $\rho_\pm(u) = \pm i^n e^{-\pi i c} v$, $\rho_\pm(v) = \pm i^{-n} e^{\pi i c} u$, $\rho(z) = -z$. It is defined when $i^n P$ is real on $i\mathbb{R}$.

Involution for q-deformations.

- In the case of q-deformations, one way to define ρ is similarly to the previous slide: $\rho(u) = v$, $\rho(v) = u$, $\rho(Z) = Z^{-1}$.
- The map ρ is well-defined when $P(Z) = \overline{P}(Z^{-1})$, hence the roots of P are symmetric under $\alpha \mapsto \overline{\alpha}^{-1}$ and

$$P(Z) = a_m Z^m + \cdots + a_{1-m} Z^{1-m} + a_{-m} Z^{-m}.$$

- In this case there is more freedom: $\rho(Z) = Z^{-1}$, $\rho(u) = sZ^k q^k v$, $\rho(v) = s^{-1} Z^{-k} q^k u$, where k is an integer and $s \in \mathbb{C}$ satisfies $|s| = 1$.
- I mainly focused on the case of $k = 0$, but recently learned from Gaiotto that $k = m$ is a better choice. Then

Theorem (K., 2023)

Let $\mathcal{A}$ be a q-deformation, ρ be a conjugation above with $s = 1$, $k = m$. Then a positive trace is unique when it exists.

Computation of traces

Let $\mathcal{A}$ be a filtered deformation with parameter $P(z)$ of degree n. We will describe the space of traces on $\mathcal{A}$.

- We have $\mathcal{A} = \oplus_{l\in\mathbb{Z}}\mathcal{A}_l$ — an eigenspace of $\operatorname{ad} z$ decomposition. Hence any trace T is supported on $\mathcal{A}_0$.
- A map $T\colon \mathcal{A}_0 \to \mathbb{C}$ is a trace if and only if

$$T\big(P(x+\tfrac{1}{2})R(x+\tfrac{1}{2}) - P(x-\tfrac{1}{2})R(x-\tfrac{1}{2})\big) = 0$$

  for all polynomials R.

Analytic formula.

Suppose that all roots of $P(x)$ belong to the open strip $|\operatorname{Re} x| < \frac{1}{2}$. Then any trace T can be written as

$$T(R) = \int_{i\mathbb{R}} R(x)w(x)dx,$$

where

$$w(x) = \frac{Q(e^{2\pi \mathrm{i} x})}{\mathcal{P}(e^{2\pi \mathrm{i} x})},$$

$\mathcal{P}(x) = \prod_{P(\alpha)=0} (x + e^{2\pi \mathrm{i}\alpha})$, $\deg Q < n$ and $Q(0) = 0$. In general, for $\rho = \rho_{c,\pm}$ we take

$$w(x) = e^{2\pi \mathrm{i} c x} \frac{Q(e^{2\pi \mathrm{i} x})}{\mathcal{P}(e^{2\pi \mathrm{i} x})},$$

where $\deg Q < n$.

Positivity condition

- The definition of positive trace is $T(a\rho(a)) > 0$ for all nonzero $a \in \mathcal{A}$.
- It is enough to check this when $a \in \mathcal{A}_m$ for some integer m.
- We can assume that $m = 0, 1$. For $m = 0$ any $a \in \mathcal{A}_0$ is $a = R(z)$ and we have $a\rho(a) = R(z)\overline{R}(-z)$.
- We have
$$T(S(x)\overline{S}(-x)) > 0$$
for all nonzero polynomials S if and only if $w(x) \geq 0$ on $i\mathbb{R}$.
- This means that $Q(x) \geq 0$ when $x > 0$.
- Similarly, considering elements of degree one, we get a condition on $w(x)$ on $i\mathbb{R} + \frac{1}{2}$, hence a condition on $Q(x)$ when $x < 0$.

The answer

- It follows shortly that the convex cone of positive forms is isomorphic to the convex cone of nonnegative polynomials of degree $d \leq d_0$ for certain d_0 from $n-1$ to $n-4$ depending on parity of n, choice of $\rho_\pm$ and whether $\rho_\pm^2$ is identity.
- In the case when not all roots of $P(x)$ belong to $|\operatorname{Re} x| < \frac{1}{2}$ analytic formulas for traces become harder: instead of $i\mathbb{R}$ we should take slightly different contour.
- It can be proved that roots α with $|\operatorname{Re} \alpha| > \frac{1}{2}$ do not change the cone of positive forms, and each pair of roots $\alpha \pm \frac{1}{2}$ with $\operatorname{Re} \alpha = 0$ multiplies the cone by $\mathbb{R}_{\geq 0}$: there exists trace $R \mapsto R(\alpha)$.

The answer, ctd.

Theorem (Etingof, K., Rains, Stryker, 2021)

Let $\mathcal{A} = \mathcal{A}_P$ be a filtered quantization of A with parameter P equipped with a conjugation ρ such that $\rho^2 = g_t$. Let ℓ be the number of roots α of P such that $|\operatorname{Re}\alpha| < \frac{1}{2}$ counted with multiplicities and r be the number of distinct roots α of P with $\operatorname{Re}\alpha = -\frac{1}{2}$. Then the cone $\mathcal{C}_+$ of ρ-equivariant positive definite traces on $\mathcal{A}$ is isomorphic to $\mathcal{C}_+^1 \times \mathcal{C}_+^2$, where $\mathcal{C}_+^2 = \mathbb{R}_{\geq 0}^r$, and $\mathcal{C}_+^1$ is the cone of nonzero polynomials G of degree at most $d_0 \in [\ell - 4, \ell - 1]$

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- So the cases considered by Beem, Peelaers and Rastelli, namely $n \leq 4$, are the only cases where the positive trace is unique up to scaling.

Examples

Let $n = 1$, so that $P(x) = x$, the case of Weyl algebra $\mathbb{C}\langle u, v\rangle/([u, v] - 1)$. For $\rho(u) = v$, $\rho(v) = -u$, a unique (up to scaling) positive traces is given by

$$w(x) = \frac{1}{2\cos \pi x}.$$

Let $n = 2$ and $\rho(u) = v$, $\rho(v) = u$. A positive trace is unique and exists when $P(x) = x^2 + C$ satisfies $C > -\frac{1}{4}$. If $C \geq 0$ we get spherical principal series representations, if $-\frac{1}{2} < C < 0$ we get complementary series representations. Let $C = -\beta^2$. Then the corresponding weight is

$$w(x) = \frac{1}{4\cos \pi(x - \beta)\cos \pi(x + \beta)}.$$

Case of q-deformations and $k = 0$.

Suppose that $\mathcal{A}$ is q-deformation. Consider $\rho = \rho_s$ given by $\rho(u) = sv$, $\rho(v) = s^{-1}u$, $\rho(Z) = Z^{-1}$. It is well-defined when $P(x)$ is real on unit circle. Let τ be a purely imaginary number such that $q = e^{\pi i \tau}$, let w be a function such that

$$w(x) = w(x+1),$$

$$w(x+\tau) = tw(x),$$

and $w(x+\frac{\tau}{2})P(e^{2\pi \mathrm{i} x})$ is holomorphic when $q^{-1} \leq e^{2\pi i x} \leq q$. Then

$$T(R(z)) = \int_0^1 w(x)R(e^{2\pi \mathrm{i} x})dx$$

is a g_t-twisted trace. Moreover, if all roots α of $P(x)$ satisfy $q < |\alpha| < q^{-1}$ then all traces are obtained in this way.

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$$T(R(z)) = \int_0^1 w(x)R(e^{2\pi \mathrm{i} x})dx$$

is a g_t-twisted trace. Moreover, if all roots α of $P(x)$ satisfy $q < |\alpha| < q^{-1}$ then all traces are obtained in this way. For general P this formula is modified similarly to the case of filtered deformations.

The answer for q-deformations

Theorem (K.,2022)

Let $\mathcal{A}$ be a q-deformation with parameter P. Let $s \in S^1$, ρ_s be the corresponding conjugation. Let l be the number of roots α of P such that $q < |\alpha| < q^{-1}$, r be the number of distinct roots α with $|\alpha| = q$. Then the cone $\mathcal{C}_+$ of positive traces is isomorphic to $\mathcal{C}_l \times \mathbb{R}^r_{\geq 0}$, where $\mathcal{C}_l$ is described below.

One way to describe $\mathcal{C}_l$ is as follows. Choose $\frac{l}{2}$ distinct pairs of points $z_1, \overline{z_1}, \ldots, z_{\frac{l}{2}}, \overline{z_{\frac{l}{2}}}$. Then $\mathcal{C}_l$ can be realized as the set of elliptic functions with poles at $z_j, \overline{z_j}$ positive on real line and $\mathbb{R} + \frac{\tau}{2}$.

The case of q-deformations and $k \neq 0$.

- Let $\rho(Z) = Z^{-1}$, $\rho(u) = Z^k q^k v$, $\rho(v) = Z^{-k} q^k u$ with nonzero k.

Bimodules, ctd.

Let us rewrite the definition of good root: instead of $|\mathrm{Re}\,\alpha_i| < \frac{1}{2}$ write $-1 < \alpha_i - \alpha_j < 1$. We can make a similar definition in the case of bimodules over q-deformations. Then we have

Theorem (K.,2023)

Let m be the number of good roots. Then the cone of positive definite Hermitian forms for bimodules is isomorphic to the cone of nonnegative polynomials of degree at most $d \in [m-4, m-1]$ in the case of filtered deformations and to the cone $\mathcal{C}_m$ of elliptic functions with periods 1 and $\tau \in i\mathbb{R}$ and m possible poles that are nonnegative on $\mathbb{R}$ and $\mathbb{R} + \frac{\tau}{2}$.

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In particular, for $n = 2$ we recover all unitary representations of $\mathrm{SL}_2(\mathbb{C})$.

The case of q-deformations and $k \neq 0$.

- Let $\rho(Z) = Z^{-1}$, $\rho(u) = Z^k q^k v$, $\rho(v) = Z^{-k} q^k u$ with nonzero k.
- In this case the dimension of the space of traces is larger than n.

The case of q-deformations and $k \neq 0$.

- Let $\rho(Z) = Z^{-1}$, $\rho(u) = Z^k q^k v$, $\rho(v) = Z^{-k} q^k u$ with nonzero k.
- In this case the dimension of the space of traces is larger than n.
- On the other hand, it can be shown that the maximal possible dimension of the space of positive traces is $n - 2k$ (1 if this is zero.)

Positive traces when $k \neq 0$.

- For $k \leq 0$ any trace can be written as an integral. For $k > 0$ this is not so: for some traces $\log|T(Z^l)|$ grows quadratically as l tends to infinity, so it cannot be expressed as a contour integral.
- Instead, we write $T(R(Z)) ='' \int'' R(z)w(z)\frac{dz}{z}$, where w is a two-sided formal power series, and $'' \int'' \frac{dz}{z}$ means the constant term of the power series.
- It follows from positivity that $|T(Z^l)|$ is bounded. Namely, we use that $T(a\rho(a)) > 0$ for $a = 1 + \alpha Z^l$ for $\alpha \in \mathbb{C}$.
- We use this and the trace condition to prove that w is a Laurent expansion of a function holomorphic in the annulus $r < |z| < r^{-1}$ for some $r < 1$. Moreover, we prove that $P(qz)w(z)$ is a Laurent expansion of a function holomorphic on $r < |z| < q^{-2}r^{-1}$. It follows that $w(z)$ is a quasi-periodic meromorphic function on $\mathbb{C}^\times$. After this the classification of positive traces is similar to the case of filtered deformations and q-deformations with $k = 0$.

Future directions

- Use methods of [GHRWZ] to classify traces on more general Higgs branches, starting from more general abelian Higgs branches.
- q-deformations are examples of K-theoretic Coulomb branches. Try to classify traces on more general K-theoretic Coulomb branches.

Thank you!

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