Title: Multiplicative Global Springer Theory - VIRTUAL

Speakers: Marielle Ong

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Abstract: The moduli of Higgs bundles and the Hitchin fibration are central to many thriving research areas, such as mirror symmetry, non-abelian Hodge theory and the geometric Langlands program. A group-theoretic or multiplicative version was introduced by Frenkel and Ngo in 2011 to give a geometric interpretation of orbital integrals and trace formulas from automorphic representation theory. Since then, there is an ongoing program to replicate the theory of Higgs bundles for the multiplicative case. One notable development is the study of multiplicative affine Springer fibers. Like the usual ones, they are local analogues of multiplicative Hitchin fibers. In this talk, I discuss my work in continuing this program and providing a multiplicative version of Z. Yun's global Springer theory. This involves the study of parabolic multiplicative Higgs bundles and affine Springer fibers.

Multiplicative Global Springer Theory

Marielle Ong

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January 4th 2024

Classical Springer theory

- G reductive group over an algebraically closed field k
- Lie algebra $\mathfrak{g}$, maximal torus $T \subset G$, Weyl group W
- $\mathcal{B}$ is the flag variety of G

The *Grothendieck-Springer simultaneous resolution* is the forgetful map

$$\pi : \tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} : x \in \mathfrak{b}\} \to \mathfrak{g}.$$

The *Springer fiber* of $x \in \mathfrak{g}$ is the closed subscheme of $\mathcal{B}$ given by

$$\mathcal{B}_x = \{\mathfrak{b} \in \mathcal{B} : x \in \mathfrak{b}\}.$$

Example: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

The flag variety $\mathcal{B}$ is the moduli space of full flags $0 = V_0 \subset V_1 \subset V_2 = \mathbb{C}^2$. If $x \in \mathfrak{g}$, then $\mathcal{B}_x$ contain flags such that $x(V_i) \subset V_{i-1}$.

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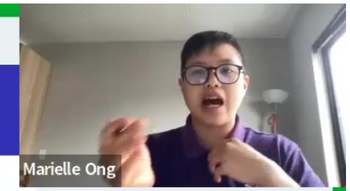
Springer Theory

- 1976: Springer constructs W-actions on $H^*(\mathcal{B}_x)$.
- 1981: Lusztig constructs W-actions on the perverse sheaf $R\pi_*\mathbb{Q}_\ell[\dim \mathfrak{g}]$, which encodes the cohomology of the Springer fibers.

Springer theories



- algebraically closed field k

	Classical	Local	Global
Field	k	local field $k((t))$	function field $k(X)$
Fiber	Springer Fiber	Affine Springer Fiber	Hitchin Fiber
Location	Flag variety	Affine Grassmannian	Hitchin moduli stack
classify some	Borel subalgebras	Higgs bundles over disc	Higgs bundles over X
Symmetries	W	$\widetilde{W} = W \ltimes X_*(T)$	$\widetilde{W}$
Rep theory	cohomology of Springer fibers	homology of parabolic ASF	cohomology of parabolic HF
sheaf-theoretic	Yes	No	Yes

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1996: Lusztig

$\widetilde{W} \curvearrowright f_*^{par} \mathbb{Q}_\ell$

$f_H : \mathcal{M}_H \longrightarrow \mathcal{A}_H$

2011: Yun

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Questions: group-theoretic = multiplicative

- What is a multiplicative version of these fibers?
- What geometric objects do they parametrize?
- Is there a parabolic version of these multiplicative fibers?
- Is there a multiplicative version of Yun's global Springer theory?

Affine Springer fibers

- G split connected reductive group over k with rank r
- Lie algebra $\mathfrak{g}$, maximal torus $T \subset G$, Weyl group W
- $F = k((t)), \mathcal{O} = k[[t]]$.

The *affine Grassmannian* is a sheaf $\mathrm{Gr}_G : \mathbf{Alg}_k^{\mathrm{op}} \to \mathbf{Set}$ for the fpqc topology,

$$R \mapsto G(R((t)))/G(R[[t]]).$$

Let $\gamma \in \mathfrak{g}^{\mathrm{rs}}(F)$. Its *affine Springer fiber* is the subfunctor of Gr_G that sends

$$R \mapsto X_\gamma(R) = \{g \in \mathrm{Gr}_G(R) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathfrak{g}(R[[t]])\}.$$

Examples

Example: $G = \mathrm{SL}_2(\mathbb{C})$

$\mathbb{C}((t))$

$\|$

The affine Grassmannian $\mathrm{Gr}_G(\mathbb{C}) = \underline{\mathrm{SL}_2(F)} / \mathrm{SL}_2(\mathcal{O})$ parametrizes projective $\mathcal{O}$-submodules

$$\Lambda = e_1\mathcal{O} \oplus e_2\mathcal{O} \subset F^2, \quad e_1, e_2 \in F^2$$

such that

1. There exists $N \geq 0$ such that $t^N(\mathcal{O}^2) \subseteq \Lambda \subseteq t^{-N}(\mathcal{O}^2)$.
2. $\bigwedge^2 \Lambda = \mathcal{O}^2$.

If $\gamma \in \mathfrak{sl}_2^{rs}(F)$, then $X_\gamma = \{\Lambda \in \mathrm{Gr}_G : \gamma(\Lambda) \subset \Lambda\}$.

Example: $G = \mathrm{SL}_2(\mathbb{C})$

If $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, then X_γ = infinite chain of $\mathbb{P}^1$'s.

✓

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If $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, then $X_\gamma =$ infinite chain of $\mathbb{P}^1$'s.

Geometric interpretation

- $D = \mathrm{Spec}(k[[t]])$ be the formal disk
- $D^\times = \mathrm{Spec}(k((t)))$ be the formal punctured disk

For $\gamma \in \mathfrak{g}^{rs}(F)$,

$$X_\gamma(k) = \{g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathfrak{g}(\mathcal{O})\}$$
$$= \left\{ (E, \phi, s) \,\middle|\, \begin{array}{l} E \text{ is a } G\text{-torsor on } D \\ \phi \in H^0(D, \mathrm{ad}(E)) \\ s \text{ trivialization of } E \text{ on } D^\times \\ (E, \phi)|_{D^\times} \cong (E_0, \gamma) \end{array} \right\},$$

where E_0 is the trivial bundle on $D^\times$.

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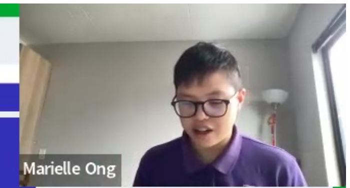

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Higgs bundles

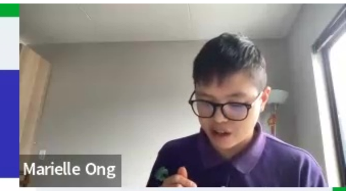
- X complete, smooth, connected curve over k of genus g
- L is a line bundle on X of degree $\deg(L) \geq 2g$.

An *L-twisted G-Higgs bundle* (E, ϕ) over X is a pair consisting of:

- a G-torsor E over X,
- a *Higgs field* $\phi \in H^0(X, \mathrm{ad}(E) \otimes L)$.

The moduli stack of Higgs bundles $\mathcal{M}_H = \mathrm{Hom}(X \times -, [\mathfrak{g}/G \times \mathbb{G}_m])$ sends

$$\begin{aligned} S \in \mathbf{Sch}_k \mapsto \mathcal{M}_H(S) &= \mathrm{Hom}(X \times S, [\mathfrak{g}/G \times \mathbb{G}_m]) \\ &= \text{Groupoid of } L\text{-twisted } G\text{-Higgs bundles over } X \times S \end{aligned}$$



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Hitchin fibration

The Chevalley morphism $\chi : \mathfrak{g} \to \mathfrak{c} := \mathfrak{t}/W \cong \mathfrak{g}//G$ induces the *Hitchin fibration*.

Example: $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$$\chi : \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{c} = \mathfrak{t}/W = \{\text{characteristic polynomials}\},$$

$$A \mapsto (a_1, \dots, a_n), \quad a_i = \mathrm{tr}(\Lambda^i A).$$

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$$\begin{array}{ccc} \mathcal{M}_H = \mathrm{Hom}(X \times -, [\mathfrak{g}/G \times \mathbb{G}_m]) & & \text{Moduli stack of } L\text{-twisted } G\text{-Higgs bundles } (E, \phi) \text{ over } X \\ \Big\downarrow f_H & & \Big\downarrow \text{Hitchin morphism} \\ \mathcal{A}_H = \mathrm{Hom}(X \times -, [\mathfrak{c}/\mathbb{G}_m]). & & \text{Hitchin base} \end{array}$$

affine

Example: $G = \mathrm{GL}_n(\mathbb{C})$

$$\overline{\mathrm{Pic}}(\widetilde{X}_a) \subseteq \left\{ \begin{matrix} E \in \mathbf{Bun}_n(X) \\ \phi \in H^0(X, \mathrm{End}(E) \otimes L) \end{matrix} \right\} \xrightarrow{f_H} \bigoplus_{i=1}^n H^0(X, L^i)$$

$$(E, \phi) \xrightarrow{(a_i = \mathrm{tr}(\Lambda^i \phi))_{i=1}^n} (a_1, \dots, a_n)$$

where $\widetilde{X}_a$ is the curve defined by the characteristic polynomial of ϕ.

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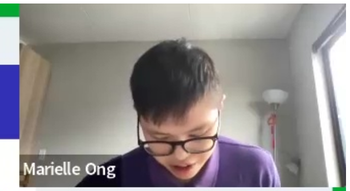
where $\underline{\widetilde{X}_a}$ is the curve defined by the characteristic polynomial of ϕ.

Local-global relationship

Product formula [Ngô 08]

For "nice" $a : X \to [\mathfrak{c}/\mathbb{G}_m]$, the set $U_a = a^{-1}([\mathfrak{c}^{\mathrm{rs}}/\mathbb{G}_m]) \subset X$ is open, dense and non-empty. There is a homeomorphism of stacks,

$$(\text{mod local Picard}) \quad \prod_{x \in X \setminus U_a} X_{\gamma_x} \simeq \mathcal{M}_a, \quad (\text{mod global Picard}).$$



Want to find a group-theoretic or multiplicative version of affine Springer fibers and Hitchin fibers.

$$X_\gamma(k) = \{g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \underline{\mathfrak{g}(\mathcal{O})}\}$$

$G(\mathcal{O})$

Cartan decomposition

Cartan decomposition

Every coweight $\lambda : \mathbb{G}_m \to T$ induces a map

$$\lambda : \mathbb{G}_m(F) \to T(F), \quad t \mapsto t^\lambda := \lambda(t)$$

Gr_G admits the Cartan decomposition into a disjoint union of Schubert cells,

$$\mathrm{Gr}_G(k) = \bigsqcup_{\lambda \in X_*(T)^+} G(\mathcal{O})t^\lambda G(\mathcal{O}), \quad \overline{G(\mathcal{O})t^\lambda G(\mathcal{O})} = \bigcup_{\substack{\mu \in X_*(T)^+ \\ \mu \leq \lambda}} G(\mathcal{O})t^\mu G(\mathcal{O})$$

Cartan Decomposition (matrix version)

For every $A \in \mathrm{GL}_n(F)$, there exists unique $\lambda \in X_*(T)^+$ and some $X, Y \in \mathrm{GL}_n(\mathcal{O})$ such that

$$XAY = \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix}, \quad (\lambda_1, \ldots, \lambda_n) \in X_*(T)^+ \cong \mathbb{Z}^n$$

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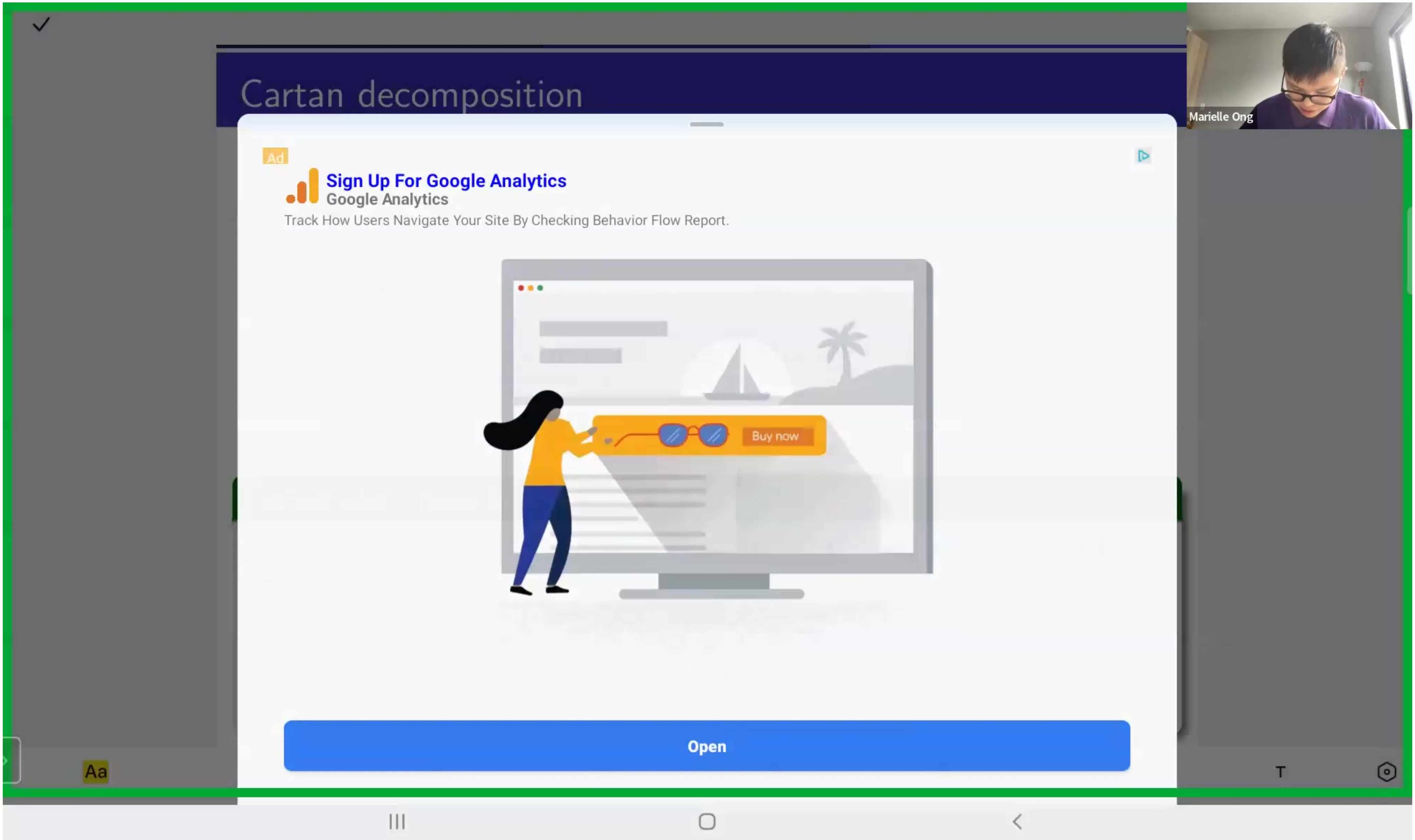
Cartan decomposition
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$$= GL_n(F)/GL_n(O)$$

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Multiplicative affine Springer fibers

Definition [Kottwitz-Viehmann 10, Bouthier 12]

Let $\gamma \in G^{\mathrm{rs}}(F)$ and $\lambda \in X_*(T)^+$. The *multiplicative affine Springer fibers* associated to (γ, λ) are sub-ind-schemes of Gr_G with k-points

$$X_\gamma^\lambda(k) = \left\{ g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in G(\mathcal{O}) t^\lambda G(\mathcal{O}) \right\},$$

$$X_\gamma^{\leq\lambda}(k) = \left\{ g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \overline{G(\mathcal{O}) t^\lambda G(\mathcal{O})} \right\}$$

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- non-σ-linear variants of affine Deligne Lusztig varieties

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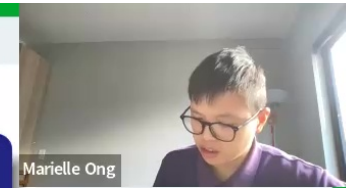
- non-σ-linear variants of affine Deligne Lusztig varieties $\quad L = \overline{\mathbb{F}_q}$

Vinberg monoid



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- non-σ-linear variants of affine Deligne Lusztig varieties

$L = \overline{\mathbb{F}}_q((t))$

$\sigma : t \longmapsto t^q$

$$\left\{ g \in \frac{G(L)}{G(\mathcal{O})} : g^{-1}\sigma(g) \in G(\mathcal{O}) t^\lambda G(\mathcal{O}) \right\}$$

Vinberg monoid

- G simply connected, semi-simple group over k with rank r
- T maximal torus, Z center of G
- simple roots α_i, fundamental weights ω_i
- $\rho_{\omega_i} : G \to \mathrm{GL}(V_{\omega_i})$ irreducible representation of highest weight ω_i

Define the *enhanced group* $G_+ = (T \times G)/Z$ where $Z \hookrightarrow T \times G$ anti-diagonally.

Definition [Vinberg, 94]

The *Vinberg monoid* V_G is the normalization of the closure of the image of

$$(\alpha_i(t), \omega_i(t)\rho_{\omega_i}(g)) : G_+ \to \left(\mathbb{G}_m^r \times \prod_{i=1}^r \mathrm{GL}(V_{\omega_i})\right) \subseteq \left(\mathbb{A}^r \times \prod_{i=1}^r \mathrm{End}(V_{\omega_i})\right).$$

It is a reductive monoid with unit group G_+. The *non-degenerate locus* V_G^0 is the inverse image of $(\mathbb{A}^r \times \prod_{i=1}^r \mathrm{End}(V_{\omega_i}) \setminus \{0\})$. It is a smooth, open, dense subvariety of V_G.

Abelianization

The Vinberg monoid admits an abelianization morphism

$$\alpha_G : V_G \to \underline{A_G = V_G//(G \times G)} \cong \mathbb{A}^r,$$

where A_G is a commutative monoid with unit group T^{ad}.

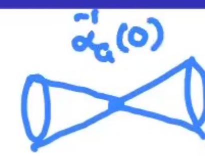
$$\alpha_G^{-1}(1) = G, \quad \alpha_G^{-1}(0) \text{ is the asymptotic monoid.}$$

V_G is the universal monoid satisfying flat α_G with reduced, irreducible fibers.
$\Longrightarrow$ multiparameter contraction of the $(G \times G)$-action on G to $\alpha_G^{-1}(0)$.

subvariety of V_G.

Abelianization

$G_+ = \frac{T \times G}{Z} \longrightarrow T_{ad} = T/Z$



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Examples

Example: $G = \mathrm{SL}_2(\mathbb{C})$

The enhanced group is given by $G_+ = \mathrm{GL}_2(\mathbb{C})$ and the Vinberg monoid is

$$V_G = \{(t, A) \in \mathbb{A}^1 \times \mathrm{End}(\mathbb{C}^2) : \det(A) = t\} \cong \mathrm{End}(\mathbb{C}^2) \xrightarrow{\det} A_G \cong \mathbb{A}^1.$$

Example: $G = \mathrm{SL}_3(\mathbb{C})$

The enhanced group is given by

$$G_+ = \frac{(T \times \mathrm{SL}_3(\mathbb{C}))}{\{\lambda I : \lambda^3 = 1\}}.$$

$$\begin{array}{c} V_G = \{(x, y, A, B) \in \mathbb{A}^2 \times \mathrm{End}(\mathbb{C}^3)^2 : A^T B = AB^T = xyI, \Lambda^2 A = xB, \Lambda^2 B = yA\} \\ \big\downarrow \alpha_G \\ A_G \cong \mathbb{A}^2 \end{array}$$

Aa Aa Aa T

Example: $G = \mathrm{SL}_2(\mathbb{C})$

The enhanced group is given by $G_+ = \mathrm{GL}_2(\mathbb{C})$ and the Vinberg monoid is

$$V_G = \{(t, A) \in \mathbb{A}^1 \times \mathrm{End}(\mathbb{C}^2) : \det(A) = t\} \cong \mathrm{End}(\mathbb{C}^2) \overset{\alpha_G}{\xrightarrow{\det}} A_G \cong \mathbb{A}^1.$$

Example: $G = \mathrm{SL}_3(\mathbb{C})$

The enhanced group is given by

$$G_+ = \frac{(T \times \mathrm{SL}_3(\mathbb{C}))}{\{\lambda I : \lambda^3 = 1\}}.$$

$$\begin{array}{c} V_G = \{(x, y, A, B) \in \mathbb{A}^2 \times \mathrm{End}(\mathbb{C}^3)^2 : A^T B = AB^T = xyI, \Lambda^2 A = xB, \Lambda^2 B = yA\} \\ \Big\downarrow \alpha_G \\ A_G \cong \mathbb{A}^2 \end{array}$$

A variant

Define the affine scheme V_G^λ by the pullback

$$\begin{tikzcd}
V_G^\lambda \arrow[r] \arrow[d] & V_G \times T^{\mathrm{ad}} \arrow[d, "{(x,y) \mapsto \alpha_G(x)y}"] \\
\mathrm{Spec}(\mathcal{O}) \arrow[r, "t^{-w_0(\lambda_{\mathrm{ad}})}"] & A_G
\end{tikzcd}$$

Define $V_G^{\lambda,0}$ similarly by replacing V_G with V_G^0.

$$V_G^\lambda(\mathcal{O}) = \bigcup_{\substack{\mu \in X_*(T_{\mathrm{ad}})_+ \\ \mu \leq \lambda_{\mathrm{ad}}}} G_+(\mathcal{O}) t^{(-w_0(\lambda_{\mathrm{ad}}),\mu)} G_+(\mathcal{O}),$$

$$V_G^{\lambda,0}(\mathcal{O}) = G_+(\mathcal{O}) t^{(-w_0(\lambda_{\mathrm{ad}}),\lambda_{\mathrm{ad}})} G_+(\mathcal{O}).$$

Geometric interpretation

The Kottwitz homomorphism κ_G is the composition

$$G(F) \xrightarrow{\quad r_G \quad} X_*(T) \xrightarrow{p_G} \pi_1(G) = X_*(T)/\mathbb{Z}\Phi^\vee$$

where $r_G(\gamma) = \mu \in X_*(T)$ such that $\gamma \in G(\mathcal{O})t^\mu G(\mathcal{O})$.

If $\kappa_G(\gamma) = p_G(\lambda)$, there exists $\gamma_\lambda \in G_+(F)$ such that

$$\begin{aligned} X_\gamma^\lambda(k) &= \left\{g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in G(\mathcal{O})t^\lambda G(\mathcal{O})\right\}, \\ &= \{g \in \mathrm{Gr}_G : \mathrm{Ad}_{g^{-1}}(\gamma_\lambda) \in V_G^{\lambda,0}(\mathcal{O})\} \\ &= \left\{ (E,\phi,s) \;\middle|\; \begin{array}{l} E \text{ is a } G\text{-torsor on } D \\ \phi \in H^0(D, E \times^G V_G^0) \\ s \text{ trivialization of } E \text{ on } D^\times \\ (E,\phi)|_{D^\times} \cong (E_0, \gamma_\lambda) \end{array} \right\} \end{aligned}$$

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The Kottwitz homomorphism κ_G is the composition

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$$= \{g \in \mathrm{Gr}_G : \mathrm{Ad}_{g^{-1}}(\gamma_\lambda) \in V_G^{\lambda,0}(\mathcal{O})\} \; *$$

$$= \left\{ (E, \phi, s) \;\middle|\; \begin{array}{l} E \text{ is a } G\text{-torsor on } D \\ \phi \in H^0(D, E \times^G V_G^0) \\ s \text{ trivialization of } E \text{ on } D^\times \\ (E, \phi)|_{D^\times} \cong (E_0, \gamma_\lambda) \end{array} \right\}$$

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If $\underline{\kappa_G(\gamma)} = \underline{p_G(\lambda)}$, there exists $\gamma_\lambda \in G_+(F)$ such that

$$X_\gamma^{\leq\lambda} \quad X_\gamma^\lambda(k) = \left\{ g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \overbrace{G(\mathcal{O})t^\lambda G(\mathcal{O})} \right\},$$

$$= \{ g \in \mathrm{Gr}_G : \mathrm{Ad}_{g^{-1}}(\gamma_\lambda) \in V_G^{\lambda,0}(\mathcal{O}) \} \quad *$$

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Geometric interpretation

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where $r_G(\gamma) = \mu \in X_*(T)$ such that $\underline{\gamma \in G(\mathcal{O})t^\mu G(\mathcal{O})}$.

$$X_*(T) \longrightarrow X_*(T_{ad})$$
$$\lambda \longmapsto \lambda_{ad}$$

If $\underline{\kappa_G(\gamma)} = \underline{p_G(\lambda)}$, there exists $\gamma_\lambda \in G_+(F)$ such that

$$X_\gamma^{\leq\lambda} \quad X_\gamma^\lambda(k) = \{g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in G(\mathcal{O})t^\lambda G(\mathcal{O})\}, \quad (-w_0(\lambda)_{ad}, \lambda_{ad})$$

$$= \{g \in \mathrm{Gr}_G : \mathrm{Ad}_{g^{-1}}(\gamma_\lambda) \in V_G^{\lambda,0}(\mathcal{O})\} \quad * \quad G_+(\mathcal{O})+ \quad G_+(\mathcal{O})$$

$$= \left\{ (E, \phi, s) \;\middle|\; \begin{array}{l} E \text{ is a } G\text{-torsor on } D \\ \phi \in H^0(D, E \times^G V_G^0) \quad V_G \\ s \text{ trivialization of } E \text{ on } D^\times \\ (E, \phi)|_{D^\times} \cong (E_0, \gamma_\lambda) \end{array} \right\}$$

Geometric interpretation

The Kottwitz homomorphism κ_G is the composition

$$\overset{\gamma}{G(F)} \xrightarrow{\ r_G\ } \overset{\mu}{X_*(T)} \xrightarrow{\ p_G\ } \overset{[\mu]}{\pi_1(G)} = X_*(T)/\mathbb{Z}\Phi^\vee$$

$X_* = \{(\lambda, \mu) \in X_*(T_{ad}) \times X_*(T_{ad}) : \lambda + \mu \in X_*(T)\}$

where $r_G(\gamma) = \mu \in X_*(T)$ such that $\underline{\gamma \in G(\mathcal{O}) t^\mu G(\mathcal{O})}$.

$X_*(T) \longrightarrow X_*(T_{ad})$

$\lambda \longmapsto \lambda_{ad}$

If $\underline{\kappa_G(\gamma)} = \underline{p_G(\lambda)}$, there exists $\gamma_\lambda \in G_+(F)$ such that

$(-w_0(\lambda)_{ad}, \lambda_{ad})$

$$X^{\le\lambda}_\gamma \qquad X^\lambda_\gamma(k) = \left\{ g \in \mathrm{Gr}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in G(\mathcal{O}) t^\lambda G(\mathcal{O}) \right\},$$

$$= \{ g \in \mathrm{Gr}_G : \mathrm{Ad}_{g^{-1}}(\gamma_\lambda) \in V_G^{\lambda,0}(\mathcal{O}) \} \quad *$$

$G_+(O)+$ $\qquad G_+(O)$

$$= \left\{ (E, \phi, s) \;\middle|\; \begin{array}{l} E \text{ is a } G\text{-torsor on } D \\ \phi \in H^0(D, E \times^G V_G^0) \quad V_G \\ s \text{ trivialization of } E \text{ on } D^\times \\ (E, \phi)|_{D^\times} \cong (E_0, \gamma_\lambda) \end{array} \right\}$$

Multiplicative Higgs bundles

- X complete smooth connected curve over k of genus g.
- L is a T-torsor on X.

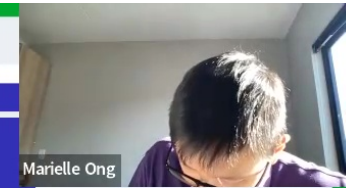
Definition

A *multiplicative Higgs bundle* (E, ϕ) is a pair consisting of:

- a G-torsor E over X,
- a *multiplicative Higgs field* $\phi \in H^0(X, E \times^G V_G \otimes L)$.

Define the moduli stack of multiplicative Higgs bundles as

$$\mathcal{M} = \mathrm{Hom}(X \times -, [V_G/G \times T]).$$



Multiplicative Hitchin fibration

The multiplicative Chevalley map $\chi_+ : V_G \to \mathfrak{C}_+ := V_G // G \cong V_T / W$ fits into

$$\begin{tikzcd} V_G \arrow[r, "\chi_+"] \arrow[rr, bend left, "\alpha_G"] & \mathfrak{C}_+ \cong A_G \times \mathfrak{C} \cong \mathbb{A}^{2r} \arrow[r] & A_G \cong \mathbb{A}^r \end{tikzcd}$$

induce the *multiplicative Hitchin fibration*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[rr, bend left, "\alpha"] & \mathcal{A} \arrow[r] & \mathcal{B} \end{tikzcd}$$

$\mathrm{Hom}(X \times -, [V_G/G \times T])$ $\mathrm{Hom}(X \times -, [\mathfrak{C}_+/T])$ $\mathrm{Hom}(X \times -, [A_G/T])$

The multiplicative Chevalley map $\chi_+ : V_G \to \mathfrak{C}_+ := V_G/\!/G \cong V_T/W$ fits into

$\frac{T\times T}{Z}$

$(\alpha_i, \mathrm{tr}(\rho^*\omega_i))$

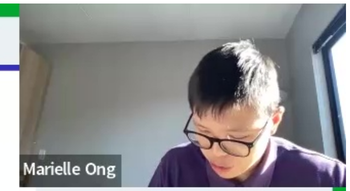
$$\begin{tikzcd}
V_G \arrow[r, "\chi_+"] \arrow[rr, bend left, "\alpha_G"] & \mathfrak{C}_+ \cong A_G \times \mathfrak{C} \cong \mathbb{A}^{2r} \arrow[r, "pr"'] & A_G \cong \mathbb{A}^r
\end{tikzcd}$$

$= G/\!/G$

induce the *multiplicative Hitchin fibration*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "f"] \arrow[rr, bend left, "\alpha"] & \mathcal{A} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

$\mathrm{Hom}(X \times -, [V_G/G \times T])$ $\quad$ $\mathrm{Hom}(X \times -, [\mathfrak{C}_+/T])$ $\quad$ $\mathrm{Hom}(X \times -, [A_G/T])$



Linear vs Multiplicative analogies

	Linear	Multiplicative
value of Higgs field	$\mathfrak{g}$	V_G
Chevalley morphism	$\chi : \mathfrak{g} \to \mathfrak{c}$	$\chi_+ : V_G \to \mathfrak{C}_+$
Section	Kostant section	Steinberg quasi-sections
regular locus of the fibers	dense	not dense
Hitchin fibration	Hitchin 87	Frenkel-Ngô 11
Local fibers	X_γ	X_γ^λ (Bouthier 12)
Product formula	Ngô 08	Chi 17
Equidimensional	Yes	Chi 17
Dimension formula	Bezrukanikov 96	Chi 17
Fundamental Lemma	Ngô 08	Wang 23
Local parabolic fiber	Yun 11	Ong 23
Global Springer theory	Yun 11	Ong 23

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	Linear	Multiplicative
value of Higgs field	$\mathfrak{g}$	V_G
Chevalley morphism	$\chi : \mathfrak{g} \to \mathfrak{c}$	$\chi_+ : V_G \to \mathfrak{C}_+$ $f \circ \chi_+ \in \mathrm{Aut}(V_G)$
Section	Kostant section	Steinberg quasi-sections
regular locus of the fibers	dense	not dense
Hitchin fibration	Hitchin 87	Frenkel-Ngô 11
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Global Springer theory	Yun 11	Ong 23

Parabolic Affine Springer fiber

Fix a Borel subgroup B of G. Define the Iwahori subgroup I as

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & G(\mathcal{O}) \arrow[d, "t \mapsto 0"] \\ B(k) \arrow[r, hook] & G(k) \end{tikzcd}$$

Let Fl_G be the affine flag variety with k-points $G(F)/I$.

Let $\gamma \in \mathfrak{g}^{\mathrm{rs}}(F)$. A *parabolic affine Springer fiber* is a closed sub-ind-scheme of Fl_G with

$$X_\gamma^{\mathrm{par}}(k) = \{g \in \mathrm{Fl}_G(R) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathrm{Lie}(I)\}.$$

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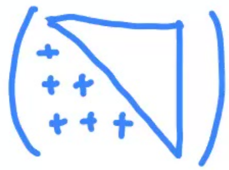

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & G(\mathcal{O}) = G(k[[t]]) \arrow[d, "t \mapsto 0"] \\ B(k) \arrow[r, hook] & G(k) \end{tikzcd}$$

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Parabolic Multiplicative Affine Springer fiber

Recall that $\widetilde{W} = W \ltimes X_*(T)$. The affine flag variety admits a Bruhat decomposition:

$$\mathrm{Fl}_G(k) = \bigcup_{w \in \widetilde{W}} IwI, \quad \overline{IwI} = \bigcup_{\substack{v \in \widetilde{W} \\ v \leq w}} IvI$$

Definition [Ong 23]

Let $\gamma \in G^{\mathrm{rs}}(F)$ and $\lambda \in X_*(T)^+$. The *parabolic multiplicative affine Springer fibers* are given by

$$X_\gamma^{\lambda,\mathrm{par}}(k) = \left\{ g \in \mathrm{Fl}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \bigcup_{\mu \in W(\lambda)} It^\mu I \right\}$$

$$X_\gamma^{\leq\lambda,\mathrm{par}}(k) = \left\{ g \in \mathrm{Fl}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \bigcup_{\mu \in W(\lambda)} \bigcup_{w \leq t^\mu} IwI \right\}$$

decomposition:

$$\mathrm{Fl}_G(k) = \bigcup_{w \in \widetilde{W}} IwI, \quad \overline{IwI} = \bigcup_{\substack{v \in \widetilde{W} \\ v \leq w}} IvI$$

Definition [Ong 23]

Let $\gamma \in G^{\mathrm{rs}}(F)$ and $\lambda \in X_*(T)^+$. The *parabolic multiplicative affine Springer fibers* are given by

$w(\lambda)$

$w\lambda w^{-1} =$

$$X_\gamma^{\lambda,\mathrm{par}}(k) = \left\{ g \in \mathrm{Fl}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \boxed{\bigcup_{\mu \in W(\lambda)} It^\mu I} \right\} = \bigcup I + I$$

$$X_\gamma^{\leq\lambda,\mathrm{par}}(k) = \left\{ g \in \mathrm{Fl}_G(k) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \bigcup_{\mu \in W(\lambda)} \bigcup_{w \leq t^\mu} IwI \right\}$$

Admissible set

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Admissible set

Definition

In $\widetilde{W} = W \ltimes X_*(T) = \{wt_\lambda : w \in W, \lambda \in X_*(T)\}$, the λ-*admissible set* is

$$\mathrm{Adm}(\lambda) = \{w \in \widetilde{W} : w \leq t_{x(\lambda)} \text{ for some } x \in W\},$$

where $\leq$ is an extension of the Bruhat order on $W_a = W \ltimes \Phi^\vee$.

Connections to arithmetic geometry and number theory:

- $\mathrm{Adm}(\lambda)$ parametrizes the cells in the special fiber of local models - proejctive schemes that model the singularities of Shimura varieties.
- the σ-linear variant of $X_\gamma^{\leq\lambda,\mathrm{par}}$ is related to the Newton strata of the special fiber.
- the support of the Bernstein function, which forms a basis of the center of the Iwahori-Hecke algebra.

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$$X_\gamma^{\leq \lambda, \mathrm{par}} = \{ g \in Fl_G : Ad_g^{-1}(\gamma) \in \bigcup_{w \in \mathrm{Adm}(\lambda)} IwI \}$$

Connections to arithmetic geometry and number theory:

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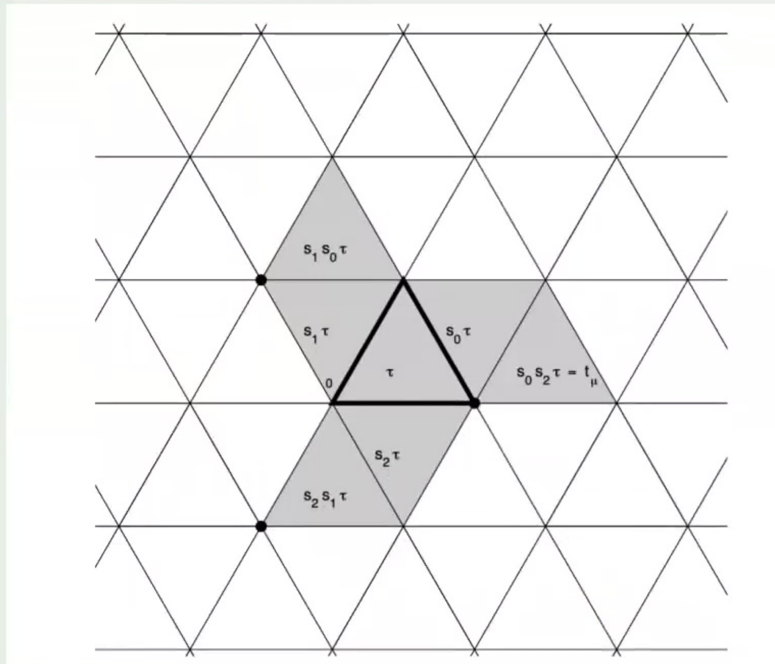
- the σ-linear variant of $X_\gamma^{\leq \lambda, \mathrm{par}}$ is related to the Newton strata of the special fiber.
- the support of the Bernstein function, which forms a basis of the center of the Iwahori-Hecke algebra.

There is a correspondence between elements of W_a and alcoves.

Example: $G = \mathrm{GL}_3$, $\lambda = (-1, 0, 0)$



$\mathrm{Adm}(\lambda)$, where τ is the base alcove.

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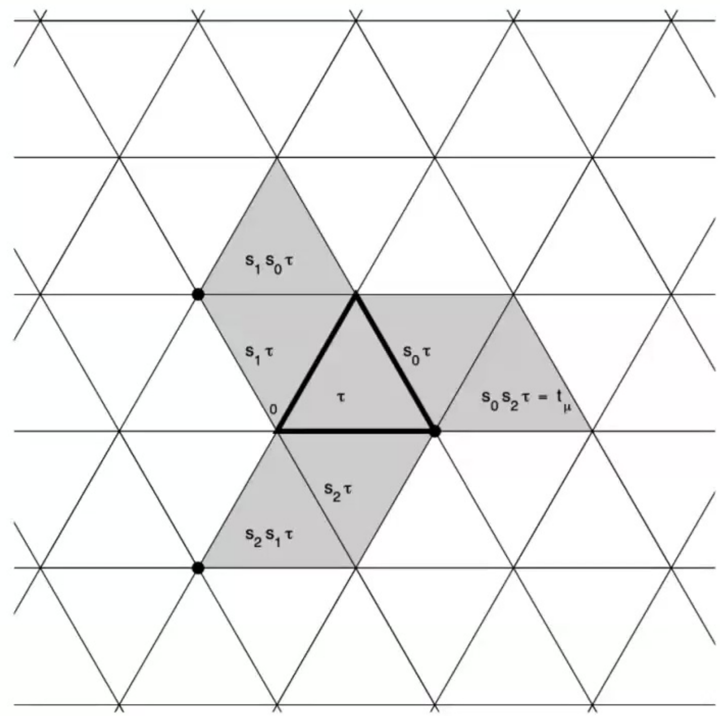
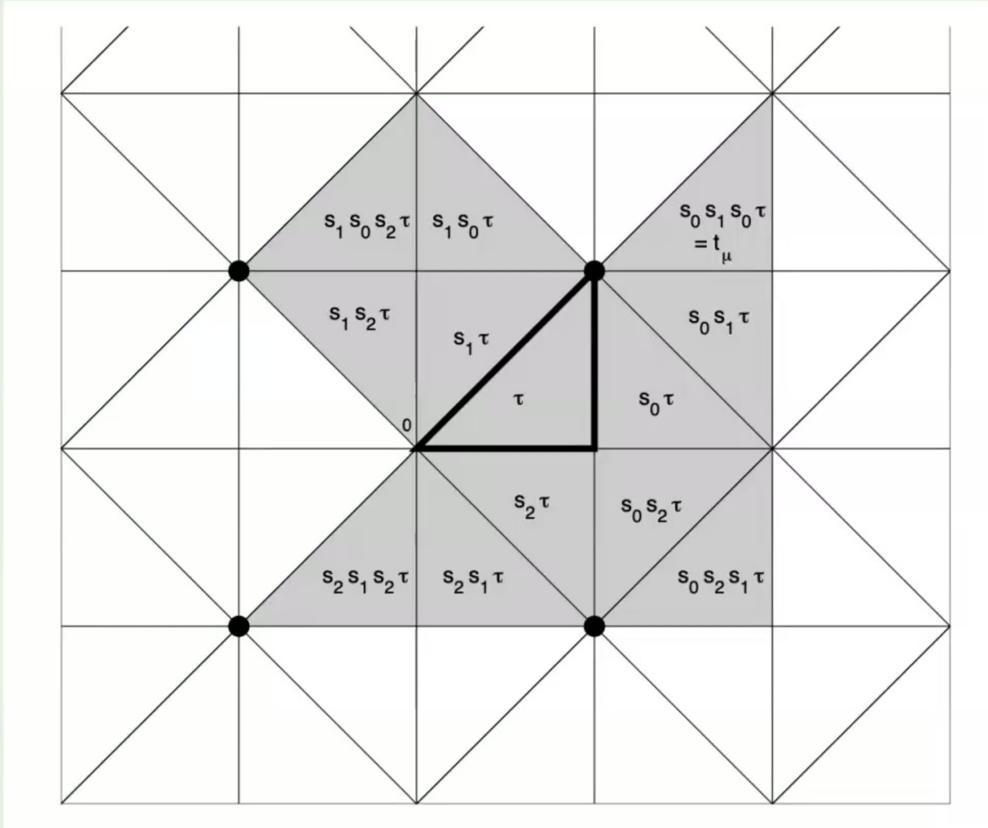
"$W \ltimes \mathbb{Z}\Phi^\vee \subseteq \widetilde{W}$



Example: $G = \mathrm{GSp}_4$, $\lambda = (-1, -1, 0, 0)$



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Parabolic multiplicative Higgs bundles

Let L be a T-torsor on X.

Definition [Ong 23]

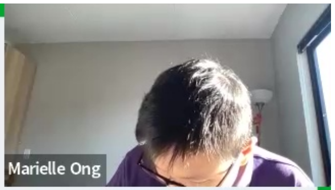
A *parabolic multiplicative Higgs bundle* is a quadruple (x, E, ϕ, E_x^B) consisting of

- $x \in X$
- (E, ϕ) is a multiplicative Higgs bundle
- E_x^B is a B-reduction of E_x

such that ϕ is compatible with the B-reduction.

There is a monoid version of the Grothendieck-Springer simultaneous resolution,

$$\begin{tikzcd} \widetilde{V}_G = \{(x, B) \in V_G \times \mathcal{B} : x \in V_B\} \arrow[r] \arrow[d, "\pi"'] & V_T \arrow[d, "q"] \\ V_G \arrow[r, "\chi_+"] & \mathfrak{C}_+ \end{tikzcd}$$



Parabolic multiplicative Hitchin fibration

$\mathcal{M}^{\text{par}}$ is the moduli stack of parabolic multiplicative Higgs bundles.

$$\begin{tikzcd}
\mathcal{M}^{\text{par}} \arrow[r, "\text{ev}"] \arrow[d] & {[\widetilde{V}_G/G \times T] = [V_B/B \times T]} \arrow[d, "\pi \leftarrow \text{induced GS resolut}"] \\
\mathcal{M} \times X \arrow[r, "\text{ev}"] & {[V_G/G \times T]}
\end{tikzcd}$$

The *parabolic multiplicative Hitchin fibration* is

$$f^{\text{par}} : \mathcal{M}^{\text{par}} \to \mathcal{A} \times X,$$
$$(x, E, \varphi, E_x^B) \mapsto (f(E, \varphi), x),$$

where f is the multiplicative Hitchin fibration.

Parabolic multiplicative Hitchin fibration

$\mathcal{M}^{\text{par}}$ is the moduli stack of parabolic multiplicative Higgs bundles.

$$\begin{tikzcd} (x,E,\phi,E^B_x) \arrow[d, mapsto] & \mathcal{M}^{\text{par}} \arrow[r, "\text{ev}"] \arrow[d] & {[\widetilde{V}_G/G\times T] = [V_B/B\times T]} \arrow[d, "\pi"] \\ (E,\phi,x) & \mathcal{M}\times X \arrow[r, "\text{ev}"] & {[V_G/G\times T]} \end{tikzcd}$$

← induced GS resolution

The *parabolic multiplicative Hitchin fibration* is

$$f^{\text{par}} : \mathcal{M}^{\text{par}} \to \mathcal{A}\times X,$$
$$(x,E,\varphi,E^B_x) \mapsto (f(E,\varphi),x),$$

where f is the multiplicative Hitchin fibration.

Local results [Ong 23]

- Non-emptiness pattern:
  - $X_\gamma^{\lambda,\mathrm{par}}$ is non-empty iff X_γ^λ is non-empty.
- Relation to the Iwahori monoids $I^{\lambda_{\mathrm{ad}}}$ and $I^{\lambda_{\mathrm{ad}},0}$:
  - $X_\gamma^{\lambda,\mathrm{par}} = \{g \in \mathrm{Fl}_G : \mathrm{Ad}_{g^{-1}}(\gamma_\lambda) \in I^{\lambda_{\mathrm{ad}},0}\}$.
- Dimension of parabolic MASFs
  - Combining the results of [He-Yu 20], [Sadhukhan 22] and [He 23] gives

$$\dim X_\gamma^{\leq\lambda,\mathrm{par}} = \dim X_\gamma^{\leq\lambda} + \mathcal{B}_{u(w_0)},$$

where $\mathcal{B}_{u(w_0)}$ is the Springer fiber of a unipotent element associated to w_0.

  - [Ong 23] geometrizes the above formula and extends it to dominant coweights.

$$\dim X_\gamma^{\leq\lambda,\mathrm{par}} = \dim X_\gamma^{\leq\lambda} + \dim \mathcal{B}_{e_\lambda},$$

where $e_\lambda \in \overline{T}_+$ is the idempotent associated to e_λ.

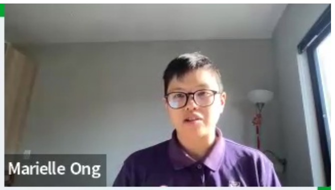
- Equidimensionality of parabolic MASFs
- Product formula

Hecke correspondences

$$
\begin{tikzcd}
 & \mathcal{H}^{\mathrm{par}} \arrow[dl, "\overleftarrow{f_b}"'] \arrow[dr, "\overrightarrow{f_b}"] & \\
\mathcal{M}^{\mathrm{par}} \arrow[r, "f^{\mathrm{par}}"] & \mathcal{A} \times X & \mathcal{M}^{\mathrm{par}} \arrow[l, "f^{\mathrm{par}}"']
\end{tikzcd}
$$

Restrict to "very ample" $b \in \mathcal{B}$ and $(\mathcal{A}_b \times X)'$, where a certain codimension estimate holds.

$$
\begin{tikzcd}
 & \mathcal{H}_b^{\mathrm{par}} \arrow[dl, "\overleftarrow{f_b}"'] \arrow[dr, "\overrightarrow{f_b}"] & \\
\mathcal{M}_b^{\mathrm{par}} \arrow[r, "f^{\mathrm{par}}"] & (\mathcal{A}_b \times X)' & \mathcal{M}_b^{\mathrm{par}} \arrow[l, "f^{\mathrm{par}}"']
\end{tikzcd}
$$



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Marielle Ong | January 4th 2024

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Hecke correspondences

$$\begin{tikzcd} & \mathcal{H}^{\text{par}} \arrow[dl, "\overleftarrow{f}"'] \arrow[dr, "\overrightarrow{f}"] & \\ \mathcal{M}^{\text{par}} \arrow[r, "f^{\text{par}}"] & \mathcal{A} \times X & \mathcal{M}^{\text{par}} \arrow[l, "f^{\text{par}}"'] \end{tikzcd}$$

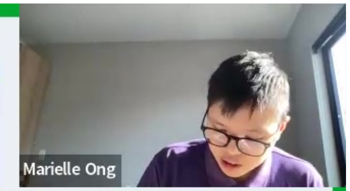
$$\begin{tikzcd} \mathcal{H}^{\text{par}} \arrow[d] \\ A \times X \arrow[d] \\ B \times X \quad b \end{tikzcd}$$

Restrict to "very ample" $b \in \mathcal{B}$ and $(\mathcal{A}_b \times X)'$, where a certain codimension estimate holds.

$$\begin{tikzcd} & \mathcal{H}_b^{\text{par}} \arrow[dl, "\overleftarrow{f_b}"'] \arrow[dr, "\overrightarrow{f_b}"] & \\ \mathcal{M}_b^{\text{par}} \arrow[r, "f^{\text{par}}"] & (\mathcal{A}_b \times X)' & \mathcal{M}_b^{\text{par}} \arrow[l, "f^{\text{par}}"'] \end{tikzcd}$$

Restricted

Parabolic multiplicative global Springer theory



Theorem [Ong 23]

There is a right action of $\widetilde{W}$ on $\mathcal{M}_b^{\mathrm{par}}$ such that $\mathcal{H}_b^{\mathrm{par}}|_{(\mathcal{A}_b \times X)^{\mathrm{rs}}}$ is a disjoint union of graphs $\mathcal{H}_w$ of this $\widetilde{W}$-action.

The fundamental class $[\mathcal{H}_w|_{(\mathcal{A}_b \times X)^{\mathrm{rs}}}]$ can be viewed as a correspondence – elements of $\mathrm{Hom}(\overrightarrow{f_b}^* \mathbb{Q}_\ell, \overleftarrow{f_b}^! \mathbb{Q}_\ell)$. Thanks to the codimension estimate, it is graph-like with respect to $(\mathcal{A}_b \times X)^{\mathrm{rs}}$ and hence, it induces a well-defined map

$$[\mathcal{H}_w]_\# \in \mathrm{End}(f_{b,*}^{\mathrm{par}} \mathbb{Q}_\ell).$$

Theorem [Ong 23]

The assignment $w \mapsto [\mathcal{H}_w]_\#$ defines an action of $\widetilde{W}$ on $f_{b,*}^{\mathrm{par}} \underline{\mathbb{Q}}_\ell$.

Future directions

- Continue generalizing Yun's global Springer theory
  - extend the action of $\widetilde{W}$ on $f^{\text{par}}_{b,*}\underline{\mathbb{Q}}_\ell$ to the action of graded DAHA.
  - give a second construction of the $\widetilde{W}$-action on $f^{\text{par}}_{b,*}\underline{\mathbb{Q}}_\ell$ via Coxeter presentations.
  - study the endoscopic decomposition of $f^{\text{par}}_{b,*}\underline{\mathbb{Q}}_\ell$
- Multiplicative non-abelian Hodge theory
- Connection between multiplicative Higgs bundles, periodic monopoles and twisted gauge theory. Incorporate the Vinberg monoid's geometry and study its applications.
- Point-counting of parabolics MASFs
  - orbital integrals of the Bernstein functions z_λ in the center of the Iwahori-Hecke algebra
- Deformation of global multiplicative affine Springer fibers into parabolic ones.