Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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Lecture 11 Perturbation Theory

Recall

$$\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{a\,c}$$

$$= \frac{1}{2}\left(\nabla_c \nabla_a h^c_b + \nabla_c \nabla_b h^a{}_c - \Box h_{ab} + \nabla_a \nabla_b h\right)$$

where

$$g_{ab} = g_{0ab} + h_{ab}$$

Using the Riemann identity

$$\nabla_c \nabla_a h^c{}_b = \nabla_a \nabla_c h^c{}_b + R^c{}_{dca} h^d{}_b$$
$$+ R_{bdca} h^{cd}$$

get $$\delta R_{ab} = -\frac{1}{2}\left[\Box h_{ab} + 2R_{acbd} h^{cd} - 2R_{(a}{}^d h_{b)d} - 2\nabla_{(a} \nabla^e \bar{h}_{b)e}\right]$$

$$= -\frac{1}{2} \Delta_L h_{ab} \qquad \left(\bar{h}_{ab} = h_{ab} - \frac{1}{2} h g_{0ab}\right)$$

Lichnerowicz op.

Δ_L is the curved space wave eqn for grav. ptns - gravitons.

$\cdots{}_{(a}{}^{d} h_{b)d} - 2\nabla_{(a}\nabla^{e}\bar{h}_{b)e}]$

$(\bar{h}_{ab} = h_{ab} - \frac{1}{2} h g_{0ab})$

<u>Gauge</u>

Under a g.t. $X^a \to X^a + \xi^{a}_{(\varepsilon)}$

$$g_{ab} \to g_{ab} + \underbrace{\mathcal{L}_\xi g_{ab}}_{h_{\xi ab}}$$

$$h_{\xi ab} = \nabla_a \xi_b + \nabla_b \xi_a$$

$O(\epsilon)$

$+\xi^a$

ab

ab

$\nabla_b \xi_a$

Changing coords generates a perturbation

$$\Rightarrow \nabla^a \bar{h}_{\xi ab} = \Box \xi_b + \nabla^a \nabla_b \xi_a - \nabla_b (\nabla^a \xi_a)$$

$$= \Box \xi_b + R^a{}_b \, \xi_a$$

– well posed diff eqn. Therefore if $\nabla^a \bar{h}_{ab}$ is not zero initially, can solve for ξ to transform to new gauge.

$\nabla_a \bar{h}^a{}_b = 0$. De Donder gauge.

TTF (transverse tracefree) $h = 0 = \nabla^a h_{ab}$.

Remaining gauge freedom: $X^a \to X^a + \chi^a$

s.t. $\Box \chi^a + R^a{}_b \chi^b = 0$ ← D solns.

De

Degrees of freedom:

$$\underbrace{\frac{D(D+1)}{2}}_{\text{cpts } h_{ab}} - \underbrace{D}_{\substack{\text{gauge constraint} \\ \nabla^a \bar{h}_{ab}=0}} - \underbrace{D}_{\text{remaining freedom}}$$

$$\frac{D(D-3)}{2}$$

$D=4 \quad N=2$

$D=5 \quad N=5$

Black hole perturbations – Regge-Wheeler 1957.

– SCH ptns. Phys Rev 108

decomposed h_{ab} into tensor spherical harmonics

Solns have even & odd parity, but no s-wave –

Grav Radn is quadrupolar.

e.g. odd

canonical

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} e^{-i\omega t} \sin\theta \, \partial_\theta P_\ell(\cos\theta)$$

Lecture 11 Perturbation Theory

Recall

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$$= \frac{1}{2}\left(\nabla_c \nabla_a h_b^{\ c} + \nabla_c \nabla_b h^c_{\ a} - \Box h_{ab} + \nabla_a \nabla_b h\right)$$

where

$$g_{ab} = g^0_{ab} + h_{ab}$$

The fns h_i satisfy an eqn of the form

$$\frac{d^2}{dr^{*2}}\left(\frac{fh}{r}\right) + \omega^2 h - \underbrace{V_{eM}(r)}_{\left(\ell(\ell+1) - \frac{6GM}{r}\right)\frac{f}{r^2}} h = 0$$

$$f = 1 - \frac{2GM}{r}$$

Application - Black String Instability

From thermodynamics, string is unstable for $L \gtrsim \frac{27\pi}{16} r_4$

For $ds^2 = ds^2_{4\,sch} - dz^2$, the "Fourier" modes are

$$h^{(4)}_{\mu\nu} e^{i\mu z} \quad (\mu L = 2\pi n)$$

The simplest mode is the s-wave

$$h_{ab} = \boxed{h_{ab}(r)\, e^{\Omega t}}\, e^{i\mu z}$$

$$\Delta_L^{(4)} h_{ab} + \mu^2 h_{ab} = 0.$$

TENSOR

VECTOR

$$\left(\begin{array}{cc|cc|c} h_{tt} & h_{tr} & 0 & & h_{ts} \\ h_{tr} & h_{rr} & & & h_{rs} \\ \hline 0 & & K & 0 & 0 \\ & & 0 & K\sin^2\theta & \\ \hline h_{ts} & h_{rs} & 0 & & h_{ss} \end{array}\right)$$

SCALAR

Looking for an instability

$$\Omega = -i\omega.$$

$+\mu^2) h_{ss}$

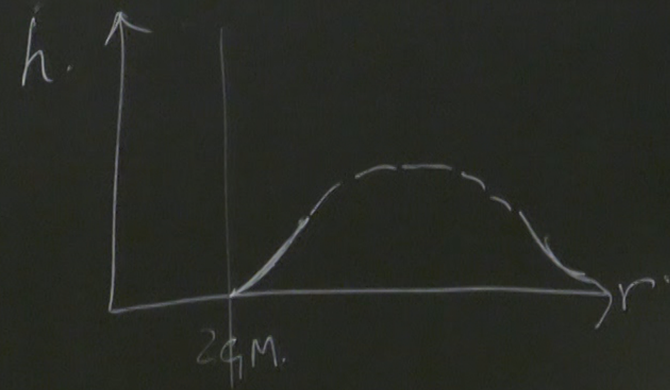
$h''_{ss} - \frac{2}{r^2}(r-GM) h'_{ss}$

$^2 + \frac{\Omega^2}{f}) h_{ss} = 0$

$e^{\pm\sqrt{\Omega^2+\mu^2}\, r}$

$(r-2GM)^{\pm 2GM\Omega}$

For regular soln, choose one branch at each boundary.



If $\exists$ soln to h, then $\exists$ max of h

$h' = 0$. E.O.M. $\Rightarrow h'' > 0$.

Thus $h_{ss} = 0$.

Thus $h_{55} = 0$. Similar argument gives $h_{t5} = h_{r5} = 0$.

Left with "tensor" mode: $\Delta_L^{(4)} h_{\mu\nu} + \mu^2 h_{\mu\nu} = 0$.

Note, for a gauge mode, $h_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$, $\Delta_L^{(4)} h_{\xi\mu\nu} = 0$.

Thus any soln is physical.

Use TTF to remove degrees of freedom. e.g.

$h^\lambda_\lambda = 0$ removes K

Either 2^{nd} order ODE in 1 var

or 2 coupled 1^{st} order in 2 vars

$$H_{\pm} = \frac{h_{tt}}{f} \pm f h_{rr} \qquad H = h_{tr}$$

$$H' = \frac{\Omega}{2f}(H_+ + H_-) - \frac{(1+f)}{rf} H$$

$$H_-' = \frac{\mu^2}{\Omega} H + \frac{H_+}{r} + \frac{(1-5f)}{2rf} H_-$$

$$H_+ = H_+(H_-, H).$$

1 var

vars

$H = h_{tr}$

$f) \; H$

$\frac{f}{f}$

$\frac{-Sf)}{2rf} \; H_-$

$$H_+ = H_+(H_-, H).$$

As $r \to \infty$

$$H \sim \pm\sqrt{\mu^2+\Omega^2}\, e^{\pm\sqrt{\mu^2+\Omega^2}\, r}$$

$$H_- \sim \frac{\mu^2}{\Omega} e^{\pm\sqrt{\mu^2+\Omega^2}\, r}$$

$r \to 2GM = r_+$

$$H \sim \left(\pm\Omega r_+ - \frac{1}{2}\right)(r-r_+)^{\pm\Omega r_+ - 1}$$

$$H_- \sim \left(\frac{\mu^2}{\Omega} \pm \frac{2}{r_+}\right)(r-r_+)^{\pm\Omega r_+}$$

$$H_+ \sim \pm 2H$$

As $r \to r_f$, loo

H_- are singular

As $r \to r_+$, looks like both branches of H_- are singular for $\Omega r_+ < 1$. However, cannot remove both solns:

$$T = 2e^{r^*/2r_+}\sinh\left(\frac{t}{2r_+}\right) \quad R = 2e^{r^*/2r_+}\cosh\left(\frac{t}{2r_+}\right)$$

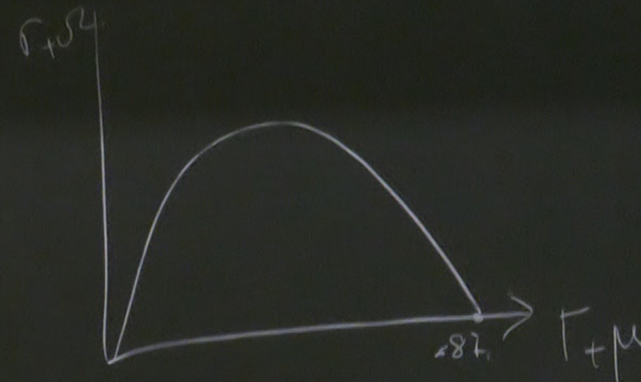
then eg $$h_{TT} \propto \frac{1}{R^2-T^2}\left(\frac{R^2+T^2}{2}H_+ - 2RTH\right) + \frac{H_-}{2}$$

as $r \to r_+$ $\quad R \to T \quad H_+ \to \pm 2H$

$H_- = \frac{\mu}{\Omega} H + \frac{H_+}{r} + \frac{(1-3f)}{2rf} H_-$

$H_+ \propto \pm 2H$

$$h_{TT} \propto \frac{H}{R^2-T^2}\Big(\underbrace{\pm(R^2+T^2)-2RT}_{\pm(R\mp T)^2}\Big)+\frac{H}{2}$$



h_{TT} regular for upper root

Finding solns becomes an eval

TD gave $r_+\mu < \frac{32}{27}$

$\Rightarrow r_+\mu$

$\frac{3^3}{27}$

This is an example of a more general perturbation analysis.

$$Q'' + \omega^2 Q = V_{eff}\, Q$$

$$Q \sim e^{-i\omega(t-r)} \quad \text{at } \infty$$

$$e^{-i\omega(t+r_*)} \quad \text{at } r_+$$

Scalar sketch:

$$\Box \Phi = \frac{\ddot{\Phi}}{f} - \frac{1}{r^2}(r^2 f \Phi')' - \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\, \Phi_{,\theta})$$

$$- \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2}$$

$$\Phi = \frac{1}{r}\sum \phi_{\omega\ell}(r)\, Y_{\ell m}(\theta, \varphi) e^{-i\omega t} e^{im\varphi}$$

$$\frac{d^2\phi}{dr_*^2} + \omega^2 \phi - f\left[\frac{\ell(\ell+1)}{r^2} + \frac{r_+}{r^3}\right]\phi = 0$$

Analytic approx (Blome + Mashoon)

takes inverted Poschl Teller pot$^{\text{ll}}$

$$V_{PT} = \frac{V_0}{\cosh^2 \alpha (r_* - r_{*0})}$$

$$V'_{\ell m}(r_{0*}) = 0. \qquad V_0 = V_{\ell m}(r_{0*})$$

$$\alpha^2 = -V''_{\ell m}(r_{0*}) / V_0$$

Mashoon)

ller potl

$-r_{*0})$

$V_0 = V_{lm}(r_{0*})$

V_0

Redefine

$$u = \tanh \alpha (r^* - r^*_0)$$

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4V_0^2}{\alpha^2}}$$

$$\mu = -i\omega/\alpha$$

get

$$\frac{d}{du}\left((1-u^2)\frac{d\phi}{du}\right) + \lambda(\lambda+1)\phi - \frac{\mu^2\phi}{1-u^2}.$$

$\Omega \qquad r \qquad 2rf$

Legendre eqn. b.c's pick out

$$\mu = \lambda_\pm - n$$

$$\omega = \pm\sqrt{V_0 - \alpha^2/4} - i\alpha(n + 1/2)$$

eg $l \to \infty$ (eikonal) $V' = 0$ at $\frac{3r_s}{2}$

$$\alpha = 2/\sqrt{3}GM \qquad \omega \leftrightarrow M.$$