Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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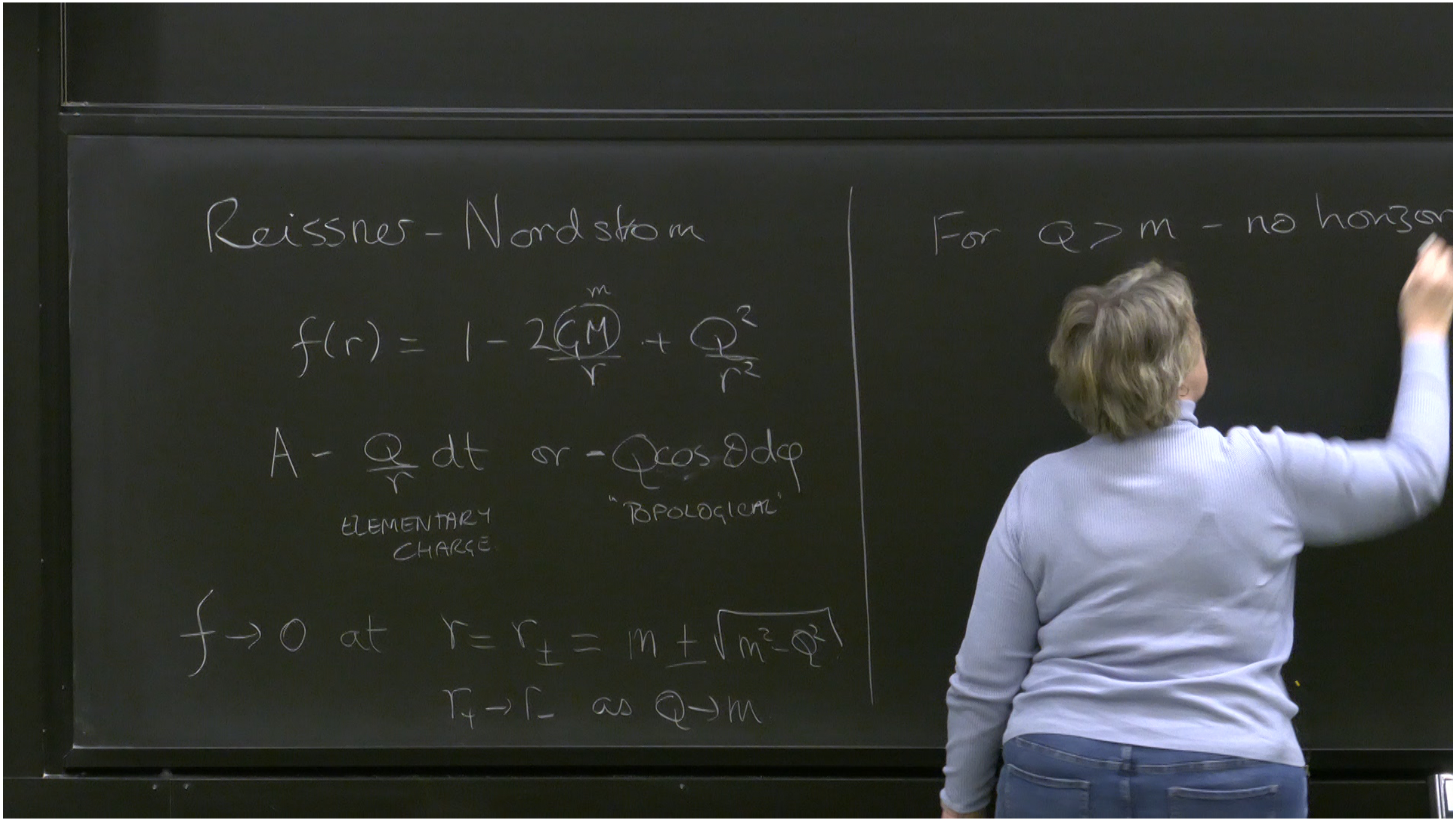
L10 (MORE) THERMODYNAMICS

$$ds^2 = f\,dt^2 - \frac{dr^2}{f} - r^2\,d\Omega_{\mathrm{II}}^2$$


Reissner-Nordstrom
$f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$
m
$A - \frac{Q}{r}dt$ or $- Q\cos\theta d\varphi$
ELEMENTARY CHARGE
"TOPOLOGICAL"
$f \to 0$ at $r = r_\pm = m \pm \sqrt{m^2 - Q^2}$
$r_+ \to r_-$ as $Q \to m$
For $Q > m$ - no horizon

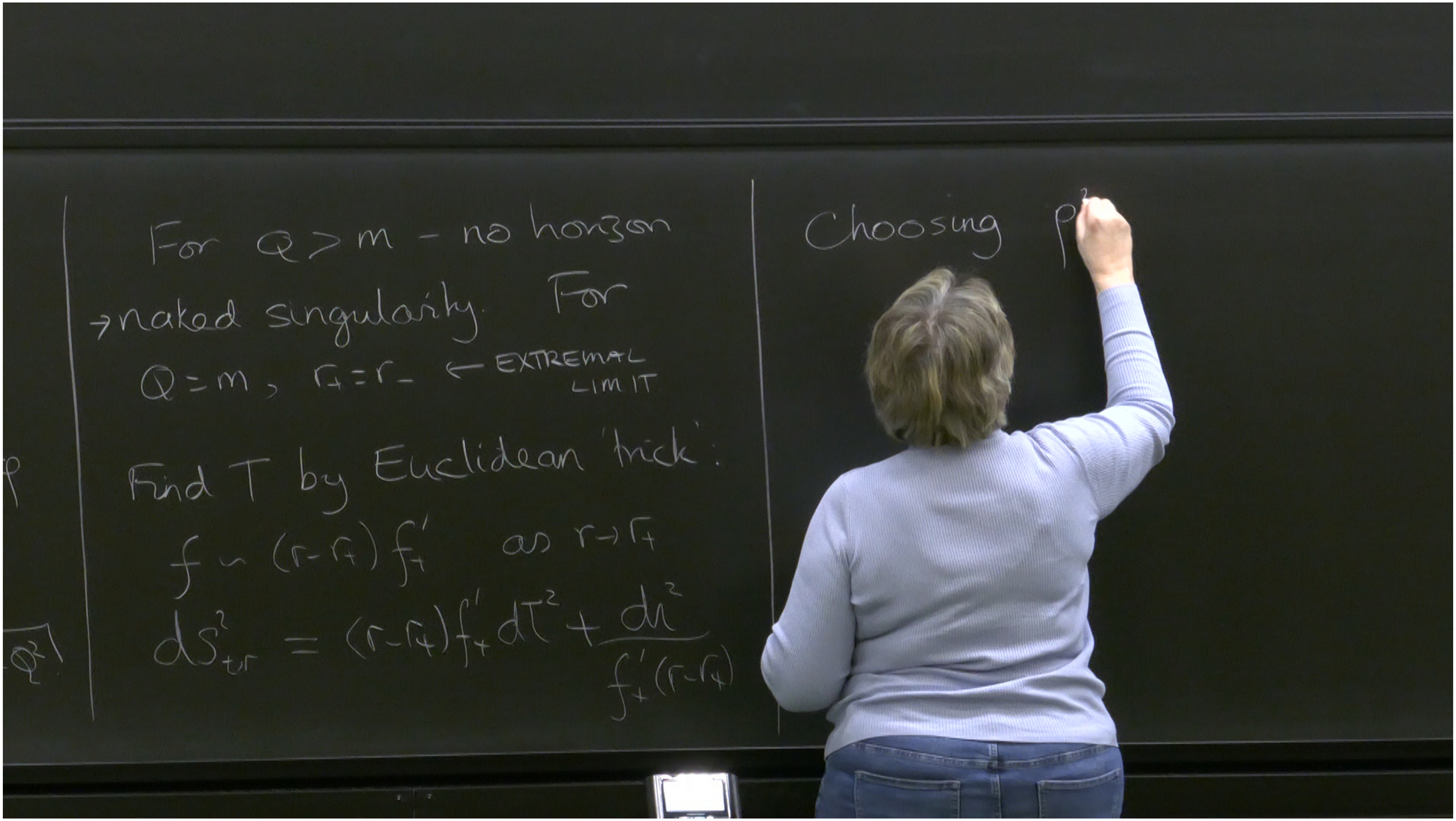
For $Q > m$ - no horizon

$\rightarrow$ naked singularity. For

$Q = m$, $r_+ = r_-$ $\leftarrow$ EXTREMAL LIMIT

Find T by Euclidean 'trick':

$f \sim (r - r_+) f'_+$ as $r \to r_+$


For $Q > m$ - no horizon
→ naked singularity. For
$Q = m$, $r_+ = r_-$ ← EXTREMAL LIMIT
Find T by Euclidean 'trick':
$f \sim (r - r_+) f'_+$ as $r \to r_+$
$ds^2_{t,r} = (r - r_+) f'_+ d\tau^2 + \frac{dr^2}{f'_+ (r - r_+)}$
Choosing ρ^2

Choosing $\rho^2 = \frac{4}{f'_+}(r - r_+)$

gives

$$ds_{tr}^2 = d\rho^2 + \rho^2 d\left(\frac{f'_+ \tau}{2}\right)^2$$

Suggests $\Delta\tau = \beta = \frac{4\pi}{f'_+}$

Check: SCH: $f'_+ = \frac{2GM}{r_+^2} = \frac{1}{2GM}$

$$\Rightarrow \beta_{SCH} = 8\pi GM$$

Choosing $\rho^2 = \frac{4}{f'_+}(r-r_+)$

gives

$$ds^2_{tr} = d\rho^2 + \rho^2 \, d\left(\frac{f'_+ \tau}{2}\right)^2$$

Suggests $\Delta\tau = \beta = \frac{4\pi}{f'_+}$

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$$\Rightarrow \beta_{SCH} = 8\pi G M$$

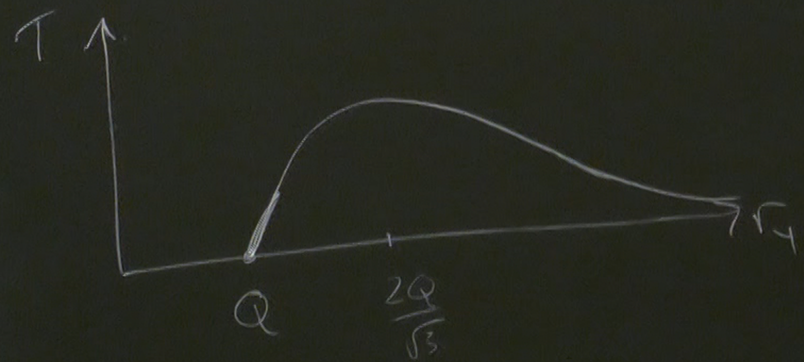
For RN $f' = \frac{2m}{r^2} - \frac{2Q^2}{r^3}$

$r \to r_+$ $\quad f'_+ = \frac{1}{r_+}\left[1 - \frac{Q^2}{r_+^2}\right]$

$$T = \frac{1}{\beta} = \frac{f'_+}{4\pi} = \frac{1}{4\pi r_+}\left[1 - \frac{Q^2}{r_+^2}\right]$$

- Parametrising degrees of freedom by r_+ & Q.

At fixed Q, $r_+ \in [Q, \infty)$



Add $\Lambda < 0$ $\quad \frac{1}{\ell^2} = -\frac{\Lambda}{3}$

$$f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}$$

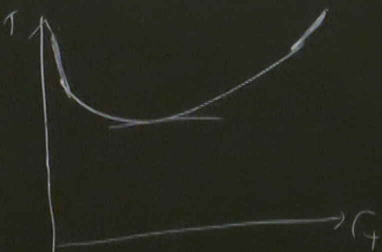
Add $\Lambda < 0$ $\quad \frac{1}{\ell^2} = -\frac{\Lambda}{3}$

$$f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}$$

$$4\pi T = f'_+ = \frac{2m}{r_+^2} - \frac{2Q^2}{r_+^3} + \frac{2r_+}{\ell^2}$$

$$= \frac{1}{r_+}\left[1 + 3\frac{r_+^2}{\ell^2} - \frac{Q^2}{r_+^2}\right]$$

Note. if $Q=0$ $\quad T = \frac{1}{4\pi r_+}\left[1 + \frac{3r_+^2}{\ell^2}\right]$



T has a minimum!

ℓ is the scale that sets min

Large black holes have the specific heat. HAWKING-PAGE.

$\frac{r_+}{\ell^2}$

$\frac{Q^2}{r_+^2}\Big]$

Note, if $Q=0$ $\quad T = \frac{1}{4\pi r_+}\left[1 + \frac{3r_+^2}{\ell^2}\right]$

T has a minimum!

ℓ is the scale that sets min

Large black holes have the specific heat. HAWKING-PAGE.

With charge:

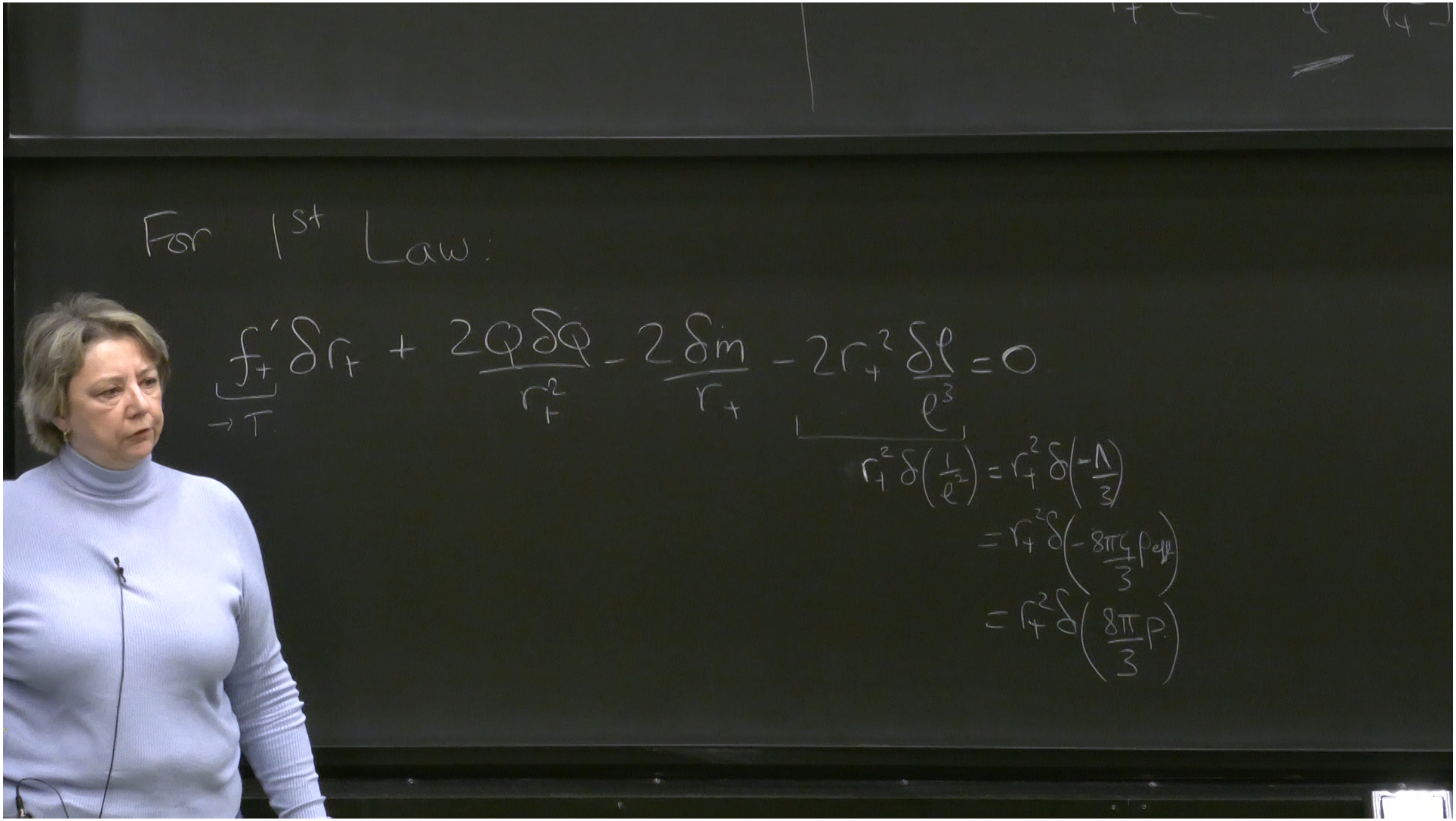
$$\frac{\partial T}{\partial r_+} = \frac{1}{4\pi r_+^2}\left(-1 + \frac{3r_+^2}{\ell^2} + \frac{3Q^2}{r_+^2}\right)$$

$= 0$ at

$$r_+^2 = \frac{\ell^2}{6}\left[1 \pm \sqrt{1 - \boxed{\frac{36Q^2}{\ell^2}}}\right]$$

$6Q < \ell$ turning pts.

$6Q > \ell$ no " "


For 1st Law:
$\underbrace{f_+'}_{\to T}\delta r_+ + \frac{2Q\delta Q}{r_+^2} - \frac{2\delta m}{r_+} - \underbrace{2r_+^2\frac{\delta \ell}{\ell^3}}=0$
$r_+^2\delta\left(\frac{1}{\ell^2}\right) = r_+^2\delta\left(-\frac{\Lambda}{3}\right)$
$= r_+^2\delta\left(-\frac{8\pi G}{3}\rho_{eff}\right)$
$= r_+^2\delta\left(\frac{8\pi}{3}P\right)$

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$$ds^2 = f\,dt^2 - \frac{dr^2}{f} - r^2 d\Omega_{\mathrm{II}}^2$$

Fluid

$$T_{\Lambda}{}^{a}{}_{b} = \frac{\Lambda}{8\pi G}\,\delta^a{}_b. \qquad \rho = -p.$$

For 1^{st} Law:

$$\underbrace{f_+'}_{\to T}\delta r_+ + \frac{2Q\delta Q}{r_+^2} - \frac{2\delta m}{r_+} - \underbrace{2r_+^2\frac{\delta \ell}{\ell^3}}_{} = 0$$

$$r_+^2\,\delta\left(\frac{1}{\ell^2}\right) = r_+^2\,\delta\left(-\frac{\Lambda}{3}\right)$$

$$= r_+^2\,\delta\left(-\frac{8\pi G}{3}\rho_{eff}\right)$$

$$= r_+^2\,\delta\left(\frac{8\pi}{3}P\right)$$

$$\delta m = \underline{\pi T\,\delta(r_+^2)} + \frac{Q}{r_+}\delta Q + \frac{r_+^3}{2}\delta\left(\frac{8\pi}{3}P\right)$$

$$T\delta S + \Phi\,\delta Q$$

$$\Phi = \frac{Q}{r_+}$$

For 1^{st} Law:

$$\underbrace{f'_+}_{\to T}\,\delta r_+ + \frac{2Q\,\delta Q}{r_+^2} - \frac{2\,\delta m}{r_+} - \underbrace{2r_+^2\frac{\delta \ell}{\ell^3}}_{} = 0$$

$$r_+^2\,\delta\left(\frac{1}{\ell^2}\right) = r_+^2\,\delta\left(-\frac{\Lambda}{3}\right)$$
$$= r_+^2\,\delta\left(-\frac{8\pi G}{3}\rho_{eff}\right)$$
$$= r_+^2\,\delta\left(\frac{8\pi}{3}P\right)$$

$$\delta m = \underbrace{\pi T\,\delta(r_+^2)}_{T\delta S} + \underbrace{\frac{Q}{r_+}\delta Q}_{\Phi\,\delta Q} + \underbrace{\frac{r_+^3}{2}\delta\left(\frac{8\pi}{3}P\right)}_{\frac{4}{3}\pi r_+^3\delta P = V\delta P}$$

$\Phi =$

δm

heat. HAWKING-PAGE.

$6Q < \ell$ turning pts.

$6Q > \ell$ no " "

$\Phi = \frac{Q}{r_+}$ - electrostatic pot^l at horizon

$\delta m = T\delta S + \Phi \delta Q + V\delta P$

cf $dU = TdS - pdV$

$\rightarrow m \quad U + pV$ - enthalpy.

heat. HAWKING-PAGE.

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Kastor + Traschen

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Kubiznak, Mann, Teo

heat. HAWKING-PAGE.

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$\frac{Q}{r_+}$ - electrostatic pot^l at horizon

$n = T\delta S + \Phi \delta Q + V \delta P$

$dU = TdS - pdV$

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BH Thermo often shows quantities in terms of geometry. However, should be able to write purely in terms of charges. "CHEMISTRY"

heat. HAWKING-PAGE.

$6Q < \ell$ turning pts.

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$\frac{Q}{r_+}$ - electrostatic pot$^{\ell}$ at horizon

$m = T\delta S + \Phi \delta Q + V \delta P$

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Kastor + Traschen

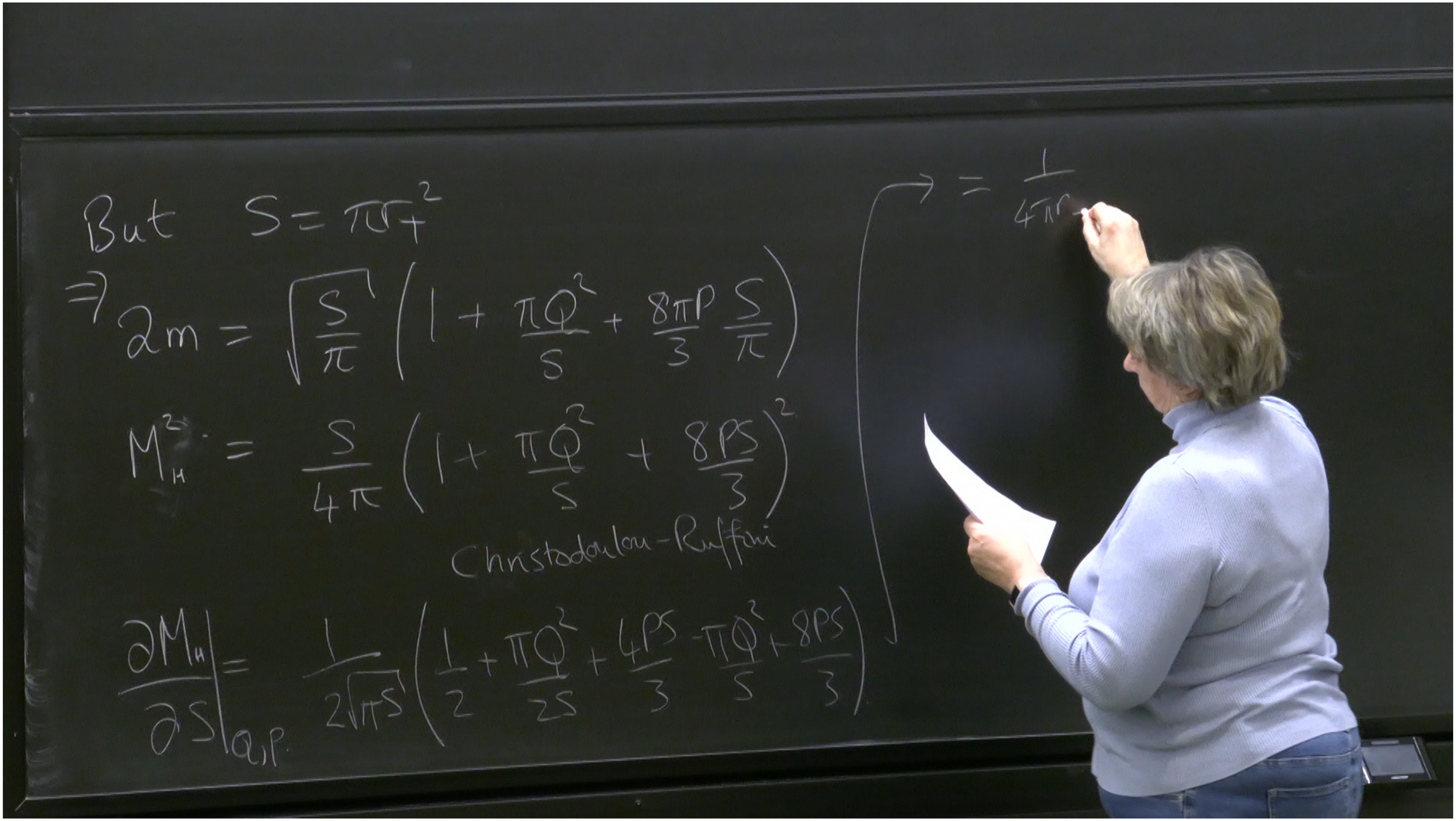
$U + pV$ - enthalpy.

Kubiznak, Mann, Teo

BH Thermo often shows quantities in terms of geometry. However, should be able to write purely in terms of charges. "CHEMISTRY"

From $f(r_+) = 0$

$$2m = r_+ \left(1 + \frac{Q^2}{r_+^2} + \frac{r_+^2}{\ell^2} \right)$$

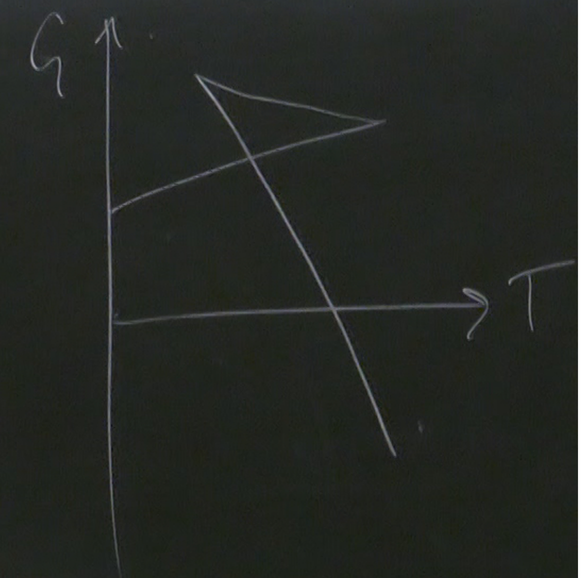

But $S = \pi r_+^2$
$\Rightarrow 2m = \sqrt{\frac{S}{\pi}}\left(1 + \frac{\pi Q^2}{S} + \frac{8\pi P}{3}\frac{S}{\pi}\right)$
$M_H^2 = \frac{S}{4\pi}\left(1 + \frac{\pi Q^2}{S} + \frac{8PS}{3}\right)^2$
Christodoulou-Ruffini
$\left.\frac{\partial M_H}{\partial S}\right|_{Q,P} = \frac{1}{2\sqrt{\pi S}}\left(\frac{1}{2} + \frac{\pi Q^2}{2S} + \frac{4PS}{3} - \frac{\pi Q^2}{S} + \frac{8PS}{3}\right)$
$= \frac{1}{4\pi r}$

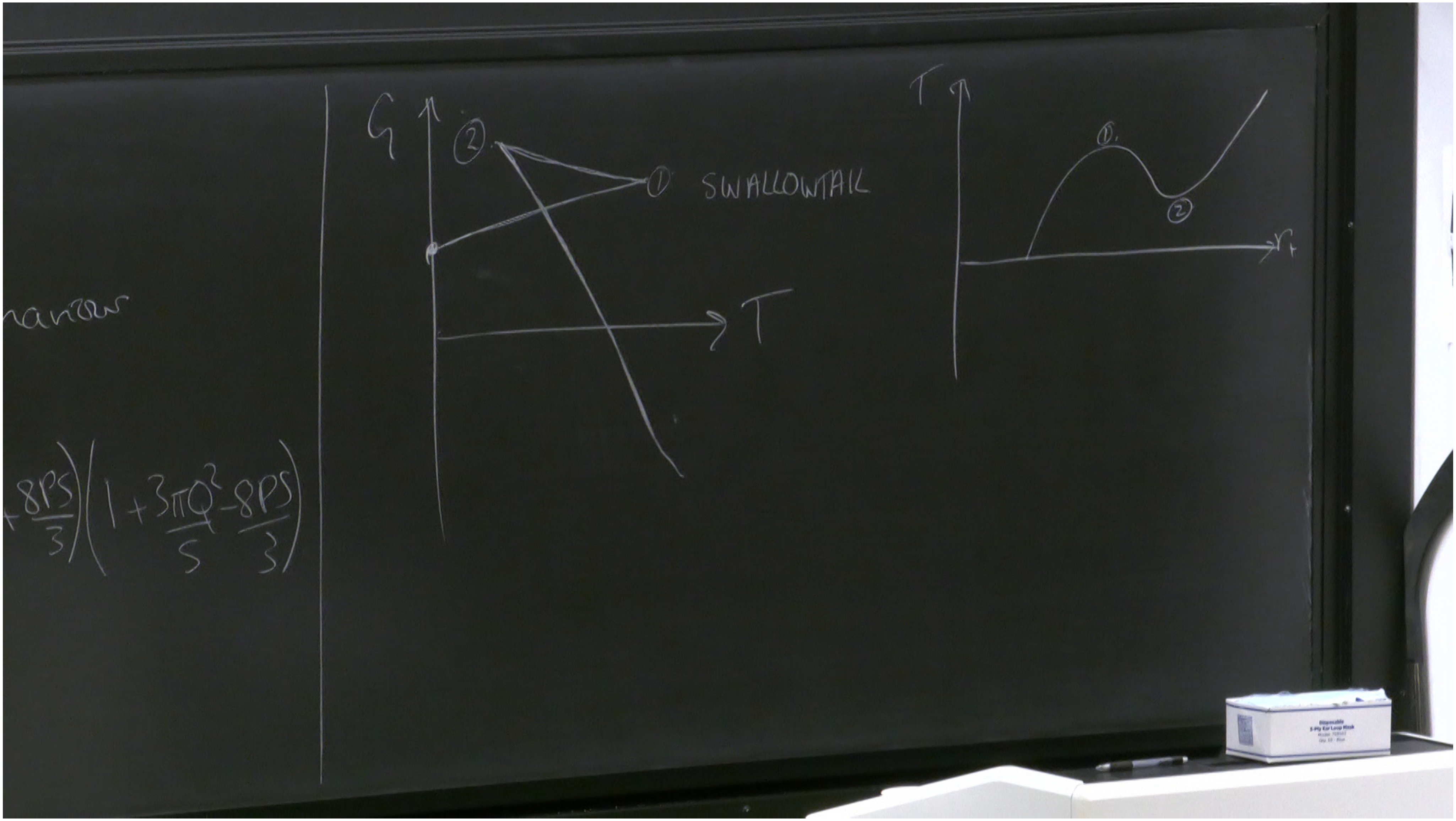
$$= \frac{1}{4\pi r_+}\left(1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{\ell^2}\right) = T.$$

Can pick out phase behaviour more directly.

$$G = M_H - TS = \frac{S}{8\pi M_H}\left(1 + \frac{\pi Q^2}{S} + \frac{8PS}{3}\right)\left(1 + \frac{3\pi Q^2}{S} - \frac{8PS}{3}\right)$$

$$+\frac{8PS}{3}\Big)$$




G
T
SWALLOWTAIL
T
r_+
$\frac{8PS}{3}\Big)\Big(1+\frac{3\pi Q^2}{5}-\frac{8PS}{3}\Big)$

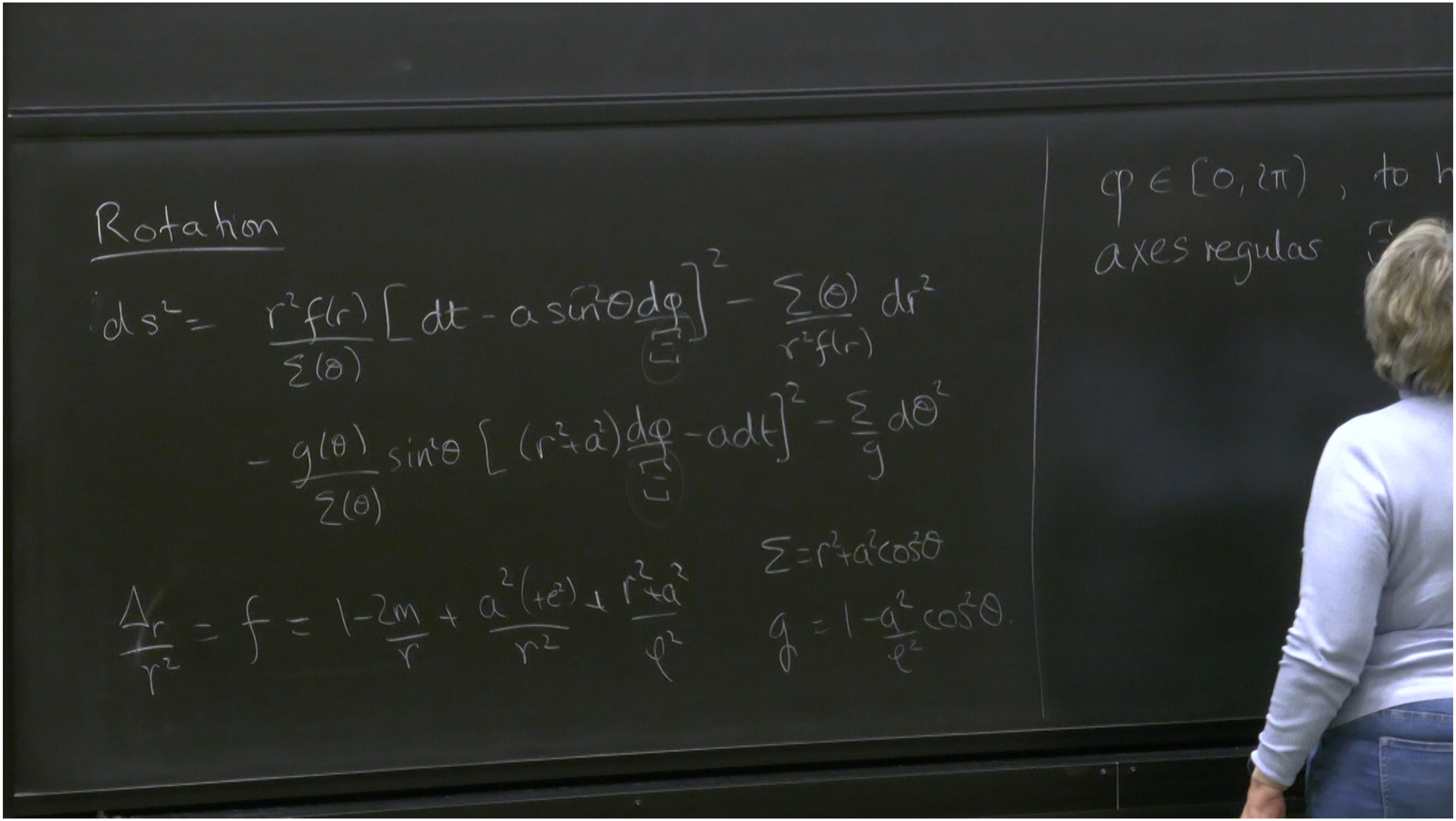
Rotation

$$ds^2 = \frac{r^2 f(r)}{\Sigma(\theta)}\left[dt - a\sin^2\theta \frac{d\varphi}{\Xi}\right]^2 - \frac{\Sigma(\theta)}{r^2 f(r)}\,dr^2$$

$$- \frac{g(\theta)}{\Sigma(\theta)}\sin^2\theta\left[(r^2+a^2)\frac{d\varphi}{\Xi} - a\,dt\right]^2 - \frac{\Sigma}{g}\,d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\frac{\Delta_r}{r^2} = f = 1 - \frac{2m}{r} + \frac{a^2(+e^2)}{r^2} + \frac{r^2+a^2}{\ell^2} \qquad g = 1 - \frac{a^2}{\ell^2}\cos^2\theta.$$



Rotation
$ds^2 = \frac{r^2 f(r)}{\Sigma(\theta)}\left[dt - a\sin^2\theta \frac{d\varphi}{\Xi}\right]^2 - \frac{\Sigma(\theta)}{r^2 f(r)}\,dr^2$
$- \frac{g(\theta)}{\Sigma(\theta)}\sin^2\theta\left[(r^2+a^2)\frac{d\varphi}{\Xi} - a\,dt\right]^2 - \frac{\Sigma}{g}\,d\theta^2$
$\Sigma = r^2 + a^2\cos^2\theta$
$\frac{\Delta_r}{r^2} = f = 1 - \frac{2m}{r} + \frac{a^2(+e^2)}{r^2} + \frac{r^2+a^2}{\ell^2}$
$g = 1 - \frac{a^2}{\ell^2}\cos^2\theta$
$\varphi \in [0, 2\pi)$, to h
axes regular ∃

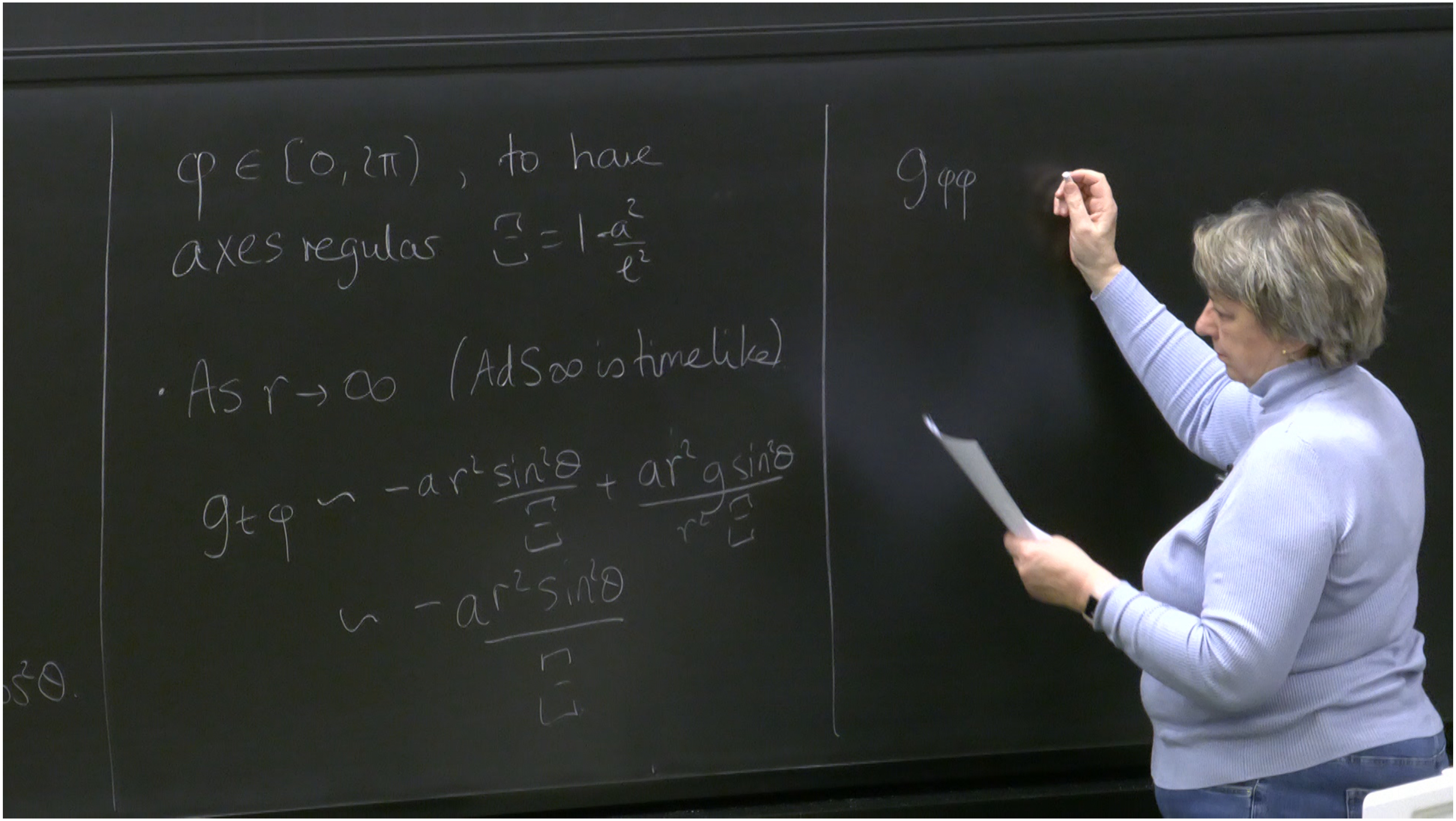
$\varphi \in [0, 2\pi)$, to have

axes regular $\Xi = 1 - \frac{a^2}{\ell^2}$

- As $r \to \infty$ (AdS ∞ is timelike)

$$g_{t\varphi} \sim -a r^2 \frac{\sin^2\theta}{\Xi} + \frac{a r^2 g \sin^2\theta}{r^2 \Xi}$$

$s^2\theta.$


$\varphi \in [0, 2\pi)$, to have
axes regular $\Xi = 1 - \frac{a^2}{\ell^2}$
· As $r \to \infty$ (AdS$_\infty$ is timelike)
$g_{t\varphi} \sim -a r^2 \frac{\sin^2\theta}{\Xi} + \frac{a r^2 g \sin^2\theta}{r^2 \Xi}$
$\sim -\frac{a r^2 \sin^2\theta}{\Xi}$
$g_{\varphi\varphi}$

to have

$\Xi = 1 - \frac{a^2}{\ell^2}$

AdS∞ is timelike)

$\frac{\sin^2\theta}{\Xi} + \frac{a r^2 g \sin^2\theta}{r^2 \Xi}$

$\frac{r^2 \sin^2\theta}{\Xi}$

$$g_{\varphi\varphi} \sim - \frac{r^2 \sin^2\theta}{\Xi^2}\left(1 - a^2/\ell^2\right) = - \frac{r^2 \sin^2\theta}{\Xi}$$

i.e. $\Omega_\infty = \lim_{r\to\infty} \frac{g_{t\varphi}}{g_{\varphi\varphi}} \sim \frac{a}{\ell^2}$

Boundary rotates in AdS.

to have

$$\Xi = 1 - \frac{a^2}{\ell^2}$$

AdS∞ is timelike)

$$\frac{\sin^2\theta}{\Xi} + \frac{ar^2 g \sin^2\theta}{r^2 \Xi}$$

$$\frac{ar^2 \sin^2\theta}{\Xi}$$

$$g_{\varphi\varphi} \sim -\frac{r^2\sin^2\theta}{\Xi^2}\left(1 - a^2/\ell^2\right) = -\frac{r^2\sin^2\theta}{\Xi}.$$

i.e. $\Omega_\infty = \lim_{r\to\infty} \frac{g_{t\varphi}}{g_{\varphi\varphi}} \sim \frac{a}{\ell^2}$ Boundary rotates in AdS.

Other difference to asymptotically flat case is $a < \ell$ required

Thermo params.

$$M_H = \frac{m}{\Xi^2} \qquad J = a M_H$$

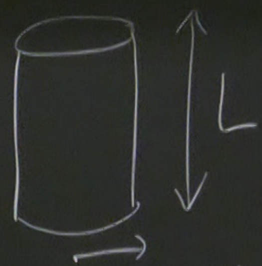
$$\Omega = \Omega_H - \Omega_\infty = \frac{a}{\ell^2} + \frac{a\,\Xi}{r_+^2 + a^2}$$

$$V = \frac{4\pi}{3\Xi} r_+ (r_+^2 + a^2) + \frac{4\pi}{3} M a^2.$$

$$+\frac{a}{r_+^2+a^2}$$

Higher dim soln: Black string in 5D

$SCH \times \mathbb{R}$



$r_4 = 2GM$

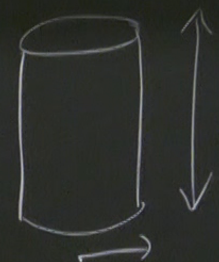
Mass of cylinder $M = \frac{r_4 L}{2G_5}$

$G_4 = G_5/L \qquad r_+ = \frac{2G_5 M}{L}$

$+\frac{a\,\Xi}{r_+^2+a^2}$

Higher dim soln: Black string in 5D

$SCH \times \mathbb{R}$



L

$r_4 = 2GM$

Mass of cylinder $M = \frac{r_4 L}{2G_5}$

Entropy $S_{5T} = \frac{4\pi r_4^2 L}{4G_5}$

$= 4\pi G_4 M^2$

$G_4 = G_5/L \qquad r_4 = \frac{2G_5 M}{L}$

Black string in 5D

Mass of cylinder $M = \frac{r_4 L}{2G_5}$

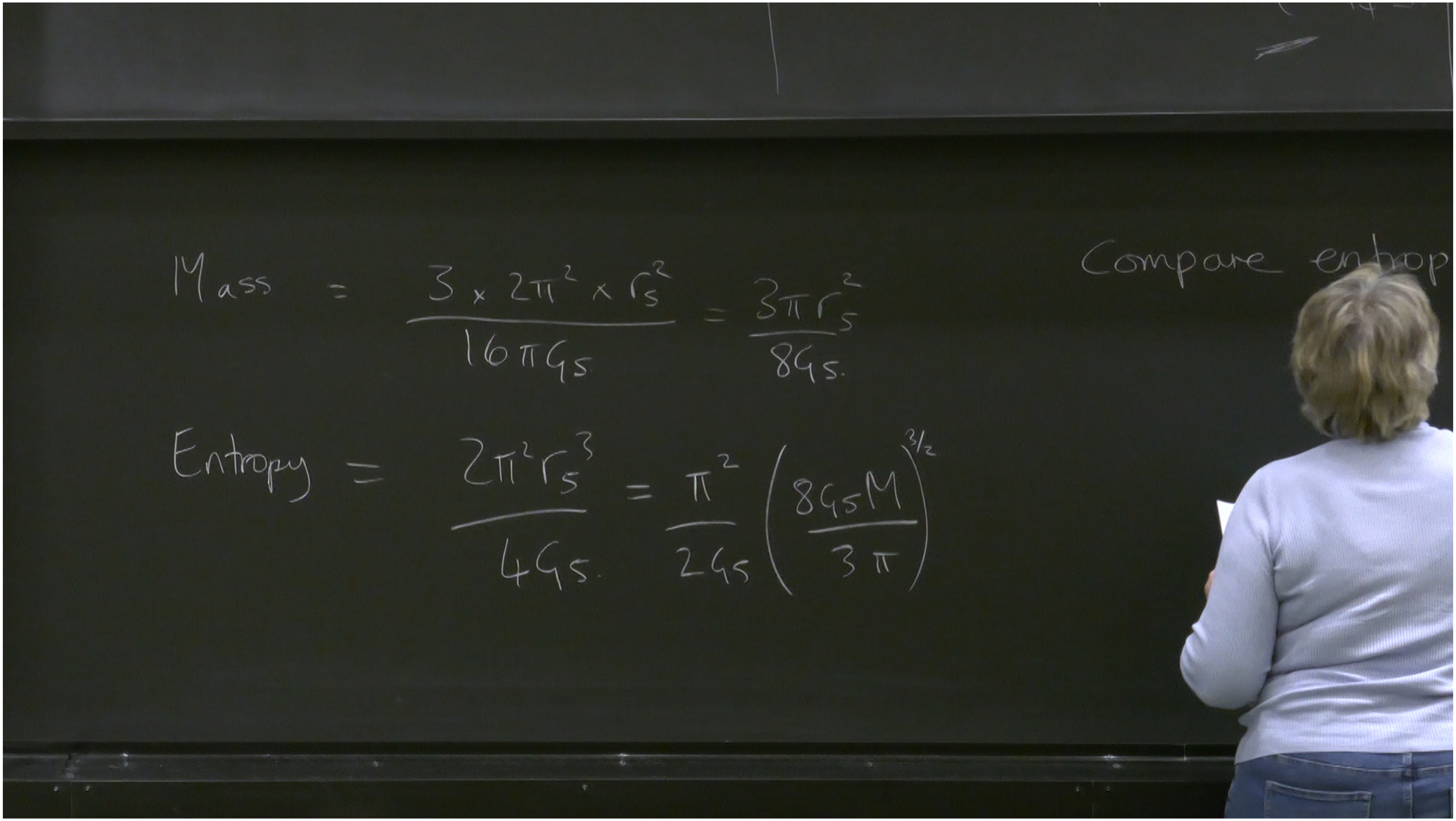
Entropy $S_{ST} = \frac{4\pi r_4^2 L}{4G_5}$

$= 4\pi G_4 M^2$

5D Black hole

$$ds^2 = \left(1 - \frac{r_5^2}{\rho^2}\right) dt^2 - \frac{d\rho^2}{1 - r_5^2/\rho^2} - \rho^2 d\Omega_{\text{III}}^2$$

$$r_5^2 = \frac{16\pi G M}{3 A_3}$$


Mass $= \frac{3 \times 2\pi^2 \times r_s^2}{16\pi G_5} = \frac{3\pi r_s^2}{8G_5}$
Entropy $= \frac{2\pi^2 r_s^3}{4G_5} = \frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3\pi}\right)^{3/2}$
Compare entrop

heat. HAWKING-PAGE.

$6Q < \ell$

$6Q > \ell$

Compare entropies, fixing mass

$$\frac{S_{BH}}{S_{BS}} = \frac{\pi^2}{2G_5}\left(\frac{8G_5M}{3\pi}\right)^{3/2} \Big/ 4\pi G_4 M^2$$

heat. HAWKING-PAGE.

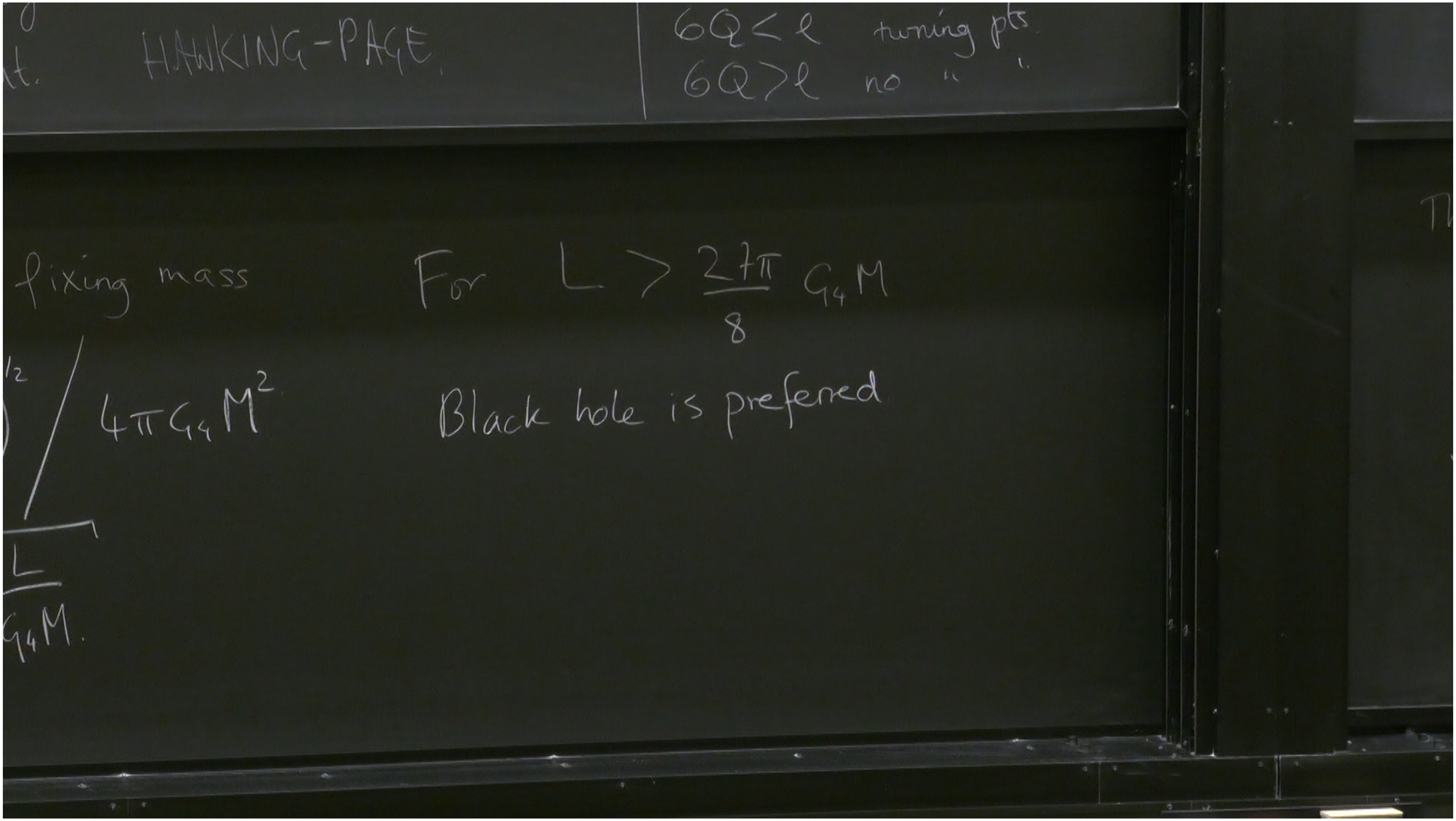
$6Q < \ell$

$6Q > \ell$

Compare entropies, fixing mass

$$\frac{S_{BH}}{S_{BS}} = \frac{\pi^2}{2G_5}\left(\frac{8G_5M}{3\pi}\right)^{3/2} \Big/ 4\pi G_4 M^2$$

$$= \sqrt{\frac{8}{27\pi}}\sqrt{\frac{L}{G_4 M}}.$$


HAWKING-PAGE.
6Q < ℓ turning pts.
6Q > ℓ no " "
fixing mass
For L > 27π/8 G₄M
4π G₄ M²
Black hole is prefered
L
G₄M.