Title: Gravitational Physics Lecture

Speakers: Ruth Gregory

Collection: Gravitational Physics

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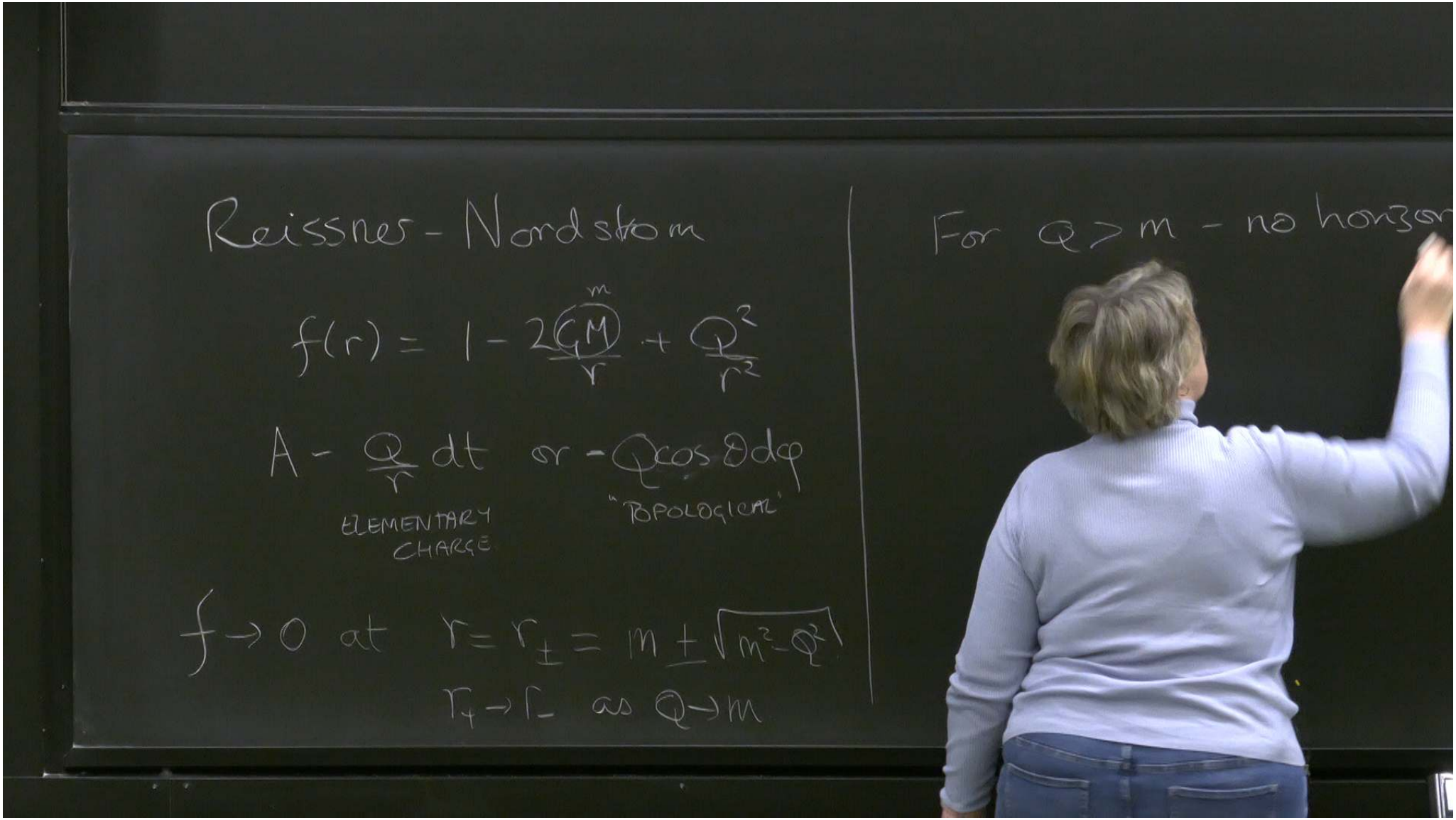
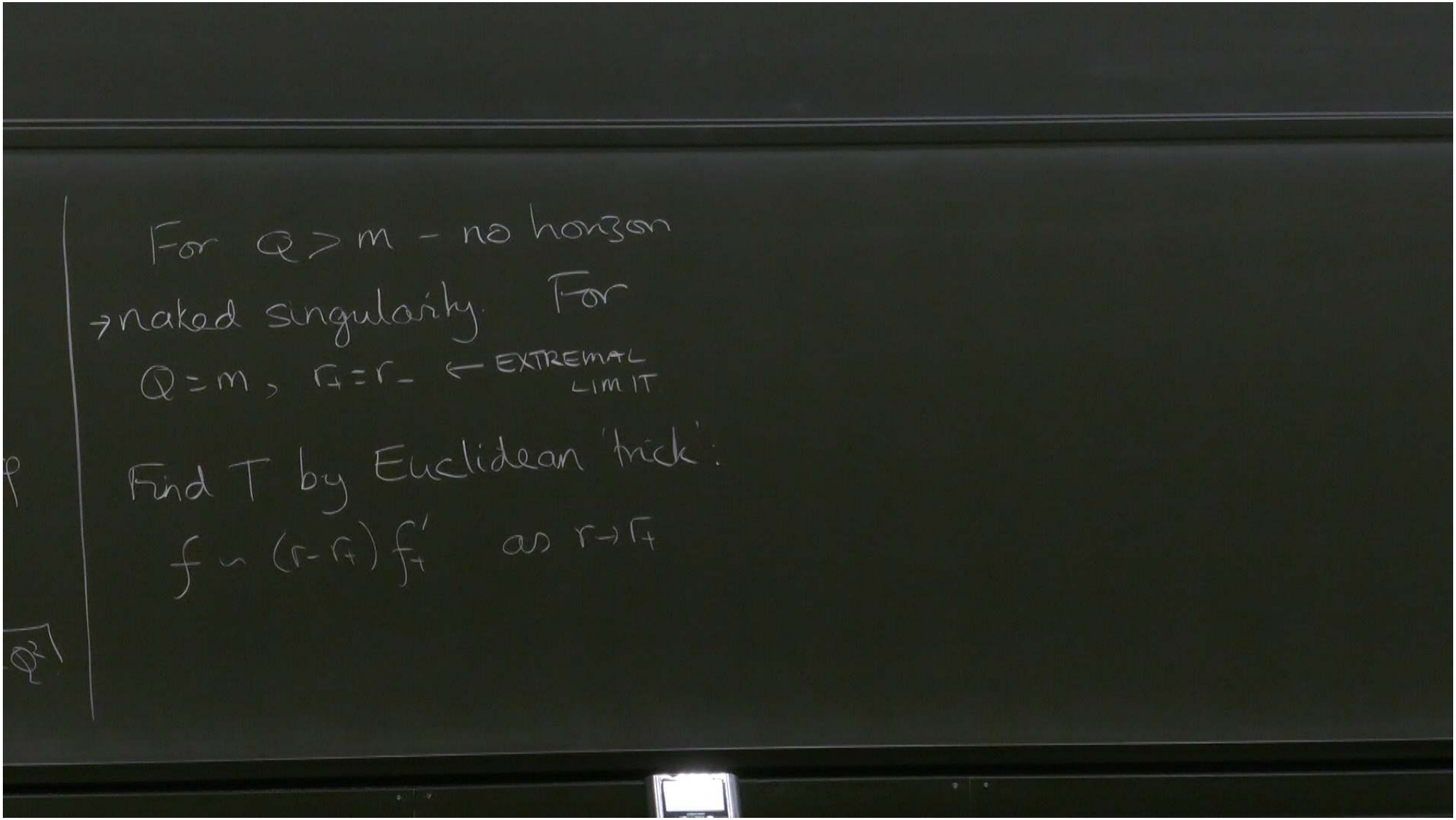
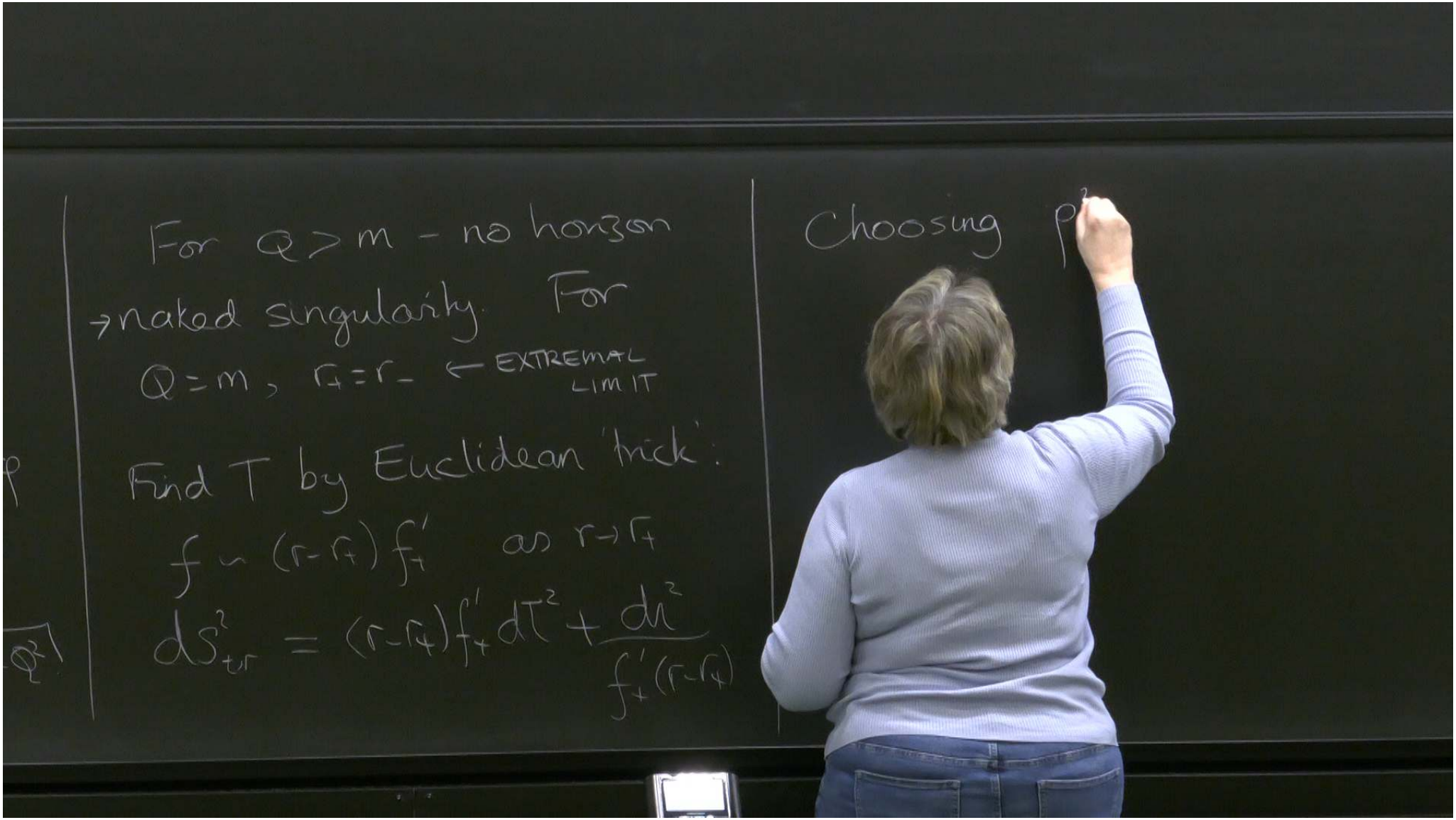
L10 (More) Thermodynamics

$$ds^2 = f dt^2 - \frac{dr^2}{f} - r^2 d\Omega_{II}^2$$


Reissner - Nordstrom
$f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$
$M \to m$
$A - \frac{Q}{r}\,dt$ or $-Q\cos\theta\,d\varphi$
ELEMENTARY CHARGE
"TOPOLOGICAL"
$f \to 0$ at $r = r_\pm = m \pm \sqrt{m^2 - Q^2}$
$r_+ \to r_-$ as $Q \to m$
For $Q > m$ - no horizon


For $Q > m$ - no horizon
→ naked singularity. For
$Q = m$, $r_+ = r_-$ ← EXTREMAL LIMIT
Find T by Euclidean 'trick':
$f \sim (r - r_+) f'_+$ as $r \to r_+$


For Q > m - no horizon
→ naked singularity. For
Q = m, r+ = r- ← EXTREMAL LIMIT
Find T by Euclidean 'trick':
f ~ (r - r+) f+' as r → r+
ds²tr = (r - r+) f+' dT² + dr² / f+'(r - r+)
Choosing p

Choosing $\rho^2 = \frac{4}{f_+'}(r - r_+)$

gives

$$ds_{1r}^2 = d\rho^2 + \rho^2 d\left(f_+' \frac{\tau}{2}\right)^2$$

Suggests $\Delta\tau = \beta = \frac{4\pi}{f_+'}$

Check: SCH: $f_+' = \frac{2GM}{r_+^2} = \frac{1}{2GM}$

$\Rightarrow \beta_{SCH} = 8\pi GM$ ✓

Choosing $\rho^2 = \frac{4}{f_+'}(r - r_+)$

gives

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$\Rightarrow \beta_{SCH} = 8\pi GM$ ✓

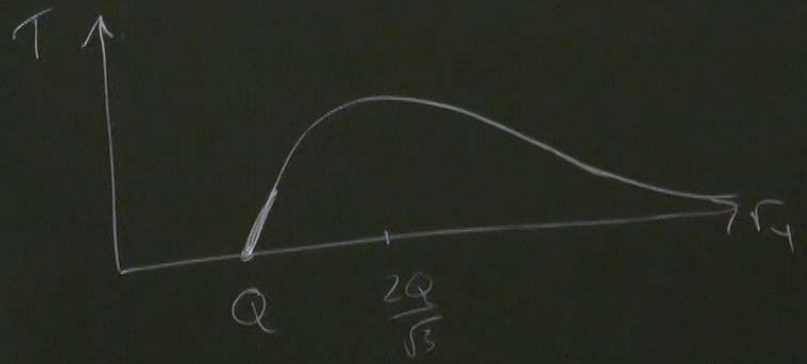
For RN $f' = \frac{2m}{r^2} - \frac{2Q^2}{r^3}$

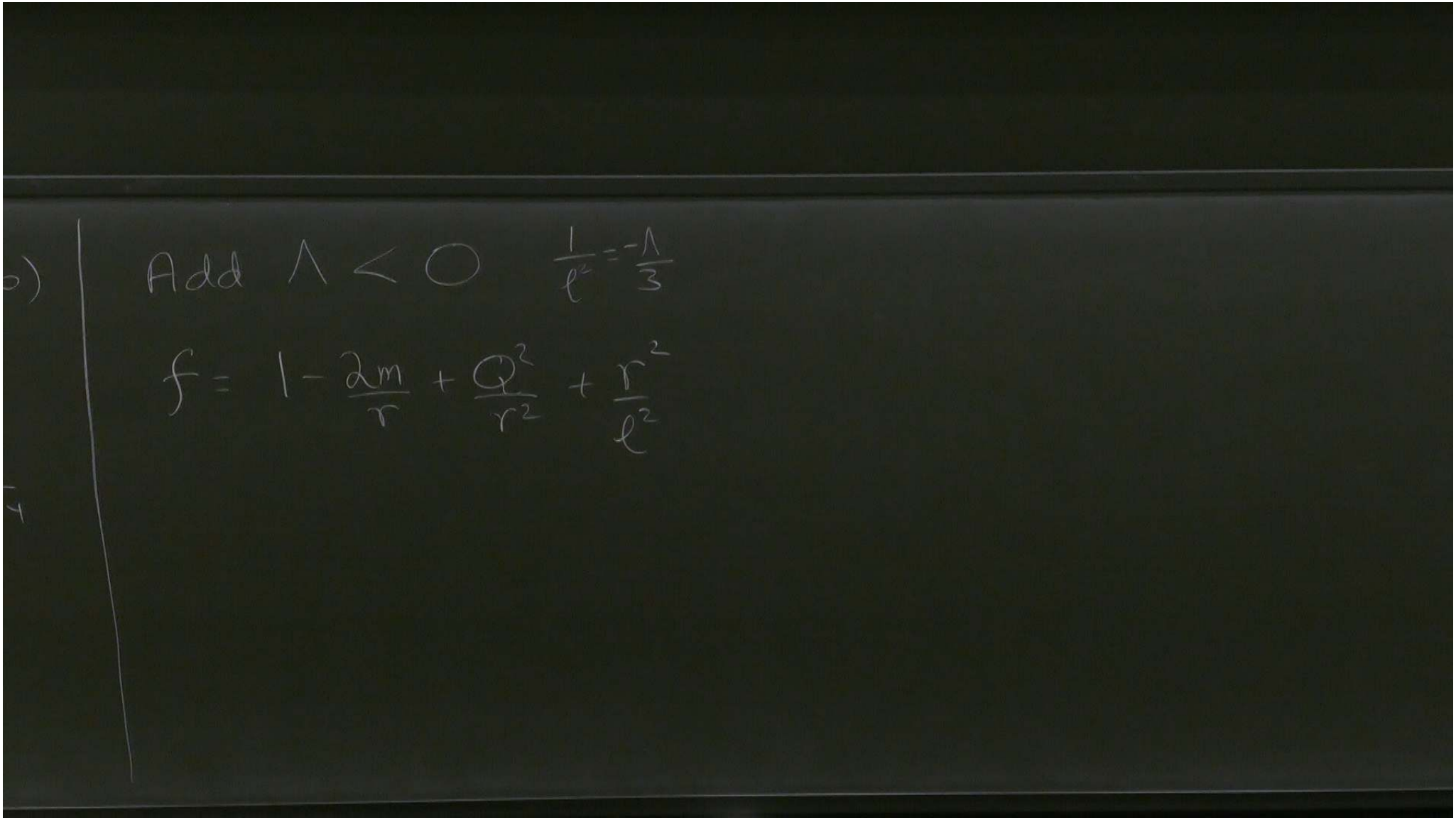
$r \to r_+$ $f_+' = \frac{1}{r_+}\left[1 - \frac{Q^2}{r_+^2}\right]$

$$T = \frac{1}{\beta} = \frac{f_+'}{4\pi} = \frac{1}{4\pi r_+}\left[1 - \frac{Q^2}{r_+^2}\right]$$

- Parametrising degrees of freedom by r_+ & Q.

At fixed Q, $r_+ \in [Q, \infty)$




Add $\Lambda < 0$ $\frac{1}{\ell^2} = \frac{-\Lambda}{3}$
$f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}$

Add $\Lambda < 0$ $\quad \frac{1}{\ell^2} = \frac{-\Lambda}{3}$

$$f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}$$

$$4\pi T = f_+' = \frac{2m}{r_+^2} - 2\frac{Q^2}{r_+^3} + 2\frac{r_+}{\ell^2}$$

$$= \frac{1}{r_+}\left[1 + 3\frac{r_+^2}{\ell^2} - \frac{Q^2}{r_+^2}\right]$$

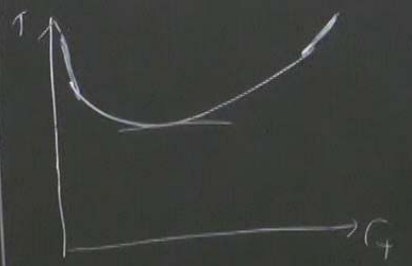
Note, if $Q=0$ $\quad T = \frac{1}{4\pi r_+}\left[1 + \frac{3r_+^2}{\ell^2}\right]$

T has a minimum!

ℓ is the scale that sets min

Large black holes have the specific heat. HAWKING-PAGE.

Note, if $Q=0$ $T = \frac{1}{4\pi r_+ L}\left[1 + \frac{3r_+^2}{\ell^2}\right]$



T has a minimum!

ℓ is the scale that sets min

Large black holes have the specific heat. HAWKING-PAGE.

With charge:

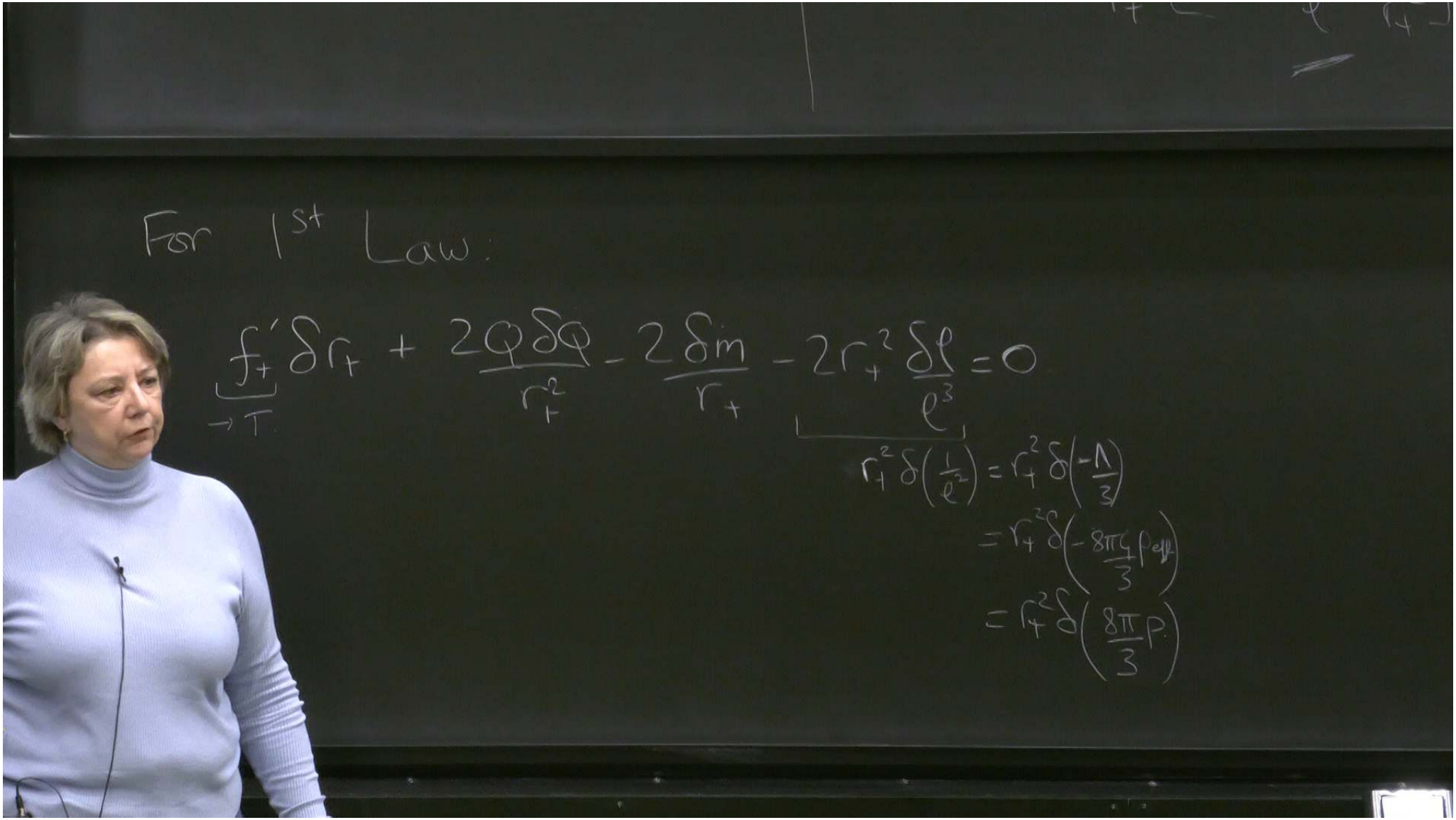
$$\frac{\partial T}{\partial r_+} = \frac{1}{4\pi r_+^2}\left(-1 + \frac{3r_+^2}{\ell^2} + \frac{3Q^2}{r_+^2}\right)$$

$= 0$ at

$$r_+^2 = \frac{\ell^2}{6}\left[1 \pm \sqrt{1 - \frac{36Q^2}{\ell^2}}\right]$$

$6Q < \ell$ turning pts.

$6Q > \ell$ no " "


For 1st Law:
$\underbrace{f_+'}_{\to T}\delta r_+ + \frac{2Q\delta Q}{r_+^2} - \frac{2\delta m}{r_+} - 2r_+^3\frac{\delta \ell}{\ell^3} = 0$
$r_+^2\delta\left(\frac{1}{\ell^2}\right) = r_+^2\delta\left(-\frac{\Lambda}{3}\right)$
$= r_+^2\delta\left(-\frac{8\pi G}{3}\rho_{eff}\right)$
$= r_+^2\delta\left(\frac{8\pi}{3}p\right)$

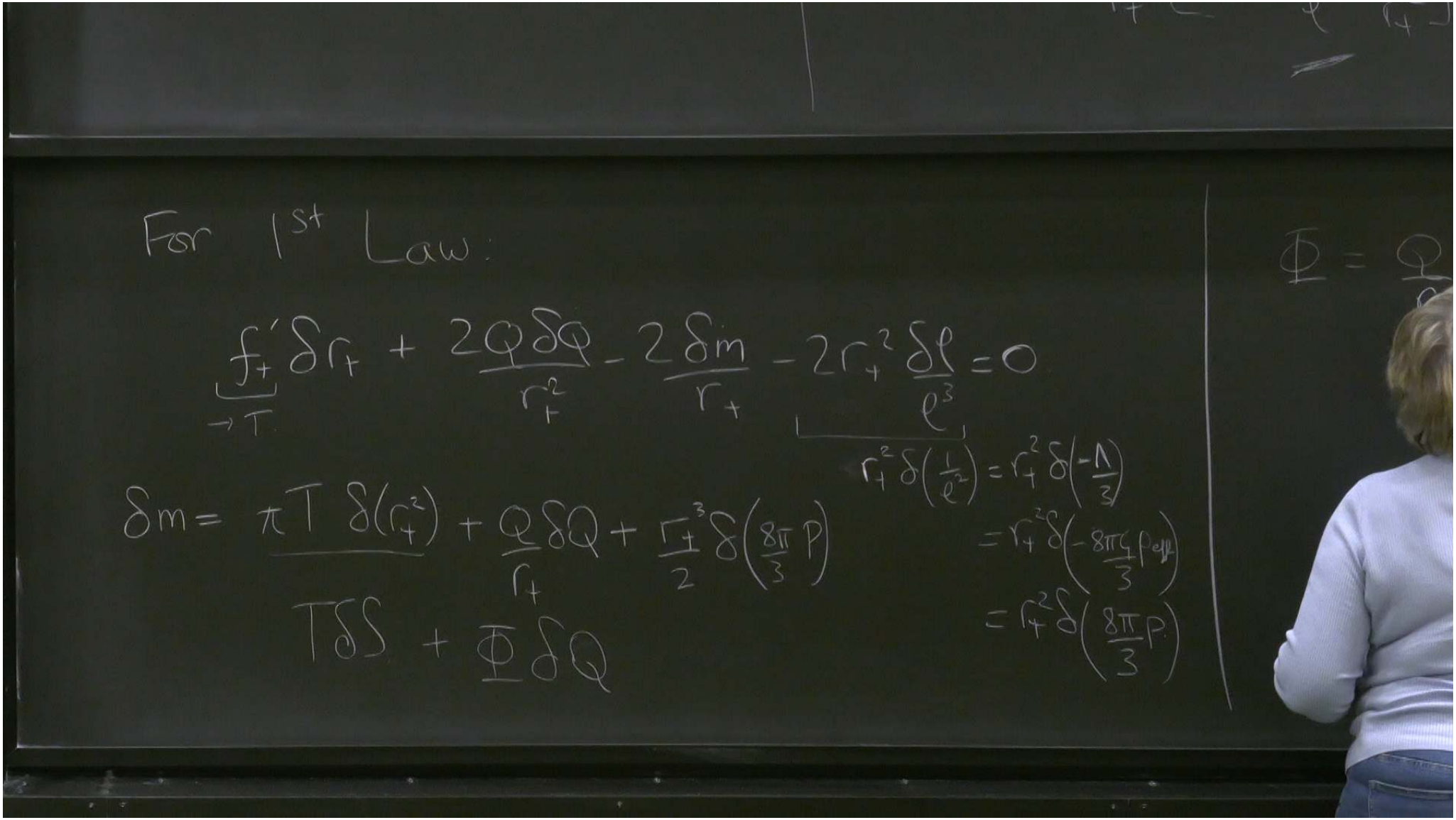
L10 (More) Thermodynamics

$$ds^2 = f dt^2 - \frac{dr^2}{f} - r^2 d\Omega_{II}^2$$

$$T^a_{\ b} = \frac{\Lambda}{8\pi G}\delta^a_{\ b}$$

Fluid

$$\rho = -p$$


For 1st Law:
$\underbrace{f_+'}_{\to T}\delta r_+ + \frac{2Q\delta Q}{r_+^2} - \frac{2\delta m}{r_+} - \underbrace{\frac{2r_+^2\delta\ell}{\ell^3}}_{r_+^2\delta\left(\frac{1}{\ell^2}\right) = r_+^2\delta\left(-\frac{\Lambda}{3}\right)} = 0$
$= r_+^2\delta\left(-\frac{8\pi}{3}\rho_{eff}\right)$
$= r_+^2\delta\left(\frac{8\pi}{3}p\right)$
$\delta m = \underbrace{\frac{\pi T\,\delta(r_+^2)}{T\delta S}} + \frac{Q}{r_+}\delta Q + \frac{r_+^3}{2}\delta\left(\frac{8\pi}{3}p\right)$
$+ \Phi\delta Q$
$\Phi = \frac{Q}{}$

For 1st Law:

$$\underbrace{f_+'}_{\to T}\delta r_+ + \frac{2Q\delta Q}{r_+^2} - \frac{2\delta m}{r_+} - \underbrace{2r_+^2\frac{\delta\ell}{\ell^3}}_{r_+^2\delta\left(\frac{1}{\ell^2}\right) = r_+^2\delta\left(-\frac{\Lambda}{3}\right)} = 0$$

$$= r_+^2\delta\left(-\frac{8\pi}{3}\rho_{eff}\right)$$

$$= r_+^2\delta\left(\frac{8\pi}{3}P\right)$$

$$\delta m = \underbrace{\pi T\,\delta(r_+^2)}_{T\delta S} + \underbrace{\frac{Q}{r_+}\delta Q}_{\Phi\delta Q} + \underbrace{\frac{r_+^3}{2}\delta\left(\frac{8\pi}{3}P\right)}_{\frac{4}{3}\pi r_+^3\delta P = V\delta P}$$

$\Phi =$

δm

heat. HAWKING-PAGE.

$6Q < \ell$ turning pts
$6Q > \ell$ no " "

$\Phi = \frac{Q}{r_+}$ - electrostatic pot^l at horizon

$\delta m = T\delta S + \Phi \delta Q + V\delta P$

cf $dU = TdS - pdV$

$\rightarrow$ m $U + pV$ - enthalpy.

heat. HAWKING-PAGE.

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Kubiznak, Mann, Teo

heat. HAWKING-PAGE.

$\delta Q < \ell$ turning pts.
$\delta Q > \ell$ no " "

$\frac{Q}{r_+}$ - electrostatic pot^l at horizon

$n = T\delta S + \Phi\delta Q + V\delta P$

$dU = TdS - pdV$

Kastor + Traschen

$U + pV$ - enthalpy.

Kubiznak, Mann, Teo

BH Thermo often shows quantities in terms of geometry. However, should be able to write purely in terms of charges. "CHEMISTRY"

heat. HAWKING-PAGE.

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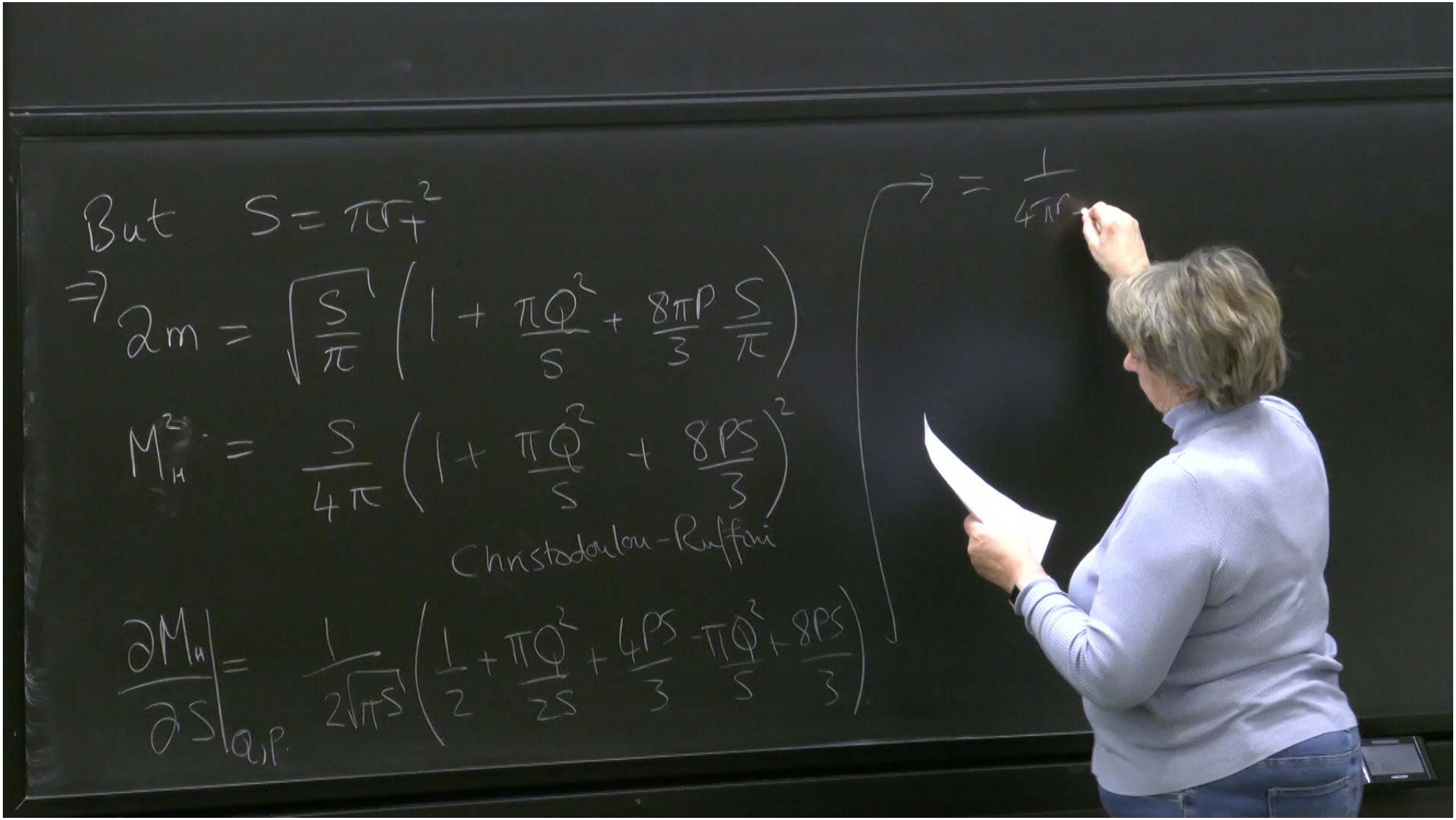
Kubiznak, Mann, Teo

BH Thermo often shows quantities in terms of geometry. However, should be able to write purely in terms of charges. "CHEMISTRY"

From $f(r_+) = 0$

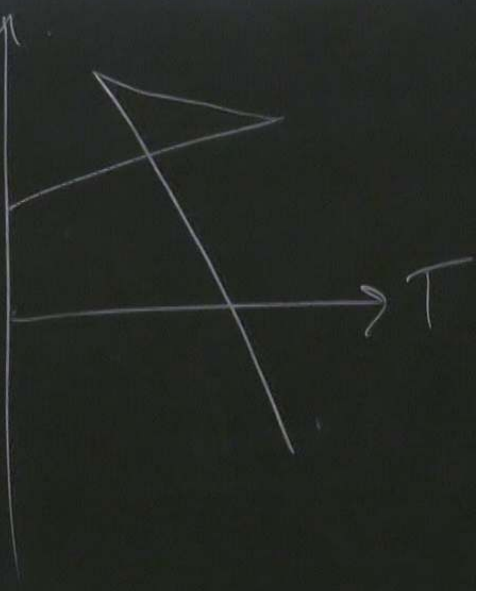
$$2m = r_+\left(1 + \frac{Q^2}{r_+^2} + \frac{r_+^2}{\ell^2}\right)$$

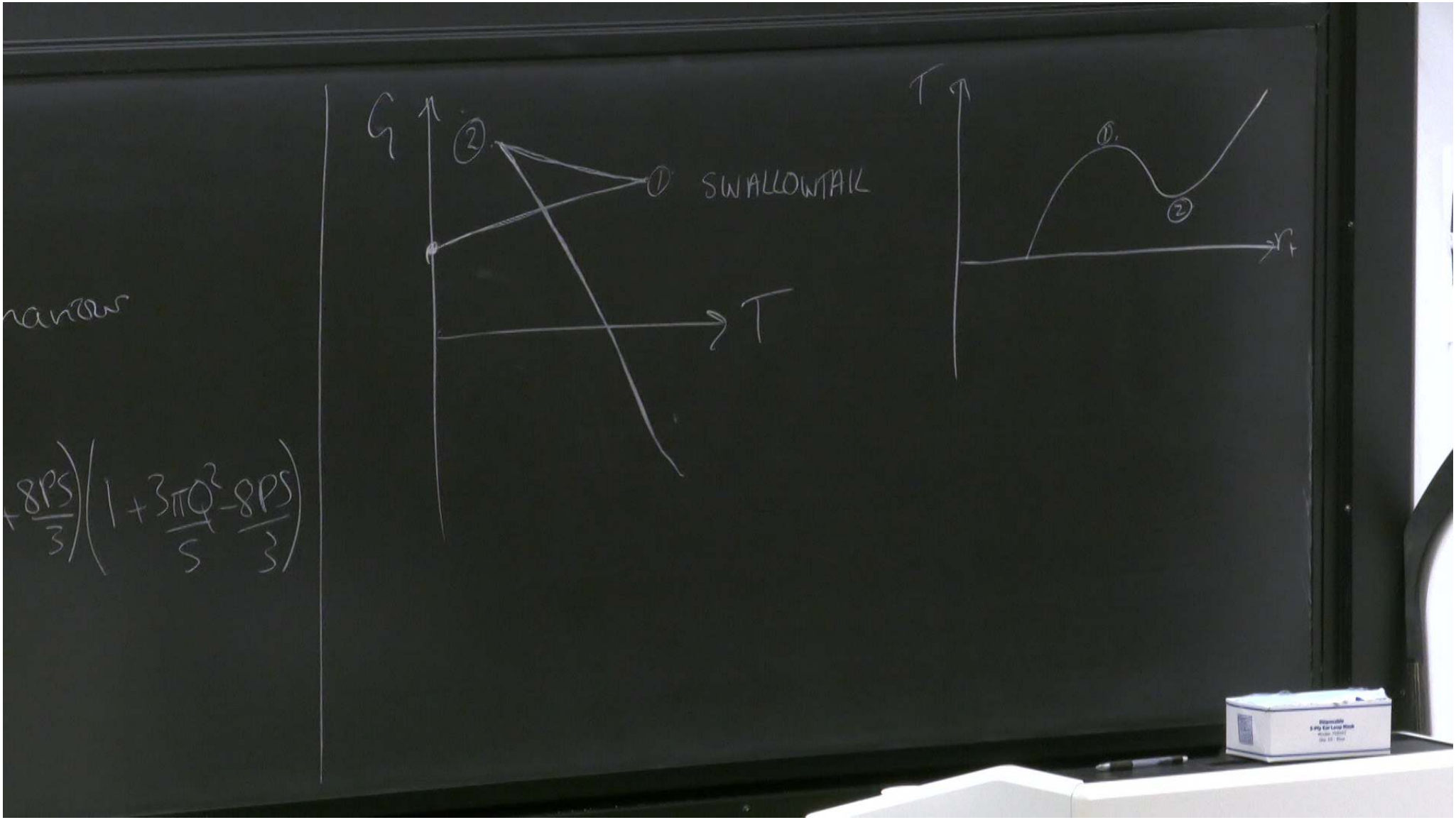

But $S = \pi r_+^2$
$\Rightarrow 2m = \sqrt{\frac{S}{\pi}}\left(1 + \frac{\pi Q^2}{S} + \frac{8\pi P}{3}\frac{S}{\pi}\right)$
$M_H^2 = \frac{S}{4\pi}\left(1 + \frac{\pi Q^2}{S} + \frac{8PS}{3}\right)^2$
Christodoulou-Ruffini
$\frac{\partial M_H}{\partial S}\Big|_{Q,P} = \frac{1}{2\sqrt{\pi S}}\left(\frac{1}{2} + \frac{\pi Q^2}{2S} + \frac{4PS}{3} - \frac{\pi Q^2}{S} + \frac{8PS}{3}\right)$
$= \frac{1}{4\pi r_+}$

$$= \frac{1}{4\pi r_+}\left(1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{\ell^2}\right) = T.$$

Can pick out phase behaviour more directly.

$$G = M_H - TS = \frac{S}{8\pi M_H}\left(1 + \frac{\pi Q^2}{S} + \frac{8PS}{3}\right)\left(1 + \frac{3\pi Q^2}{S} - \frac{8PS}{3}\right)$$




G
T
SWALLOWTAIL
T
T

Rotation

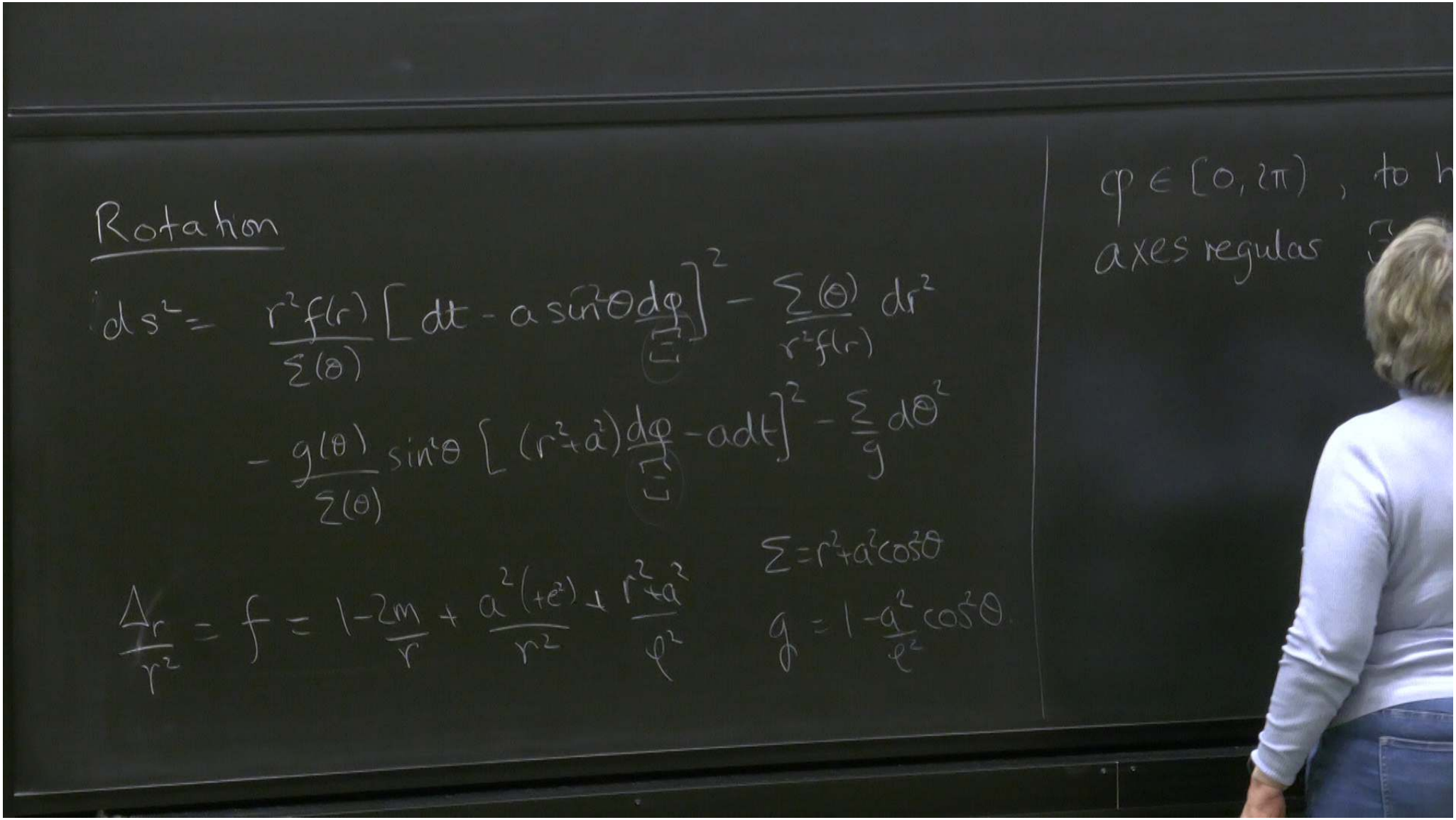
$$ds^2 = \frac{r^2 f(r)}{\Sigma(\theta)}\left[dt - a\sin^2\theta \underbrace{d\varphi}_{\Xi}\right]^2 - \frac{\Sigma(\theta)}{r^2 f(r)}dr^2$$

$$- \frac{g(\theta)}{\Sigma(\theta)}\sin^2\theta\left[(r^2+a^2)\underbrace{d\varphi}_{\Xi} - a\,dt\right]^2 - \frac{\Sigma}{g}d\theta^2$$

$$\frac{\Delta_r}{r^2} = f = 1 - \frac{2m}{r} + \frac{a^2(+e^2)}{r^2} + \frac{r^2+a^2}{\varphi^2}$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$g = 1 - \frac{a^2}{\varphi^2}\cos^2\theta.$$

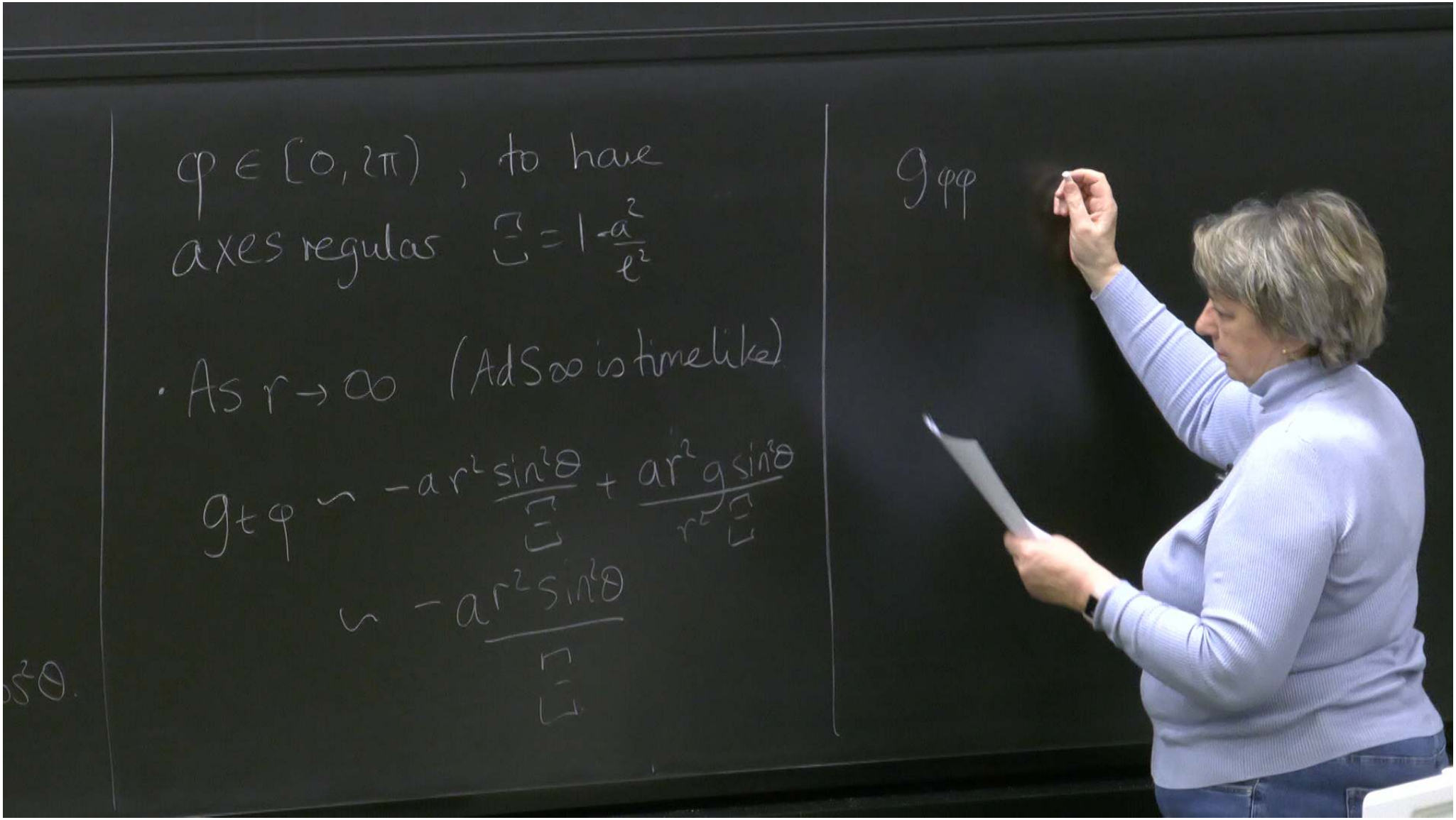

Rotation
$ds^2 = \frac{r^2 f(r)}{\Sigma(\theta)}\left[dt - a\sin^2\theta\, d\varphi\right]^2 - \frac{\Sigma(\theta)}{r^2 f(r)}dr^2$
$- \frac{g(\theta)}{\Sigma(\theta)}\sin^2\theta\left[(r^2+a^2)d\varphi - a\,dt\right]^2 - \frac{\Sigma}{g}d\theta^2$
$\frac{\Delta_r}{r^2} = f = 1 - \frac{2m}{r} + \frac{a^2(+e^2)}{r^2} + \frac{r^2+a^2}{\varphi^2}$
$\Sigma = r^2 + a^2\cos^2\theta$
$g = 1 - \frac{a^2}{\varphi^2}\cos^2\theta$
$\varphi \in [0, 2\pi)$, to h
axes regulas

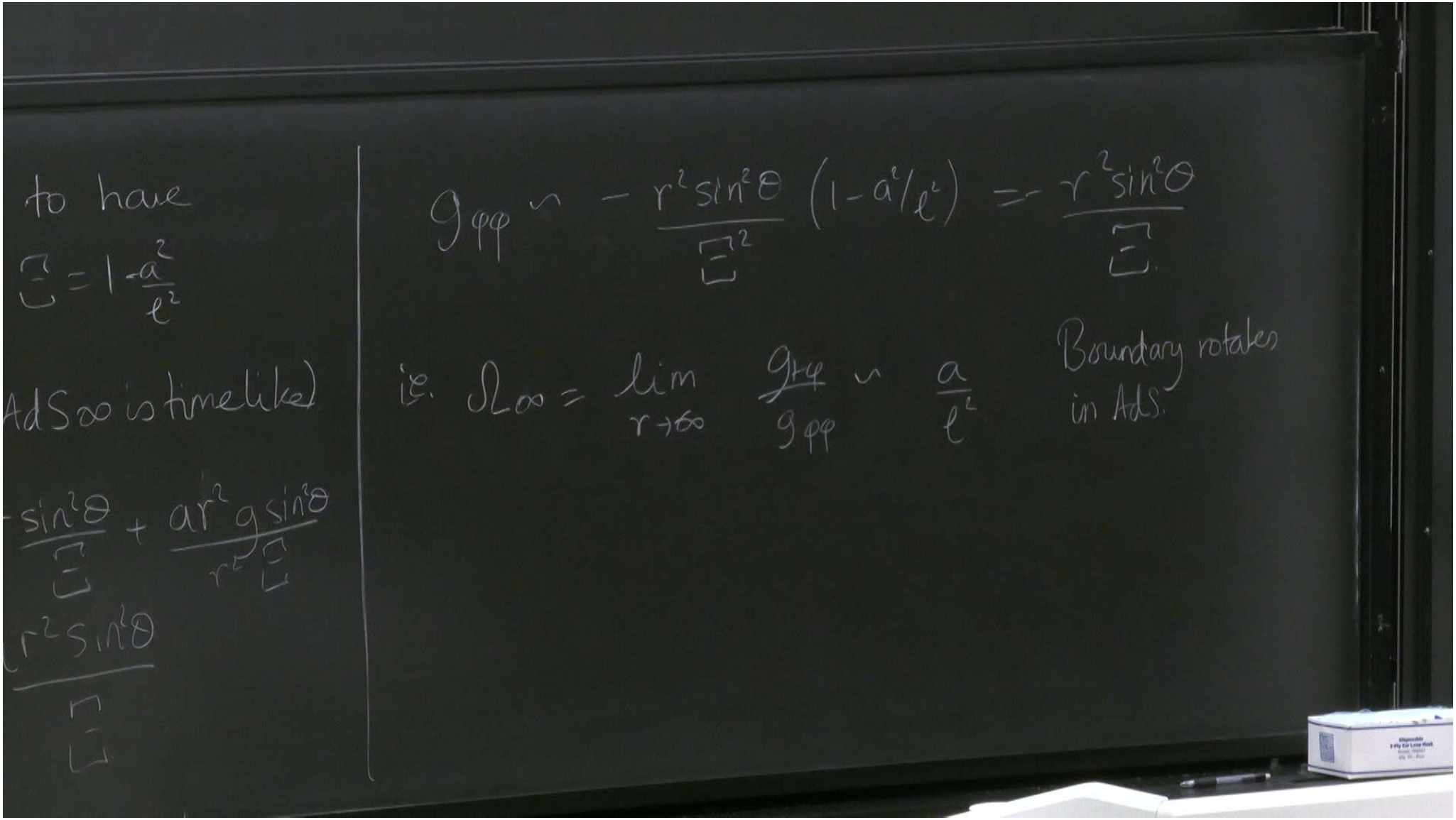
$\varphi \in [0, 2\pi)$, to have axes regular $\Xi = 1 - \frac{a^2}{\ell^2}$

- As $r \to \infty$ (AdS$_\infty$ is timelike)

$$g_{t\varphi} \sim -a r^2 \frac{\sin^2\theta}{\Xi} + \frac{a r^2 g \sin^2\theta}{r^2 \Xi}$$

$\sin^2\theta$.


$\varphi \in [0, 2\pi)$, to have
axes regular $\Xi = 1 - \frac{a^2}{\ell^2}$
· As $r \to \infty$ (AdS$_\infty$ is timelike)
$g_{t\varphi} \sim -\frac{a r^2 \sin^2\theta}{\Xi} + \frac{a r^2 g \sin^2\theta}{r^2 \Xi}$
$\sim -\frac{a r^2 \sin^2\theta}{\Xi}$
$g_{\varphi\varphi}$


to have
$\Xi = 1 - \frac{a^2}{\ell^2}$
AdS∞ is timelike)
$\frac{\sin^2\theta}{\Xi} + \frac{ar^2 g \sin^2\theta}{r^2 \Xi}$
$\frac{r^2 \sin^2\theta}{\Xi}$
$g_{\varphi\varphi} \sim -\frac{r^2 \sin^2\theta}{\Xi^2}(1 - a^2/\ell^2) = -\frac{r^2 \sin^2\theta}{\Xi}$
i.e. $\Omega_\infty = \lim_{r\to\infty} \frac{g_{t\varphi}}{g_{\varphi\varphi}} \sim \frac{a}{\ell^2}$
Boundary rotates in AdS.

to have

$\Xi = 1 - \frac{a^2}{\ell^2}$

AdS∞ is timelike)

$\frac{\sin^2\theta}{\Xi} + \frac{ar^2 g \sin^2\theta}{r^2 \Xi}$

$\frac{r^2 \sin^2\theta}{\Xi}$

$g_{\varphi\varphi} \sim -\frac{r^2 \sin^2\theta}{\Xi^2}(1 - a^2/\ell^2) = -\frac{r^2 \sin^2\theta}{\Xi}$

i.e. $\Omega_\infty = \lim_{r\to\infty} \frac{g_{t\varphi}}{g_{\varphi\varphi}} \sim \frac{a}{\ell^2}$ Boundary rotates in AdS.

Other difference to asympotically flat case is $a < \ell$ required

Thermo params:

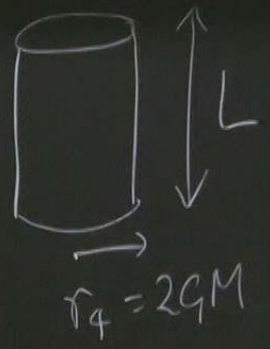
$$M_H = \frac{m}{\Xi^2} \qquad J = a M_H$$

$$\Omega = \Omega_H - \Omega_\infty = \frac{a}{\ell^2} + \frac{a\Xi}{r_+^2 + a^2}$$

$$V = \frac{4\pi}{3\Xi}\, r_+ (r_+^2 + a^2) + \frac{4\pi}{3} M a^2 .$$

Higher dim soln: Black string in 5D

SCH × $\mathbb{R}$



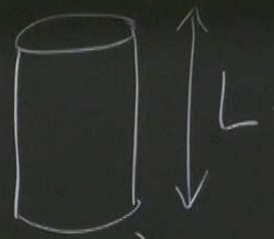
Mass of cylinder $M = \frac{r_4 L}{2G_5}$

$G_4 = G_5/L$ $\quad r_4 = \frac{2G_5 M}{L}$

$\frac{}{2} + \frac{a\Xi}{r_+^2 + a^2}$

Higher dim soln: Black string in 5D

SCH × $\mathbb{R}$



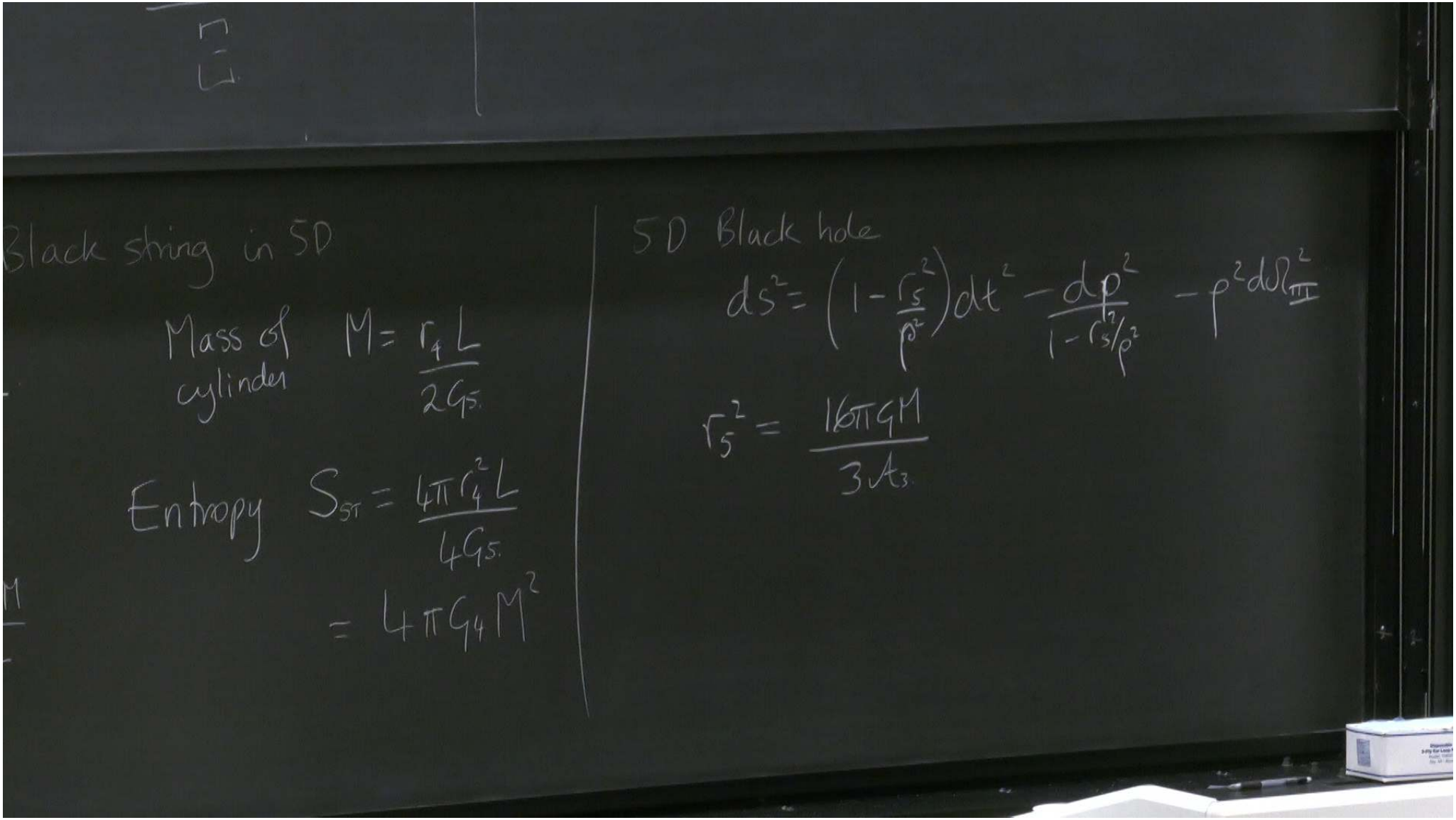
$r_4 = 2G_4 M$

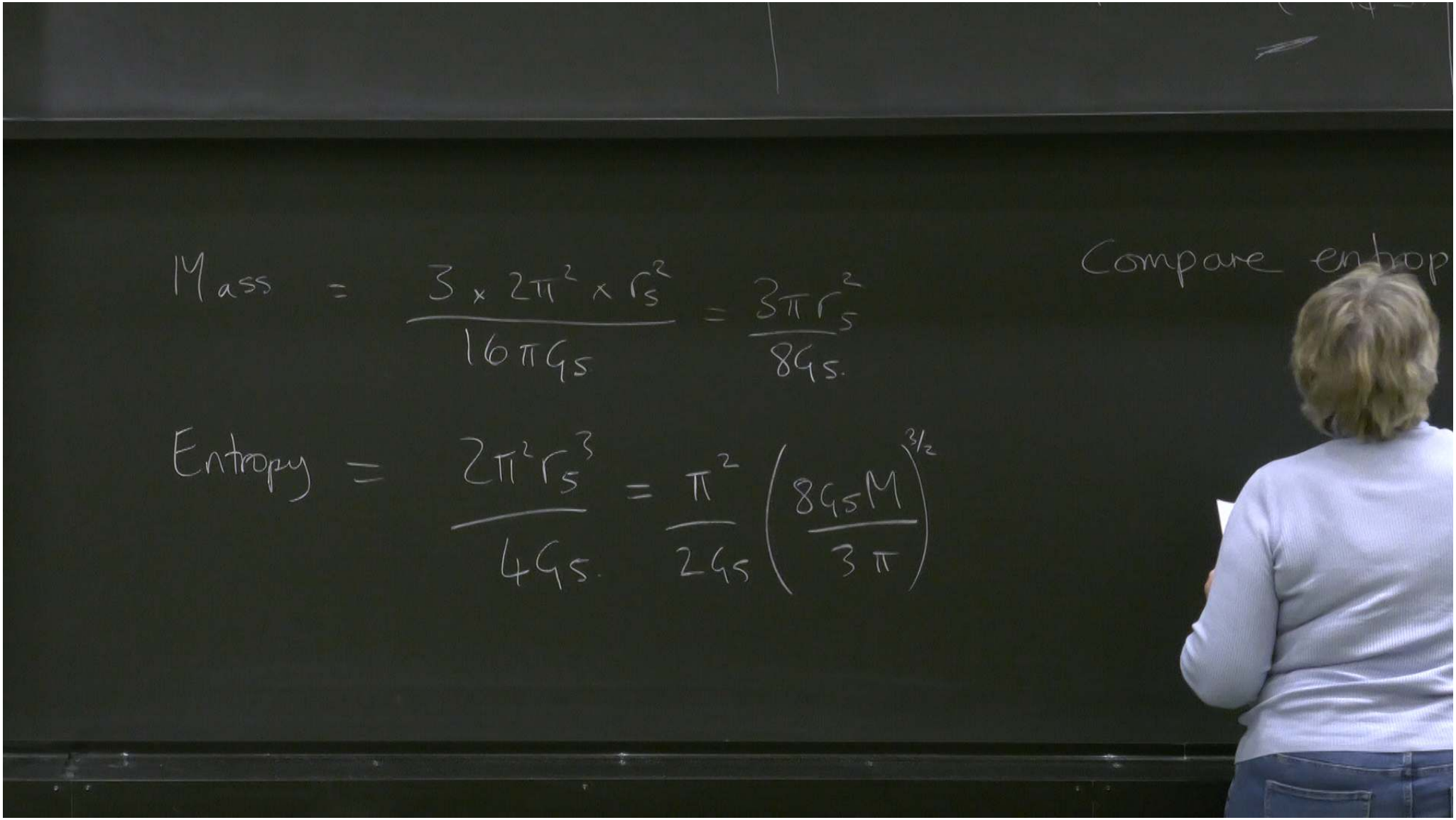
Mass of cylinder $M = \frac{r_4 L}{2G_5}$

Entropy $S_{ST} = \frac{4\pi r_4^2 L}{4G_5}$

$= 4\pi G_4 M^2$

$G_4 = G_5/L$ $\quad r_4 = \frac{2G_5 M}{L}$


Black string in 5D
Mass of cylinder $M = \frac{r_4 L}{2G_5}$
Entropy $S_{5T} = \frac{4\pi r_4^2 L}{4G_5}$
$= 4\pi G_4 M^2$
5D Black hole
$ds^2 = \left(1 - \frac{r_5^2}{\rho^2}\right)dt^2 - \frac{d\rho^2}{1 - r_5^2/\rho^2} - \rho^2 d\Omega_{\text{III}}^2$
$r_5^2 = \frac{16\pi G M}{3\mathcal{A}_3}$


Mass = $\frac{3 \times 2\pi^2 \times r_5^2}{16\pi G_5} = \frac{3\pi r_5^2}{8G_5}$
Entropy = $\frac{2\pi^2 r_5^3}{4G_5} = \frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3\pi}\right)^{3/2}$
Compare entrop

heat. HAWKING-PAGE. $6Q < \ell$ $6Q > \ell$

Compare entropies, fixing mass

$$\frac{S_{BH}}{S_{BS}} = \frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3\pi}\right)^{3/2} \Big/ 4\pi G_4 M^2$$

heat. HAWKING-PAGE.

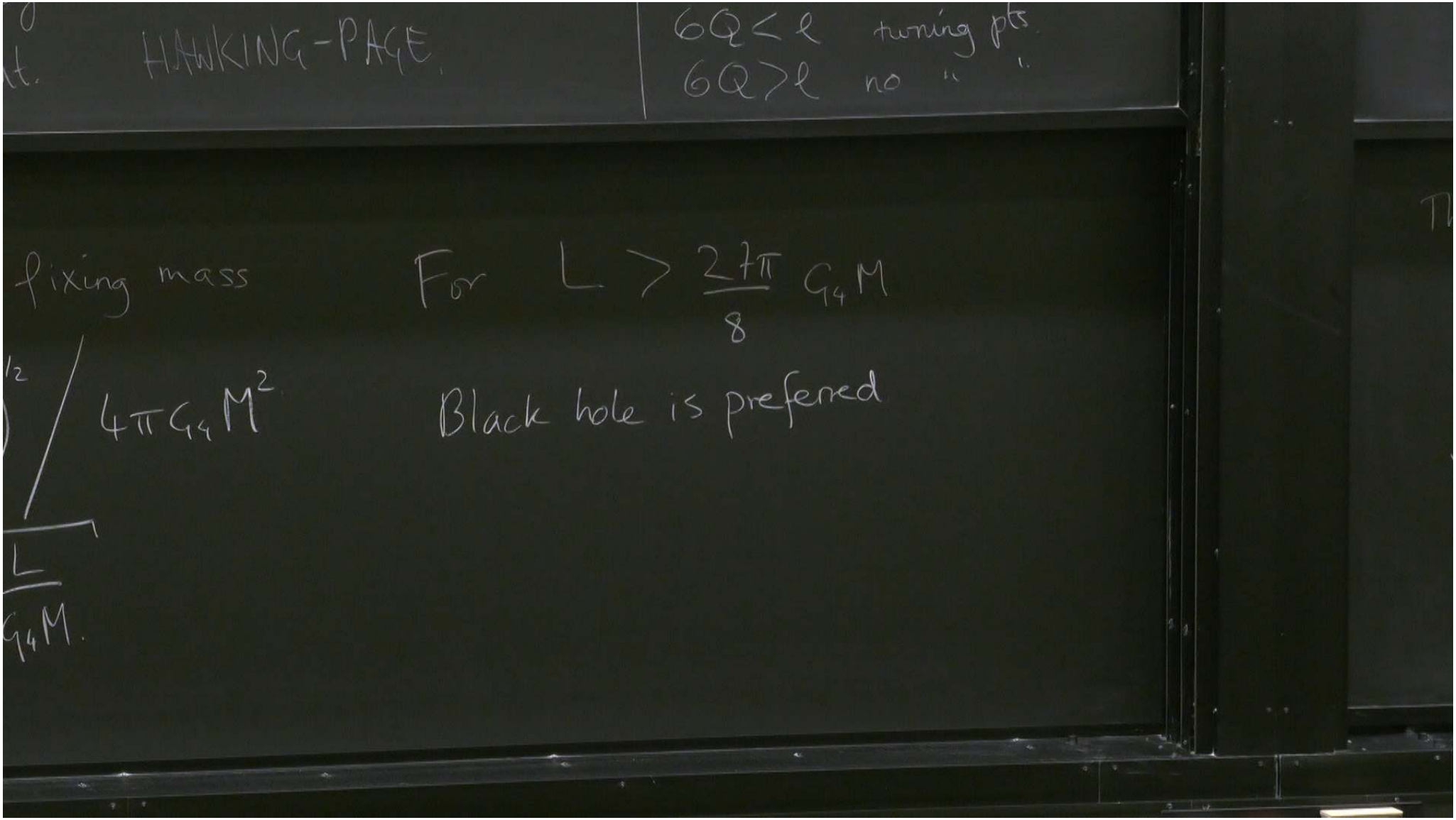
$6Q < \ell$

$6Q > \ell$

Compare entropies, fixing mass

$$\frac{S_{BH}}{S_{BS}} = \frac{\pi^2}{2G_5}\left(\frac{8G_5M}{3\pi}\right)^{3/2} \Big/ 4\pi G_4 M^2$$

$$= \sqrt{\frac{8}{27\pi}}\sqrt{\frac{L}{G_4 M}}$$


HAWKING-PAGE.
6Q<ℓ turning pts.
6Q>ℓ no " "
fixing mass
For $L > \frac{27\pi}{8} G_4 M$
$4\pi G_4 M^2$
Black hole is prefered
L
$G_4 M$.