Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

Date: January 26, 2024 - 9:00 AM

URL: https://pirsa.org/24010056

L9 The Gravitational Action

Recall the variational principle in gravity

$$S_{EH} = -\frac{1}{16\pi G}\int \underbrace{\sqrt{|g|}\, R\, d^4x}_{\text{vol el}}$$

$$\delta S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{|g|}\Big[\overbrace{R_{ab}\,\delta g^{ab} - \frac{1}{2} R\, g_{ab}\,\delta g^{ab}}^{\text{Einstein}} + \delta R_{ab}\, g^{ab}\Big]$$

$$\delta R_{ab} = \nabla_c\, \overset{\text{(TENSOR)}}{\delta\Gamma^{c}_{ab}} - \nabla_{(a}\,\delta\Gamma^{c}_{b)c}$$

$$= \frac{1}{2}\Box(\delta g^{-1})_{ab} + \frac{1}{2}\nabla_a\nabla_b\,\delta g^{-1\,c}{}_{c} - \nabla_c\nabla_{(a}\,\delta g^{-1}{}_{b)}{}^{c}$$

Derive using normal coords.
(local inertial frame : $g_{ab,c}=0$
but not 2^{nd})

$$\delta\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}\left(\overbrace{\delta g_{db,c} + \delta g_{dc,b} - \delta g_{bc,d}}^{=0 \text{ to leading order}}\right)$$

$$\overset{NC}{=} \frac{1}{2} g^{ad}\left(\nabla_c \delta g_{bd} + \nabla_b \delta g_{cd} - \nabla_d \delta g_{bc}\right)$$

– expression now covariant, so true in all frames.

ng order

$\ldots b - \delta g_{bc,d})$

$\nabla_b \delta g_{cd} - \nabla_d \delta g_{bc})$

o true in all frames.

$$\delta S_{EH} = -\frac{1}{16\pi G}\int G_{ab}\,\delta g^{ab}\sqrt{|g|}\,d^4x$$

$$-\frac{1}{16\pi G}\int \left(\Box\,\delta g^{-1} - \nabla_a\nabla_b\,\delta g^{ab}\right)\sqrt{g}\,d^4x$$

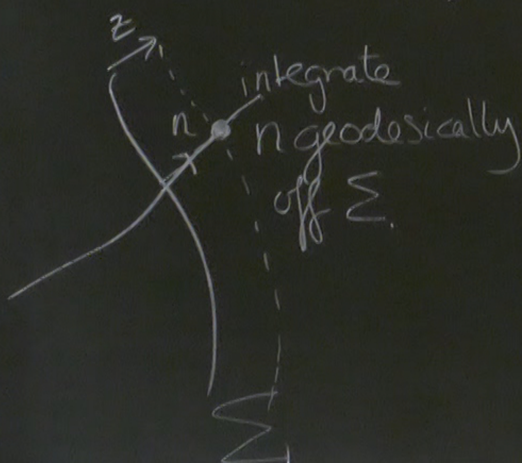
Use the expression $\nabla_a V^a = \frac{1}{\sqrt{g}}\partial_a \sqrt{g}\,V^a$.

Last term becomes

$$= \frac{1}{16\pi G}\int_{\partial M} d^3x\,\sqrt{h}\,\underset{\substack{\uparrow\\ \text{normal to boundary}}}{n_a}\left(\nabla^a\delta g^{-1} - \nabla_b\,\delta g^{ab}\right)$$

Look at boundary, work in Gaussian Normal (GN) gauge.

$$ds^2 = \boxed{h_{\mu\nu}\, dx^\mu dx^\nu} - dz^2$$



$$n = \frac{\partial}{\partial z} \qquad n^z = 1 \,,\; n_z = -1$$

n remains $\frac{\partial}{\partial z}$ under variations.

$$\delta g_{ab} = \delta h_{ab}.$$

$$K_{ab} = \nabla_a n_b = \Gamma^z_{ab}$$

$$\delta K_{ab} = \delta \Gamma^z_{ab}$$

$$= -\frac{1}{2} n^c \left(\nabla_a \delta g_{bc} + \nabla_b \delta g_{ac} - \nabla_c \delta g_{ab} \right)$$

$$= K^c{}_{(a} \delta g_{b)c} + \frac{1}{2} \nabla_n \delta g_{ab}.$$

$\nabla_c \delta g_{ab})$

$\nabla_n \delta g_{ab}.$

$$\Rightarrow h^{ab}\,\delta K_{ab} = -K_{ac}\,\delta g^{ac} - \frac{1}{2}\nabla_n \delta g^{-1}$$

Note $n_a \nabla_b \delta g^{ab} = \nabla_b\left(\cancel{n_a \delta g^{ab}}\right) - K_{ab}\,\delta g^{ab}$

(using GN)

$$K_{ab}\,\delta h^{ab} + h^{ab}\,\delta K_{ab} = -\frac{1}{2}\nabla_n \delta g^{-1}$$

$$\delta\left[g_{ab}\, g^{bc} = \delta_a{}^c\right]$$

$$\delta g\, g^{-1} + g\, \delta g^{-1} = 0$$

$\nabla_c \delta g_{ab})$

$\nabla_n \delta g_{ab}.$

$$\Rightarrow \quad h^{ab}\,\delta K_{ab} = -K_{ac}\,\delta g^{ac} - \frac{1}{2}\nabla_n \delta g^{-1\,c}{}_{c}$$

Note $\quad n_a \nabla_b \delta g^{ab} = \nabla_b(\cancel{n_a \delta g^{ab}}) - K_{ab}\,\delta g^{ab}$

(using GN)

$$K_{ab}\,\delta h^{ab} + h^{ab}\,\delta K_{ab} = -\frac{1}{2}\nabla_n (\delta g^{-1})^{c}{}_{c}$$

$$S_G = S_{EH} + \frac{1}{8\pi G}\int_{\partial M} K\sqrt{h}\, d^3x + \int \sigma\sqrt{h}\, d^3x$$

$$\xrightarrow{\delta} -\frac{1}{2}\sigma h_{ab}\,\delta h^{ab}$$

Vary:

$$\delta S_G = -\frac{1}{16\pi G}\int G_{ab}\,\delta g^{ab}\sqrt{g}\,d^4x + \frac{1}{8\pi G}\int d^3x\sqrt{h}\,\left(K_{ab}-K h_{ab}\right)\delta h^{ab}$$

In bulk Einstein gravity, can take $\delta h = 0$ on ∂M & get Einstein eqns. However, if we have boundary dynamics, get boundary eqns of motion.

Example: SCH black hole $t \to i\tau$

$$|ds^2| = \left(1-\frac{2GM}{r}\right)d\tau^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega_{II}^2$$

$r \to 2GM$

$$4\cancel{\lambda}(r-2GM)d\rho^2 = \lambda^{\cancel{2}}dr^2$$

$$\rho^2 = \lambda(r-2GM)$$

$$\frac{(2GM)dr^2}{(r-2GM)} = \frac{4d\rho^2}{\lambda}\cdot 2GM.$$

$$2\rho\, d\rho = \lambda dr$$

$\lambda = 8GM.$ ρ proper radius.

i.e.

$= \lambda^2 dr^2$

$\frac{4\,d\rho^2}{\lambda} \cdot 2GM.$

ρ proper radius.

$$\left(1-\frac{2GM}{r}\right)d\tau^2 \sim \frac{\rho^2}{8GM \cdot 2GM}\, d\tau^2$$

$$= \rho^2 d\left(\frac{\tau}{4GM}\right)^2$$

$r = 2GM$ looks like $\rho \to 0$ in polars, provided

$$\Delta\tau = 8\pi GM.$$

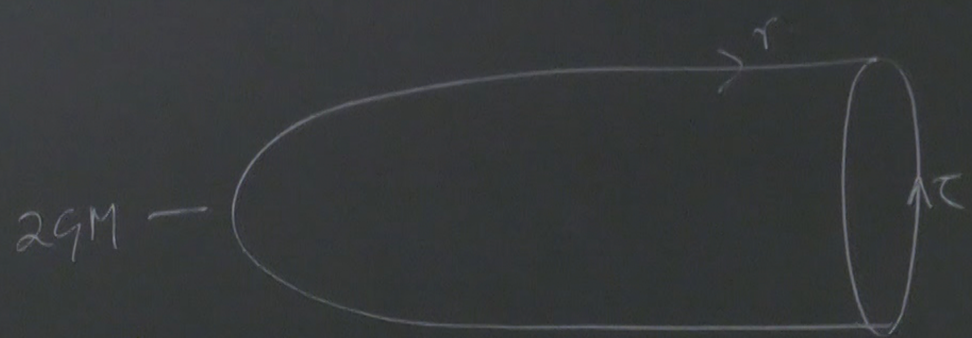
periodic time $\Delta\tau = \beta$, indicates

finite temperature T

finite temperature $T = 1/\beta \to 1/8\pi GM$

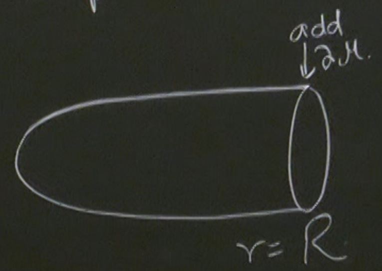
ie black holes are at finite temp

$$T = \frac{1}{8\pi GM} \times \left(\frac{\hbar c^3}{k_B}\right)$$



BLACK HOLE CIGAR

Compute action:



$$ds^2\big|_{\partial M} = \left(1 - \frac{2GM}{R}\right) d\tau^2 + R^2 d\Omega_{II}^2$$

$$n = -\sqrt{1 - \frac{2GM}{R}}\ \frac{\partial}{\partial r}$$

$$K = \nabla_a n^a = \frac{1}{r^2}\partial_r r^2 n^r$$

$$= -\frac{2}{r}\sqrt{1 - \frac{2GM}{r}} - \frac{GM}{r^2\sqrt{1 - \frac{2GM}{r}}}$$

$$\sqrt{h} = \sqrt{1 - \frac{2GM}{R}}\ R^2 \sin\theta$$

$$K\sqrt{h} = -R^2 \sin\theta \left[\frac{2}{R}\left(1 - \frac{2GM}{R}\right) + \frac{GM}{R^2}\right]$$

$$= -\sin\theta\,(2R - 3GM)$$

$$\int K\sqrt{h}\, d^3x = \int K\sqrt{h}\, d\tau\, d\theta\, d\varphi$$

$$= -4\pi\beta\,(2R - 3GM)$$

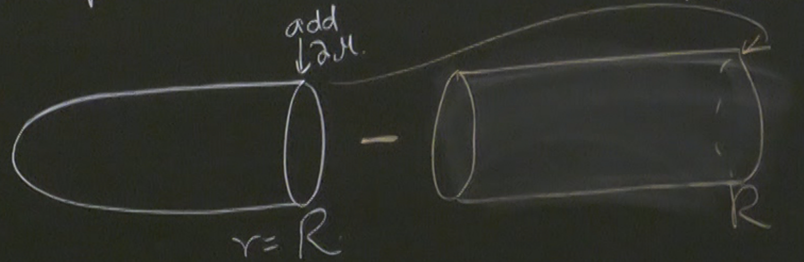
For flat space at periodicity β_0.

$$ds_0^2 = d\tau_0^2 + R^2 d\Omega_2^2$$

$$K_0 = -\frac{2}{R} \qquad \sqrt{h_0} = R^2 \sin\theta$$

$$\int K_0\sqrt{h_0} = -4\pi\beta_0 \cdot 2R.$$

Compute action:



$$ds^2\big|_{\partial M} = \left(1-\frac{2GM}{R}\right)d\tau^2 + R^2 d\Omega_{II}^2$$

$$n = -\sqrt{1-\frac{2GM}{R}}\,\frac{\partial}{\partial r}$$

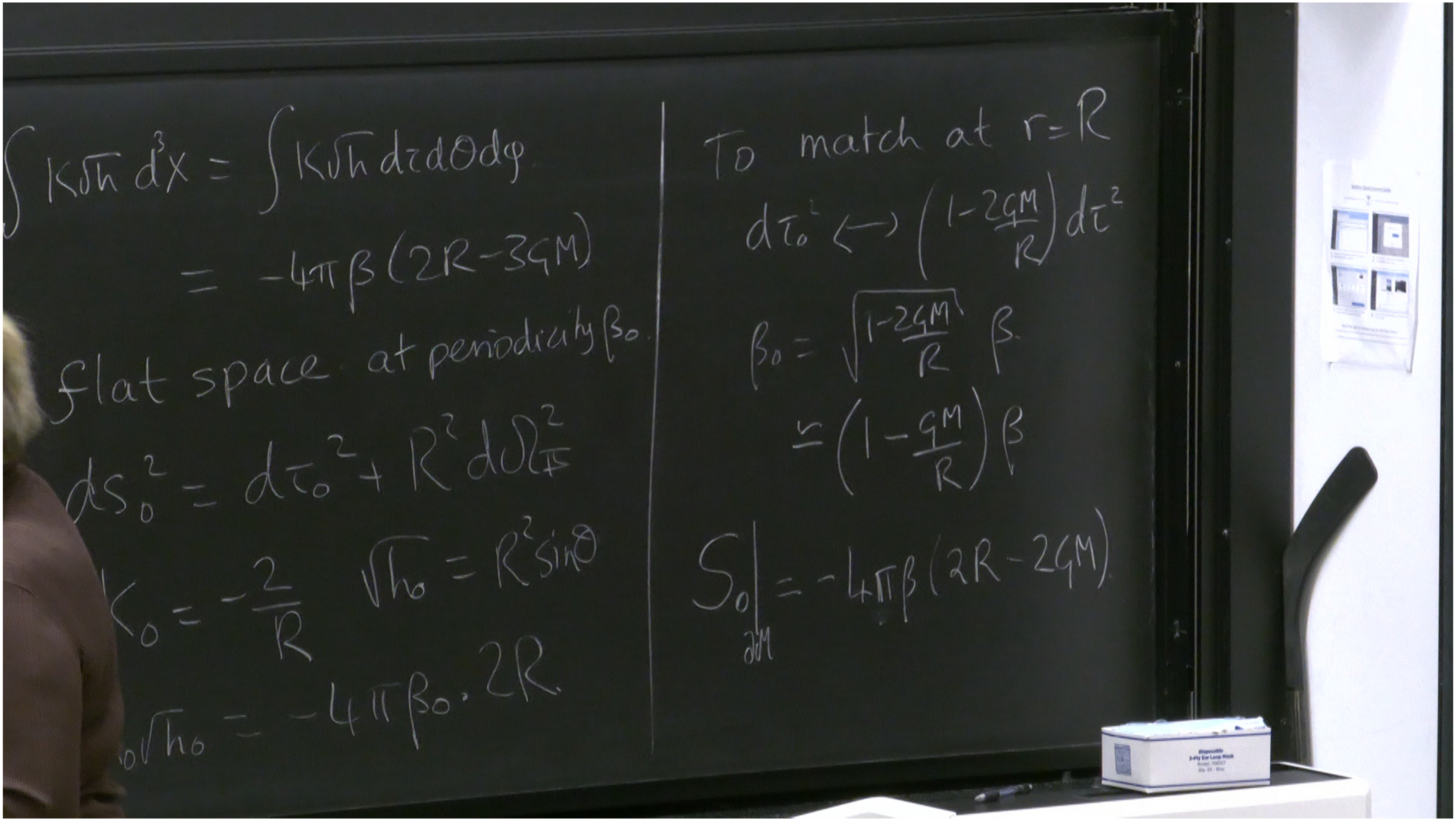
$$K = \nabla_a n^a = \frac{1}{r^2}\partial_r r^2 n^r$$

$$= -\frac{2}{r}\sqrt{1-\frac{2GM}{r}} - \frac{GM}{r^2\sqrt{1-\frac{2GM}{r}}}$$

$$\sqrt{h} = \sqrt{1-\frac{2GM}{R}}\,R^2\sin\theta$$

$$K\sqrt{h} = -R^2\sin\theta\left[\frac{2}{R}\left(1-\frac{2GM}{R}\right) + G\ldots\right.$$

$$= -\sin\theta\,(2R-3GM)$$



$\int K\sqrt{h}\, d^3x = \int K\sqrt{h}\, d\tau\, d\theta\, d\varphi$
$= -4\pi\beta(2R - 3GM)$
flat space at periodicity β_0
$ds_0^2 = d\tau_0^2 + R^2 d\Omega_2^2$
$K_0 = -\frac{2}{R}$
$\sqrt{h_0} = R^2 \sin\theta$
$\sqrt{h_0} = -4\pi\beta_0 \cdot 2R$
To match at $r=R$
$d\tau_0^2 \longleftrightarrow \left(1-\frac{2GM}{R}\right) d\tau^2$
$\beta_0 = \sqrt{1-\frac{2GM}{R}}\ \beta$
$\simeq \left(1-\frac{GM}{R}\right)\beta$
$S_0\Big|_{\partial M} = -4\pi\beta(2R - 2GM)$

Thus $$S_{BH} - S_0 = \frac{1}{8\pi G}\left[-4\pi\beta(2R-3GM) + 4\pi\beta(2R-2GM)\right]$$

$$= \frac{\beta M}{2} \qquad \left(\beta = \frac{1}{8\pi G M}\right)$$

Consider the partition fn

$$Z \sim \mathrm{tr}\, e^{-\beta H} \sim \mathrm{tr}\, e^{-\beta\left(\frac{1}{\beta}\int d^4x\, \mathcal{L}_E\right)}$$

$$= -\sin\theta\,(2R-3GM)$$

$$Z \sim \mathrm{tr}\, e^{-I_E}$$

Here entropy

$$S = \beta^2 \frac{\partial}{\partial\beta}\left[-\beta^{-1}\ln Z\right]$$

$$= \beta^2 \frac{\partial}{\partial\beta}\left(I/\beta\right)$$

$$\frac{\beta^2}{2}\frac{\partial M}{\partial\beta} \qquad \text{SCH.}$$

But $\beta\frac{\partial M}{\partial\beta} = M$

$$\Rightarrow S = \frac{\beta M}{2} = 4\pi G M^2$$

$$= \frac{4\pi(2GM)^2}{4G} = A/4G.$$