Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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How to observe?

Black holes typically have accretion disc that is luminous.
To describe disc need orbits, to describe the image we see, need light trajectories $\longrightarrow$ <u>geodesics</u>

$\ddot{X}^\mu + \Gamma$

• Let

$\frac{d}{d\tau}$

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\lambda}\dot{X}^\nu \dot{X}^\lambda = 0$$


$\dot{X}^{\mu 2} = 1$ matter

$\phantom{\dot{X}^{\mu 2} =}\ 0$ light

- Let $k^\mu$ is a Killing vec

$$\frac{d}{d\tau}\left(k_\mu \dot{X}^\mu\right) = \dot{X}^\nu \nabla_\nu \left(k_\mu \dot{X}^\mu\right)$$
$$= \dot{X}^\nu \dot{X}^\mu \nabla_\mu k_\nu$$
$$= 0$$

$k_\mu \dot{X}^\mu$ const along geodesic

SCH

$$k^\mu = \frac{\partial}{\partial t} \Rightarrow f\dot{t} = E, \text{ const.}$$

$$k^\mu = \frac{\partial}{\partial \varphi} \Rightarrow r^2\dot{\varphi} = h, \text{ const}$$

$$\ddot{X}^\theta + \Gamma^\theta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = \ddot{\theta} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}^2 + 2\Gamma^\theta_{\theta r}\dot{\theta}\dot{r}$$
$$= \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 + \frac{2\dot{\theta}\dot{r}}{r}$$
$$= 0$$

Thus $\theta = \pi/2$ with $\dot\theta = 0$ remains in $\theta = \pi/2$ plane. Geodesics lie in a plane with black hole at centre.

Finally $$\frac{d}{d\tau}\left(g_{\mu\nu}\dot X^\mu \dot X^\nu\right) = \dot X^\sigma \nabla_\sigma\left(g_{\mu\nu}\dot X^\mu \dot X^\nu\right) = 0$$

i.e. $g_{\mu\nu}\dot X^\mu \dot X^\nu$ remains 0 or 1 along geodesic.

$\ldots \theta - \sin\theta\cos\ldots$

$= 0$

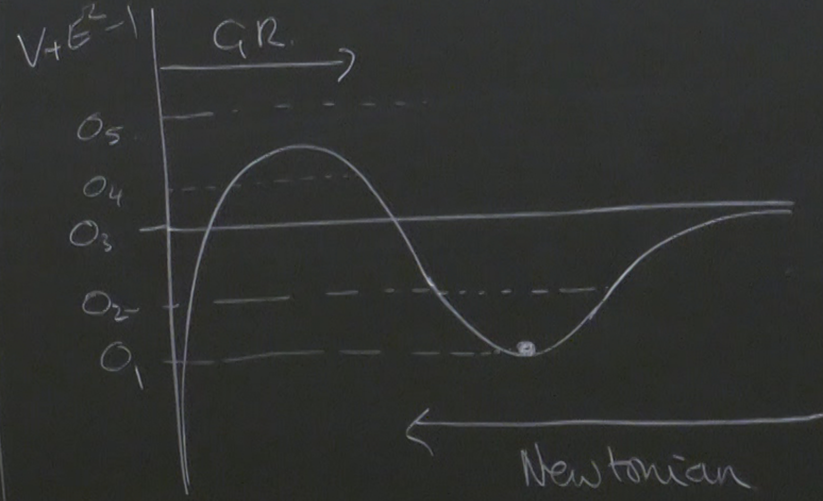
$$\dot{t}^2 f - \frac{\dot{r}^2}{f} - r^2\dot{\varphi}^2 = \eta \qquad \begin{array}{ll} 1 & \text{t-like} \\ 0 & \text{null} \end{array}$$

Can be rewritten as

$$\dot{r}^2 + V_{eff}(r) = 0$$

$$V_{eff}(r) = \eta - E^2 \underbrace{- \frac{2GM\eta}{r}}_{\text{Newtonian pole}} + \underbrace{\frac{h^2}{r^2}}_{\text{centrifugal}} \underbrace{- \frac{2GMh^2}{r^3}}_{\text{GR}}$$

$\eta = 1$ $\qquad \dot{r}^2 = E^2 - 1 - V_{eff}$



$= 0$

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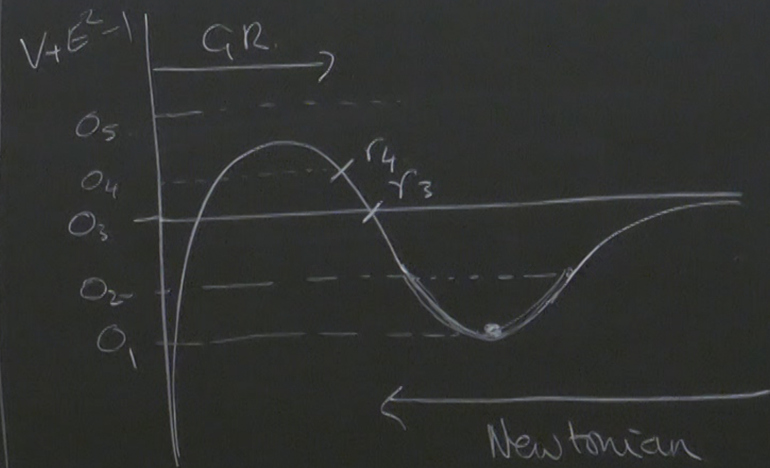


O1 is a circular orbit ($\dot r = 0$)

O2 is an elliptical ... ($r_{min}$ & $r_{max}$)

$= \ddot\theta - \sin\theta\cos\theta$

$= 0$

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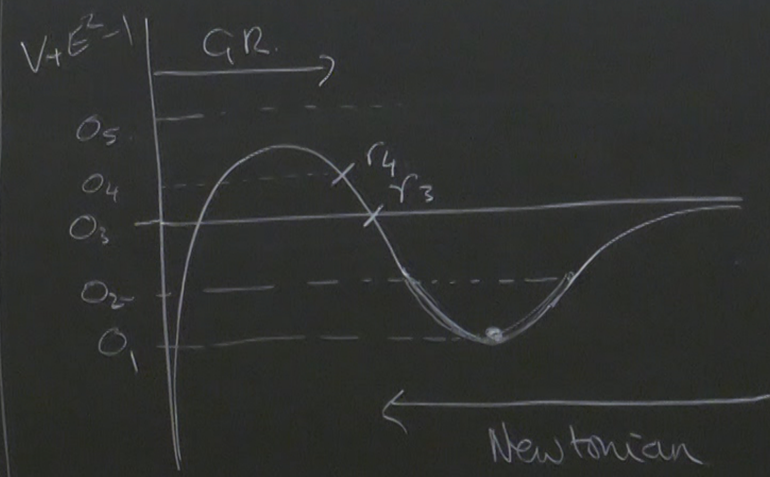



$O_1$ is a circular orbit ($\dot r = 0$)

geodesic

$O_1$: $V = V' = 0 \quad (V'' > 0)$

$$1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3} = 0$$

$$(V') \qquad \frac{2GM}{r^2} - \frac{2h^2}{r^3} + \frac{6GMh^2}{r^4} = 0$$

At turning pt $\quad r_\pm = \frac{h^2}{2GM} \pm \sqrt{\frac{h^2}{4G^2M^2} - 3}$

O1

O2

O3

O

GR/O

odesic.

Newtonian centrifugal GR.
pote

O1 is a circular orbit ($\dot{r} = 0$)

O2 is an elliptical orbit ($r_{min}$ & $r_{max}$)

O3 has $r \in [r_3, \infty)$ - parabolic

O4 has $r \in [r_4, \infty)$ - scattering

Newtonian conic sections

-3.

GR/ O5 - the plunge

$r_{max})$

Newtonian conic sections

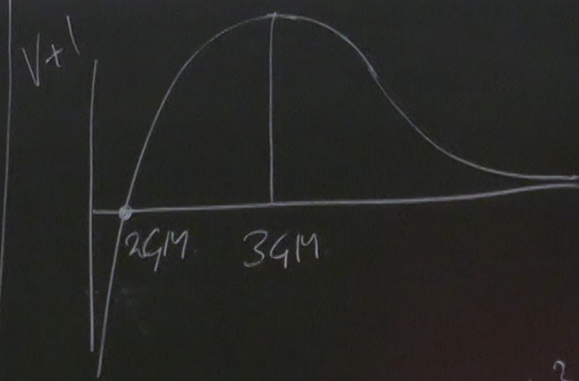
Bound orbits: As $h^2$ decreases $r_+ \to r_-$ & max & min of $V_{eff}$ merge at inflection pt.

$$h^2 = 12G^2M^2$$
$$= 2GMr_0$$
$$\Rightarrow \underline{\underline{r_0 = 6GM}}$$

This is the smallest $r$ for which $\exists$ stable orbit – the Innermost Stable Circular Orbit or ISCO. Innermost limit of acc. disc

• $\eta = 0$. (light) Can set $E = 1$

$$V_{eff} = -1 + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}.$$




$$V' = -\frac{2h^2}{r^3} + \frac{6GMh^2}{r^4} = 0$$

at $r = 3GM$.

$$V(3GM) = -1 + \frac{h^2}{27G^2M^2}$$

If $h^2 = 27G^2M^2$

then $\dot{r} = 0$. – CIRCULAR ORBIT

(but unstable)

These photon rings are characteristic feature of b.h.

More generally, let $u = 1/r$

$$\frac{du}{d\varphi} = \frac{\dot{u}}{\dot{\varphi}} = \frac{-\dot{r}/r^2}{\dot{\varphi}} = -\frac{\dot{r}}{h}$$

$$\rightarrow \left(\frac{du}{d\varphi}\right)^2 = -\frac{(\eta - E^2)}{h^2} + 2GM h^2 u - u^2 + 2GMu^3$$

$= 0$

$$V(3GM) = -1 + \frac{h^2}{27G^2M^2}$$

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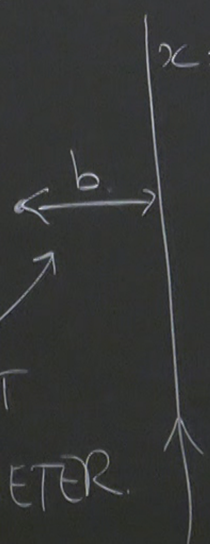
$$\xrightarrow{\eta = 0} \; = \frac{1}{h^2} - u^2 + 2GMu^3$$

To find $h$.

$x = r\cos\varphi = b$.




IMPACT PARAMETER.

$$bu = \cos\varphi$$

$$\frac{du}{d\varphi} = -\frac{\sin\varphi}{b}$$

As $u \to 0$ $\quad u'^2 + u^2 = \frac{1}{h^2}$

$$\frac{\sin^2\varphi}{b^2} + \frac{\cos^2\varphi}{b^2}$$

ie $h = b$.

Look for turning pt.

$$\frac{du}{d\varphi} = 0 \iff \underbrace{\frac{1}{b^2} - u^2 + 2GMu^3}_{} = 0$$

$u^2 - 2GMu^3$ stationary at $u = 1/3GM$.

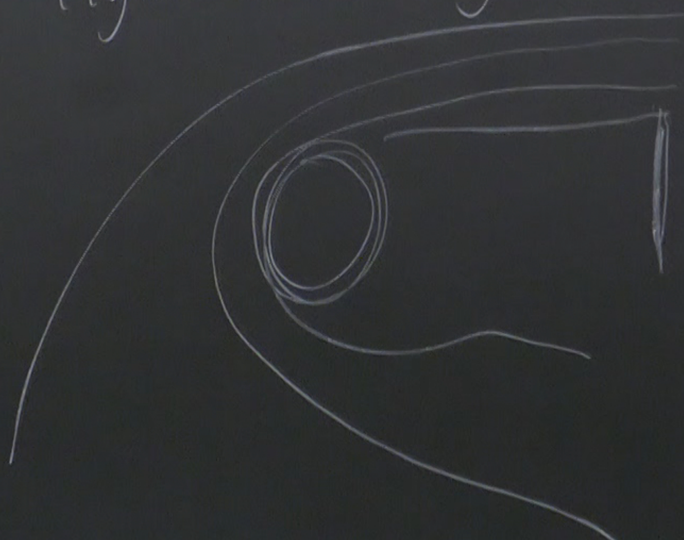
$$u^2 - 2GMu^3 = \frac{1}{27G^2M^2}$$

pt.

$= \odot$

mary

If $b^2 < 27 G^2 M^2$, $\frac{du}{d\varphi}$ can never be zero, light is always captured.



## Kerr

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right) dt^2 - \Sigma\, d\theta^2 + 4GMar\sin^2\theta\, dt\, d\varphi - \frac{\Sigma}{\Delta} dr^2 - \sin^2\theta \frac{d\varphi^2}{\Sigma}\left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]$$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad a = J/M.$$

$$\Delta = r^2 + a^2 - 2GMr$$

$b^2$ $b^2$

$$(\Sigma - 2mr) = 0 \leftrightarrow r_e = GM \pm \sqrt{G^2M^2 - a^2\cos^2\theta}$$

$\partial_t$

$g_{tt} = 0$ $\quad \left|\frac{\partial}{\partial t}\right|^2 \to 0$ - impossible to be at rest rel. to $\infty$.

ERGOSPHERE

$$\partial_t : \quad E = g_{t\mu}\dot{X}^\mu = \left(1-\frac{2GMr}{\Sigma}\right)\dot{t} + \frac{2GMar}{\Sigma}\sin^2\theta\,\dot{\varphi}$$

$$h = -g_{\varphi\mu}\dot{X}^\mu = -\frac{2GMar}{\Sigma}\sin^2\theta\,\dot{t} + \left(r^2+a^2+\frac{2GM}{\Sigma}ra^2\sin^2\theta\right)\dot{\varphi}\,\sin^2\theta$$

Take $P_\alpha(E,-h) = g_{\alpha\beta}\dot{X}^\beta \qquad \alpha,\beta = t,\varphi$

$$P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta}\dot{X}^\alpha\dot{X}^\beta = E^2 g^{tt} - 2Eh\,g^{t\varphi} + h^2 g^{\varphi\varphi}$$

i.e. $g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$ remains ... geodesic.

$V_{eff}$ ...

$$P_\alpha P_\beta g^{\alpha\beta} = \frac{E^2}{\Delta\Sigma}\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right) - 4E\frac{hGMar}{\Delta\Sigma} - \frac{h^2}{\Delta\sin^2\theta}\left(1-\frac{2GMr}{\Sigma}\right)$$

$$= \frac{\left((r^2+a^2)E - ah\right)^2}{\Sigma\Delta} - \frac{\sin^2\theta}{\Sigma}\left(aE - \frac{h}{\sin^2\theta}\right)^2$$

$\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$

Define $P_\mu = g_{\mu\nu}\dot{X}^\nu$ then $\underset{\text{or } m^2}{\eta} = -\frac{p_r^2\Delta}{\Sigma} - \frac{p_\theta^2}{\Sigma} +$

$r^2+a^2\cos^2\theta$

$$) = \eta - C = \underbrace{-\frac{2GM}{r}}_{\text{Newtonian pote}} + \underbrace{\frac{h^2}{r^2}}_{\text{centrifugal}} \underbrace{\left(-\frac{2GMh^2}{r^3}\right)}_{\text{GR}}$$

The geodesic eqn is seperable.

$$\underbrace{\frac{\left((r^2+a^2)E - ah\right)^2}{\Delta} - p_r^2\,\Delta - m^2 r^2}_{\text{fn } r \text{ const.}} = \underset{\text{CARTER CONST.}}{C} = \underbrace{p_\theta^2 + m^2\cos^2\theta + \sin^2\theta\left(aE - \frac{h}{\sin^2\theta}\right)^2}_{\text{fn } \theta\text{, const.}}$$

$S(r,\theta,\varphi,t)$. $\quad p_\mu = \frac{\partial S}{\partial x^\mu}$