Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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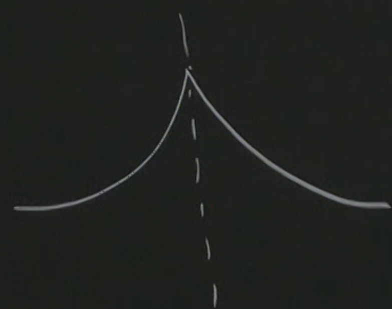
URL: https://pirsa.org/24010053

L6 GAUSS - CODAZZI

$$ds^2 = e^{-2k|z|}\eta_{\mu\nu}dx^\mu dx^\nu - dz^2$$



$z=0$
"delta fn"?

Solves AdS away from $z=0$.

cf to $$ds^2 = (1 - \underline{|r|/L})^2 \left[dt^2 - \ell^2 \cosh^2 \tfrac{t}{\ell} \, d\Omega_3^2 \right] - dr^2$$

Solves vac. Einstein

Example of warped extra dimensions

- Randall-Sundrum. "Hide" extra dims by localising physics on a <u>submanifold</u>

$] -dt^2$

Take $\Sigma \subset \mathcal{M}$ a submanifold of $\mathcal{M}$ (i.e. Σ a C^∞ manifold in its own right, but also lies in $\mathcal{M}$ & inherits structure from $\mathcal{M}$).

e.g. S^2 $\quad x^2+y^2+z^2=1 \subset \mathbb{R}^3$

$$\underline{X} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$ds_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \longleftarrow dx^2+dy^2+dz^2.$$

$] - dr^2$

Take $\Sigma \subset \mathcal{M}$ a submanifold of $\mathcal{M}$ (i.e. Σ a C^∞ manifold in its own right, but also lies in $\mathcal{M}$ & inherits structure from $\mathcal{M}$).

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$$\underline{X} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$ds_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2. \Leftarrow dx^2+dy^2+dz^2.$$

nifold
its own
herits

$\mathbb{R}^3$

inφ, $\cos\theta$)

$\Leftrightarrow dx^2+dy^2+dz^2$.

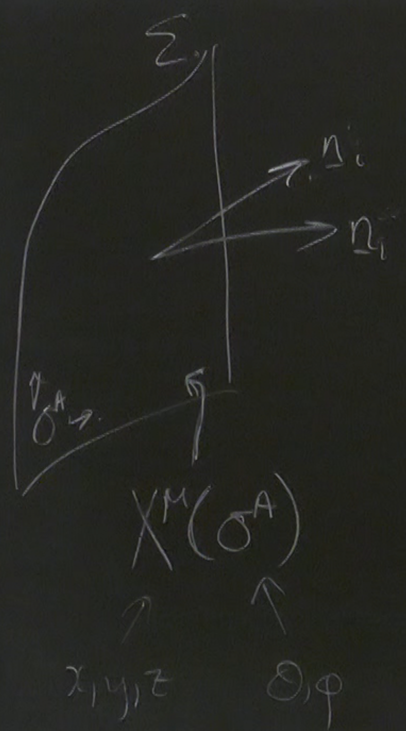
Let $\dim(M) = D$

$\dim(\Sigma) = d$

<u>Defn</u> The <u>co-dimension</u> of Σ is $n = D-d$.

i.e. $\exists$ n lin. indep. vectors, $\underline{n}_i$ normal to Σ where σ^A is a chart on Σ

$$\underline{n}_i \, \sigma^A = 0$$



Here, take $n_i^{\ \mu} n_j^{\ \nu} g_{\mu\nu} = \epsilon_i \delta_{ij}$

$$\epsilon_i = \begin{cases} 1 & \text{timelike} \\ -1 & \text{spacelike} \end{cases}$$

e.g. 3+1-formalism, $\underline{n}$ timelike, but not normalized!

Mostly, I will take $\underline{n}$ spacelike.

$(\epsilon_i)\delta_{ij}$

1 timelike

-1 spacelike

...nelike, but

$\underline{n}$ spacelike.

<u>Defn</u> The <u>1^{st} fundamental form</u> or <u>induced metric</u> of Σ is

$$h_{ab} = g_{ab} \underset{(-\epsilon_i)}{+} \underbrace{n_{ia} n_{ib}}_{\Sigma\ \text{conv.}}$$

this is the metric on Σ inherited from M.

st fundamental form

metric of Σ is

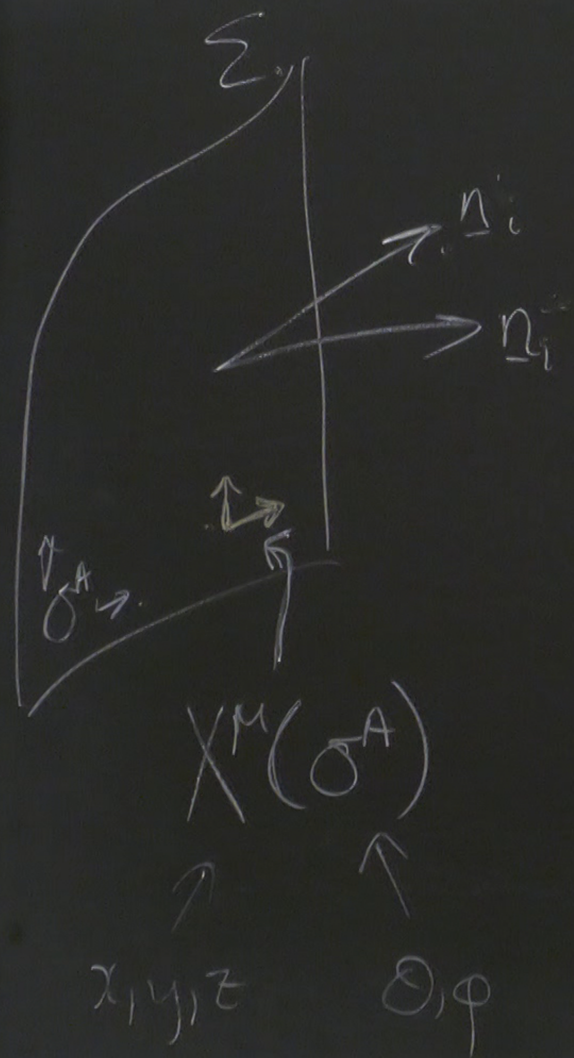
$$\underset{(-\epsilon_i)}{g_{ab} +} \underbrace{n_{ia} n_{ib}}_{\Sigma \text{ conv.}}$$

metric on Σ inherited

h_{ab} lies in (co)tgt bundle of $\mathcal{M}$.

$\frac{\partial X^\mu}{\partial \sigma^A}$ gives a map $T_p^*(\mathcal{M}) \to T_p^*(\Sigma)$

via $\omega_\mu \to \omega_A = \frac{\partial X^\mu}{\partial \sigma^A} \omega_\mu$.



Here, take $n_i^M n_j$

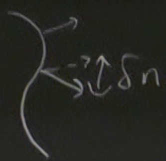
e.g 3+1 - formalism

not normalized!

Mostly, I will

$x, y, z \quad \theta, \varphi$

Intrinsic metric:



$$\gamma_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$$

Defn The 2nd fundamental form(s) or extrinsic curvature(s) of Σ is

$$K_{i\mu\nu} = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda n_{i\sigma}.$$

& measures how Σ curves in $\mathcal{M}$.

$\xi^i \uparrow \delta n$

...tal form(s)

is

$\nabla_\lambda n_{i\sigma}$.

& measures how Σ curves in $\mathcal{M}$

or $K_{iAB} = X^\mu_{,A} X^\nu_{,B)} \nabla_\mu n_{i\nu}$

$= - n_{i\mu} D_A D_B X^\mu$

where D_A is the connection of γ inherited from $\mathcal{M}$.

from M

$\nabla_\mu n_{i\nu}$.

$D_B X^\mu$

ction of γ

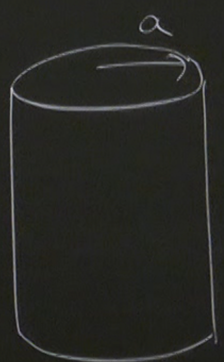
Defn The normal fundamental forms

$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j^{\ \nu}$$

measure how normals "twist" in M.

– connection on normal bundle of Σ.

e.g. Cylinder $x^2+y^2=a^2 \subset \mathbb{R}^3$

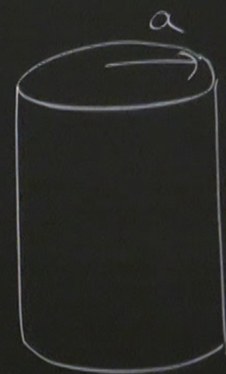


$$\underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad \text{1-normal}$$

$$h_{ab} = \delta_{ab} - n_a n_b.$$

$$= \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e.g. Cylinder $x^2+y^2=a^2 \subset \mathbb{R}^3$



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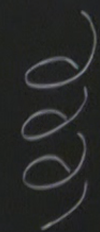
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ in cyl polars.}$$

$$\gamma_{AB} = \begin{bmatrix} a^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$K_{ab} = -\Gamma^n_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Helix</u> $\underline{X} = (\cos\lambda, \sin\lambda, \lambda)/\sqrt{2}$

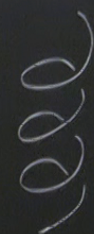


2 normals. $\underline{n}_1 = (\cos\lambda, \sin\lambda, 0)$

$\underline{n}_2 = (-\sin\lambda, \cos\lambda, -1)/\sqrt{2}.$

codim 2. & $\underline{X}' = (-\sin\lambda, \cos\lambda, 1)/\sqrt{2}.$

Helix $\underline{X} = (\cos\lambda, \sin\lambda, \lambda)/\sqrt{2}$



2 normals $\underline{n}_1 = (\cos\lambda, \sin\lambda, 0)$

$\underline{n}_2 = (-\sin\lambda, \cos\lambda, -1)/\sqrt{2}$

codim 2

& $\underline{X}' = (-\sin\lambda, \cos\lambda, 1)/\sqrt{2}$

- easier to work in intrinsic co-ords

$$K_1 = X'^a X'^b n_{a,b} = -\underline{n}\cdot\underline{X}''$$

$$= +\frac{1}{\sqrt{2}} \qquad K_2 = 0.$$

$$\underline{\beta} = n_{1a} \underline{\nabla} n_2^a$$

$$\underline{n} = \left(\frac{\sqrt{2}x}{\rho}, \frac{\sqrt{2}y}{\rho}, 0\right) \quad (\rho^2 = x^2+y^2)$$

$$\underline{m} = \left(-\frac{y}{\rho}, \frac{x}{\rho}, -\frac{1}{\sqrt{2}}\right)$$

$$\underline{\beta} = \sqrt{2}\left(\frac{x^2y}{\rho^4} + \frac{y^3}{\rho^4}, -\frac{x^3}{\rho^4} - \frac{xy^2}{\rho^4}, 0\right)$$

$$= \sqrt{2}\left(\frac{x}{\rho^2}, -\frac{y}{\rho^2}, 0\right)$$

Gauss Eqn

$$R_{\Sigma}{}^{a}{}_{bcd} = R_{\mathcal{M}}{}^{a'}{}_{b'c'd'}\, h^{a}{}_{a'}\, h^{b'}{}_{b}\, h^{c'}{}_{c}\, h^{d'}{}_{d}$$

$$+ \sum_i \epsilon_i \left[K_i{}^{a}{}_{c} K_{i\,bd} - K_i{}^{a}{}_{d} K_{i\,bc} \right]$$

Proof - codim 1 $\quad \epsilon = -1. \quad D_a V^b = h^{a'}{}_{a}\, h^{b}{}_{b'}\, \nabla_{a'} V^{b'}$

$h_{ab} = g_{ab} + n_a n_b$, take $\nabla_n n^a = 0$.

$$\nabla_a n_b = K_{ab}$$

$(V \cdot \text{tgt to } \Sigma)$

$$R_\Sigma{}^a{}_{bcd} V^b = 2 D_{[c} D_{d]} V^a$$

$$= h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \nabla_{c'} \left[h^e{}_{d'} h^{a'}{}_f \nabla_e V^f \right] - c \Leftrightarrow d.$$

$$= h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \left(\nabla_{c'} \nabla_{d'} V^{a'} \right) \quad (1)$$

$$+ h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \nabla_{d'} V^f \nabla_{c'} h^{a'}{}_f$$

$$+ h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \left(\nabla_{c'} h^e{}_{d'} \right) \nabla_e V^{a'}.$$

$bc]$

(V. tgt to Σ)

$$R_\Sigma{}^a{}_{bcd} V^b = 2 D_{[c} D_{d]} V^a$$

$$= h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \nabla_{c'}\left[h^e{}_{d'} h^{a'}{}_f \nabla_e V^f \right] - c \leftrightarrow d.$$

$$= h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \left(\nabla_{c'} \nabla_{d'} V^{a'}\right) \quad (1)$$

$$+ h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \left(\nabla_{d'} V^f\right)\left(\nabla_{c'} h^{a'}{}_f\right)$$

$$+ h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d \left(\nabla_{c'} h^e{}_{d'}\right)\left(\nabla_e V^{a'}\right)$$

$bc]$

$$\nabla_{d]} V^a$$

$$h^e{}_{d'} h^{a'}{}_f \nabla_e V^f] - c \Leftrightarrow d.$$

$$\nabla_{d'} V^{a'}) \quad (1).$$

$$(\nabla_{d'} V^f)(\nabla_{c'} h^{a'}{}_f)$$

$$(\nabla_{c'} h^e{}_{d'})(\nabla_e V^{a'})$$

$$\nabla_{c'}(n^e n_{d'})$$

$$\to h^a{}_{a'} h^c{}_{c'} h^{d'}{}_d K^{a'}{}_{c'} [\underbrace{n_f \nabla_{d'} V^f}]$$

$$- V^f \underbrace{\nabla_{d'} n_f}_{K_{d'f}}$$

$$\cancel{n^{a'} K_{c'f}} + \underline{n_f K^{a'}{}_{c'}}$$

$$= h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d R^{a'}{}_{b'c'd'} V^{b'}$$

$$- K^a{}_c K_{db} V^b + K^a{}_d K_{cb} V^b.$$

e.g. S^2 $\quad \underline{n} = \frac{\partial}{\partial r}$

$$\sigma^A = \theta, \varphi$$

$$\underset{\sim}{x}(\sigma^A) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \underline{n}$$

$$\underline{X}_{,\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\underline{X}_{,\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$X^i_{,A} X^j_{,B} \delta_{ij}$$

$$\gamma_{AB} = \underline{X}_{,A} \cdot \underline{X}_{,B}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

$$K_{AB} = -\underline{n} \cdot \underline{X}_{;AB}$$

$$\uparrow$$

$$\left(\underline{X}_{,AB} - \Gamma^C_{AB} \underline{X}_{,C}\right)$$

$\frac{\partial}{\partial r}$

$\ldots s\varphi, \sin\theta \sin\varphi, \cos\theta) = \underline{n}$

$\ldots os\varphi, \cos\theta \sin\varphi, -\sin\theta)$

$\ldots \sin\varphi, \sin\theta \cos\varphi, 0)$

$$X^i_{,A} X^j_{,B} \delta_{ij}$$

$$\gamma_{AB} = \underline{X}_{,A} \cdot \underline{X}_{,B}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

$$K_{AB} = -\underline{n} \cdot \underline{X}_{;AB}$$

$$\uparrow$$

$$\left(\underline{X}_{,AB} - \Gamma^{C}_{AB} \underline{X}_{,C} \right)$$

$$\underline{X}_{;\theta\theta} = \underline{X}_{,\theta\theta} = -\underline{n}$$

$$\underline{X}_{;\theta\varphi} = \underline{X}_{,\theta\varphi} - \Gamma^{\varphi}_{\theta\varphi} \underline{X}_{\varphi} = 0$$

$$\underline{X}_{,\varphi\varphi} = \underline{X}_{,\varphi\varphi} - \underset{?\;-\sin\theta\cos\theta}{\Gamma^{\theta}_{\varphi\varphi}} \underline{X}_{\theta}$$

$$X^i_{\,;A} X^j_{\,;B}\,\delta_{ij}$$

$B.$

$$\begin{bmatrix} \ldots & 0 \\ \ldots & \ldots n^2\theta \end{bmatrix}$$

$;AB.$

$$\ldots_{B} - \Gamma^{C}_{AB} \underline{X}_{,C} \Big)$$

$$\underline{X}_{;\theta\theta} = \underline{X}_{,\theta\theta} = -\underline{n}$$

$$\underline{X}_{;\theta\varphi} = \underline{X}_{,\theta\varphi} - \Gamma^{\varphi}_{\theta\varphi} \underline{X}_{\varphi} = 0$$

$$\underline{X}_{;\varphi\varphi} = \underline{X}_{,\varphi\varphi} - \underbrace{\Gamma^{\theta}_{\varphi\varphi}}_{-\sin\theta\cos\theta} \underline{X}_{\theta} = -\sin^2\theta\,\underline{n}$$

$$K_{AB} = -\underline{n} \cdot \begin{bmatrix} -\underline{n} & \underline{0} \\ \underline{0} & -\sin^2\theta\,\underline{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} = \gamma_{AB}$$

Hence

$$R^A{}_{BCD} = \delta^A_C \gamma_{BD} - \delta^A_D \gamma_{BC}$$

(no contribution from $\underbrace{R^a{}_{bcd}}_{\mathbb{R}^3}$)