

Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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LECTURE 5

Kaluza-Klein
compactification

Introduce an extra dimension;
consider a scalar on $\mathbb{R}^4 \times S^1$

$$\Phi(x^\mu, x) \quad ds^2 = ds_4^2 - L^2 dx^2$$

4D extra dim

$$\square_5 \Phi = \square_4 \Phi - \frac{1}{L^2} \frac{\partial^2 \Phi}{\partial x^2}$$

Fourier decompose: $\Phi = \varphi_n(x^\mu) e^{inx}$

$$\square_5 \Phi = e^{in\chi} \left[\square_4 \varphi_n + \frac{n^2}{L^2} \varphi_n \right]$$

↑
mass term

From 4D perspective, the Fourier modes have masses $\propto 1/L$. If L is "small", modes that depend on χ are very massive.

Low energy theory indep. of $\chi \Leftrightarrow \frac{\partial}{\partial \chi}$ Killing
vec

$$g_{ab} = \begin{bmatrix} \tilde{g}_{\mu\nu} & g_{\mu 5} \\ g_{\nu 5} & g_{55} \end{bmatrix} \quad g_{ab, \chi} = 0$$

Encode as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - e^{2\sigma} [d\chi + A_\mu dx^\mu]^2$$

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Encode as

$$ds^2 = g_{4\mu\nu} dx^\mu dx^\nu - e^{2\sigma} [d\chi + A_\mu dx^\mu]^2$$

$$\text{so } \det g_5 = e^{2\sigma} \det g_4$$

σ , A_μ , $g_{\mu\nu}$ depend only on X^μ .

↑ scalar ↑ vector ↑ tensor

To compute curvature, use Cartan (handout)
e.g. connection

$$\underline{\omega}^{\hat{a}} = e^{\hat{a}}{}_{\mu} dx^{\mu}$$

$$\underline{\omega}^S = e^S{}_{\mu} [dx^{\mu} + A_{\mu} dx^{\mu}]$$

off-diagonality

$$\begin{aligned} d\underline{\omega}^{\hat{a}} &= d e_{\mu}^{\hat{a}} \wedge dx^{\mu} = - \Theta_{\hat{a}}^{\hat{b}} \wedge \underline{\omega}^{\hat{b}} \\ &= - \Theta_{\hat{a}}^{\hat{b}} \wedge \underline{\omega}^{\hat{b}} - \Theta^{\hat{a}}_5 \wedge \underline{\omega}^5 \end{aligned}$$

cut)

ality

$$\begin{aligned} d\underline{\omega}^5 &= d\sigma \wedge \underline{\omega}^5 + e^{\sigma} dA \\ &= \sigma_{\hat{a}} \wedge \underline{\omega}^{\hat{a}} \wedge \underline{\omega}^5 + e^{\sigma} F \\ &= -\Theta^5_{\hat{a}} \wedge \underline{\omega}^{\hat{a}} \end{aligned}$$

$$-\partial_0 \hat{a} \hat{b}_n \omega^{\hat{b}}$$

$$\hat{a} \uparrow$$

$$s \wedge \omega^s$$

Solving simultaneous eqns:

$$\partial^s \hat{a} = \sigma_{,a} \omega^s + \frac{1}{2} e^\sigma F_{ab} \omega^{\hat{b}}$$

$$\partial^{\hat{a}} \hat{b} = \partial_0^{\hat{a}} \hat{b} + \frac{1}{2} e^\sigma F^{\hat{a}}{}_{\hat{b}} \omega^s$$

Eventually, get

$$R^s{}_s = -(\partial\sigma - \partial\sigma)^2 - \frac{1}{4} e^{2\sigma} F^2$$

$$R^{\hat{a}}{}_{\hat{b}} = R_0^{\hat{a}}{}_{\hat{b}} + \frac{1}{4} e^{2\sigma} F^{\hat{a}\hat{c}} F_{\hat{b}\hat{c}} - \nabla^{\hat{a}} \nabla_{\hat{b}} \sigma - \sigma^{\hat{a}}{}_{,\hat{b}}$$

$$R = R_0 - 2 \square \sigma - 2(\nabla_0 \sigma)^2 + \frac{1}{4} e^{2\sigma} F^2$$

$$\Rightarrow \sqrt{g_5} R_5 = \boxed{e^\sigma} \sqrt{g_4} \left[R_0 - 2e^{-\sigma} \square e^\sigma + \frac{1}{4} e^{2\sigma} F^2 \right]$$

- this is the gravitational Lagrangian
"Einstein", no kinetic term for σ , F^2

New

σ

$$2(\nabla\sigma)^2 + \frac{1}{4}e^{2\sigma}F^2$$

$$\left[R_0 - 2e^{-\sigma} \square e^{\sigma} + \frac{1}{4}e^{2\sigma}F^2 \right]$$

gravitational Lagrangian

kinetic term for σ , F^2

New 4D physics

$\sigma \leftrightarrow$ scalar, "breathing mode"
(goes size of S^1) - modulus.

$A_\mu \leftrightarrow$ vector (almost) Maxwell

New 4D physics

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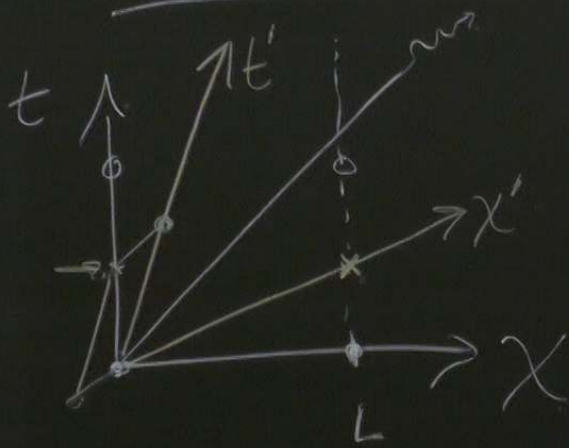
Klein fixed σ to get Einstein-Maxwell
Moduli stabilization

ω / $R_b - R_b \cdot 4 \dots$

hing mode"
dulus.
ost) Maxwell
- Einstein-Maxwell
stabilization

Action is not quite GR, it is
Scalar-tensor gravity.
MODIFIED GRAVITY

Rest Frame



x compact $\Rightarrow x \sim x + L$
at const. t

$$t' = \gamma(t - vx)$$

$$x' = \gamma(x - vt)$$

Identification is

$$t' \sim t' - v\gamma L$$

$$x' \sim x' + \gamma L$$

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A_μ cannot be
gauged away

Leads to electric & magnetic fields
e.g. SD black string

$x+L$
st. t

$$g_{zz} = -\frac{1}{1-v^2} \left[1 + \frac{2GMv^2}{r} - v^2 \right] = - \left[1 + \frac{2GMv^2}{(1-v^2)r} \right]$$

$$ds^2 = \left(\frac{1-2GM}{(1-v^2)r+2GMv^2} \right) dt^2 - \frac{dr^2}{1-\frac{2GM}{r}} - r^2 d\Omega^2$$

$$\frac{+vdt}{1-v^2}$$

$$- \left(1 + \frac{2GMv^2}{(1-v^2)r} \right) \left[dz + \frac{2GMv dt}{(1-v^2)r+2GMv^2} \right]^2$$

A.

V₅₀-0 0,1
16

LS

$$\hat{r} = r + \frac{2GMv^2}{1-v^2} \quad Q = \frac{2GMv}{1-v^2}$$

$$\hat{M} = \frac{2GM}{1-v^2}$$

$$\underline{A} = \frac{Q}{\hat{r}} \underline{dt}$$

ELECTRIC
KK B.H.

$$e^{2\sigma} = \frac{\hat{r}}{\hat{r} - vQ}$$

$$ds^2 = \left(1 - \frac{2GM}{\hat{r}}\right) dt^2 - \frac{(\hat{r} - vQ)}{\left(\hat{r} - \frac{2GM}{\hat{r}}\right)} dr^2 - (\hat{r} - vQ) d\Omega^2 - e^{2\sigma} [dz + A]^2$$

Horizon at $2r_M$,
but area of spheres
not $4\pi r^2$. Extremal
limit $v \rightarrow 1$

MAGNETIC CHARGE?

The electromagnetic e-m tensor

$$T^a_b = -F^{ac}F_{bc} + \frac{1}{4}F^c{}_c\delta^a_b$$

is invariant under $F \rightarrow *F$

$$\frac{Q}{r^2} dt \wedge dr \xrightarrow{*} Q \sin\theta d\theta \wedge d\phi$$

MAGNETIC MONOPOLE.

$$d(Q \sin\theta d\theta \wedge d\phi) = 0$$

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MAGNETIC MONOPOLE.

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Is $\underline{F} = d\underline{A}$?

$$A_\varphi = Q \cos\theta d\varphi$$

singular at $\theta = 0, \pi$.

$$A_N = Q(1 - \cos\theta) d\varphi$$

$$A_S = -Q(1 + \cos\theta) d\varphi$$

Regular at N, S pole

$\underline{F} = d\underline{A}$, but not globally.

2 gauge patches, N & S.

$$\underline{A}_S = \underline{A}_N - 2Q d\varphi$$

$$-\nabla^{\hat{a}} \nabla_{\hat{b}} \sigma - \sigma \nabla^{\hat{a}} \nabla_{\hat{b}} \sigma$$

it is

$\underline{F} = \underline{dA}$, but not globally.

2 gauge patches, N & S.

$$\underline{A}_S = \underline{A}_N - 2Q \underline{d\varphi}$$

Insert into KK perspective.

2 patch
[dφ]

$$(r-2GM)$$

$$-e^{2\sigma} [dz + A]^2$$

A_- , but not globally.
gauge patches, N & S.

$$A_S = A_N - 2Q d\phi$$

into KK perspective.

2 patches

$$[d\psi_N + Q(1 - \cos\theta) d\phi]$$

$$\& [d\psi_S - Q(1 + \cos\theta) d\phi]$$

$$\psi_S = \psi_N - 2Q\phi$$

Regular at r_+

The soln is

$$ds^2 = \left(\frac{r-r_+}{r-r_-} \right) dt^2 - \frac{dr^2}{1-\frac{r_+}{r}} - r(r-r_-) d\Omega_{II}^2 - \left(1-\frac{r_-}{r} \right) \left[d\psi_N + \sqrt{\frac{r_-}{r(r-r_-)}} (1-\cos\theta) d\varphi \right]^2$$

For coord transform on overlap of N & S patches to be well defined

$$\begin{aligned} \psi_S &= \psi_N + 2Q\varphi = \psi_N + 2Q(\varphi + 2\pi) \\ &= \psi_S + \underline{4\pi Q} \end{aligned}$$

$\theta) d\varphi]$
 $s\theta) d\varphi]$

$$\underline{\omega}^5 = e^\sigma [d\chi + A_\mu dx^\mu]$$

$$= -\Theta^5 \hat{a}$$

Extremal limit $r_+ = r_-$

$$ds^2 = dt^2 - \frac{dr^2}{1-a^2/r} - \left(1 - \frac{Q}{r}\right) \left[r^2 d\Omega_{II}^2 + Q^2 \left(\frac{d\psi}{Q} + (1 - \cos\theta) d\phi \right)^2 \right]$$

$$p^2 = 4Q(r-Q) \downarrow dp^2 \quad \underbrace{\frac{p^2}{4} \left[d\theta^2 + \sin^2\theta d\phi^2 + \left[d\chi + (1 - \cos\theta) d\psi \right]^2 \right]}_{\text{Metric on } S^3}$$

FLAT SPACETIME

Metric on S^3

Kaluza-Klein monopole

$$X + iY = \rho \cos \theta/2 e^{i\chi/2}$$

$$Z + iW = \rho \sin \theta/2 e^{i(\varphi + \chi/2)}$$

$$X^2 + Y^2 + Z^2 + W^2 = \rho^2$$

Kaluza-Klein monopole

$$\left. (1 - \cos\theta) d\varphi \right\}^z$$

$$X + iY = \rho \cos\theta/2 e^{i\chi/2}$$

$$Z + iW = \rho \sin\theta/2 e^{i(\varphi + \chi/2)}$$

$$X^2 + Y^2 + Z^2 + W^2 = \rho^2$$

$$\left. dX + (1 - \cos\theta) d\varphi \right\}^z$$

S^3

Hopf Fibration