Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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$$\underset{\sim}{\epsilon} = \frac{1}{n!}\epsilon_{\mu_1\ldots\mu_n}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$$

$$= \frac{1}{n!}\underbrace{\det\left(\frac{\partial x}{\partial x'}\right)}_{\text{weight}}\epsilon_{\mu'_1\ldots\mu'_n}\, dx'^{\mu'_1}\wedge\cdots\wedge dx'^{\mu'_n}$$

HODGE DUAL. $*: \Lambda^p \to \Lambda^{n-p}$ $\quad \underline{A} \to *\underline{A} \leftarrow \frac{1}{p!}\epsilon_{a_1\ldots a_{n-p}}{}^{b_1\ldots b_p} A_{b_1\ldots b_p}$

$\underline{d}: \Lambda^p \to \Lambda^{p+1}$ s.t. reduces to df on fn, Leibnitz $d\, A^{(p)}\wedge B^q = dA\wedge B$ $(-)^p A\wedge dB$

AND $d^2=0$ $\quad \delta = *d*$

$$\langle d\underline{\omega} \,|\, \underline{u},\underline{v}\rangle = \underline{u}(\langle\underline{\omega}|\underline{v}\rangle) - \underline{v}(\langle\underline{\omega}|\underline{u}\rangle) - \langle\underline{\omega}\,|\,[\underline{u},\underline{v}]\rangle$$

$$\underset{\sim}{\epsilon} = \frac{1}{n!}\epsilon_{\mu_1\ldots\mu_n}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$$

$$= \frac{1}{n!}\underbrace{\det\left(\frac{\partial x}{\partial x'}\right)}_{\text{weight}}\epsilon_{\mu'_1\ldots\mu'_n}\, dx'^{\mu'_1}\wedge\cdots\wedge dx'^{\mu'_n}$$

$$\underset{\sim}{\varepsilon} = \sqrt{|g|}\,\underset{\sim}{\epsilon}$$

HODGE DUAL. $* : \Lambda^p \to \Lambda^{n-p}$ $\quad \underline{A} \to *\underline{A} \leftarrow \frac{1}{p!}\epsilon_{a_1\ldots a_{n-p}}{}^{b_1\ldots b_p} A_{b_1\ldots b_p}$

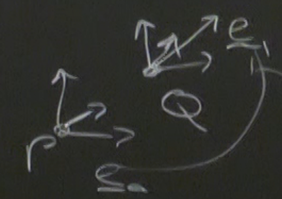
$\underline{d} : \Lambda^p \to \Lambda^{p+1}$ s.t. reduces to df on fn, Leibnitz $d\,A^{(p)}\wedge B^q = dA\wedge B + (-)^p A\wedge dB$

AND $\underline{d}^2 = 0$ $\qquad \delta = *d*$

$$\longrightarrow \langle d\underline{\omega} | \underline{u}, \underline{v}\rangle = \underline{u}(\langle\underline{\omega}|\underline{v}\rangle) - \underline{v}(\langle\underline{\omega}|\underline{u}\rangle) - \langle\underline{\omega}|[\underline{u},\underline{v}]\rangle$$

Lecture 3 Cartan's structural eqns

Recall $$\underset{\text{deriv}}{\nabla}\, \underset{\text{basis}}{\underset{\sim}{e}_b} = \underbrace{\Gamma^c{}_{ab}}_{\text{scalars}}\, \underset{\sim}{e}_c \otimes \underset{\text{co-tgt basis}}{\underset{\sim}{\omega}^a}$$



In GR we take a metric connection $\underset{\sim}{\nabla}\, g = 0$.

also usually take a torsion-free connection.

$\underline{\nabla}$ is a derivation which

i) commutes with contractions

ii) Leibnitzian

iii) reduces to d on fns.

torsion.

$$\underset{\sim}{T}(\underline{u},\underline{v}) = \underset{(u^a\nabla_a)}{\nabla_{\underline{u}}}\underline{v} - \nabla_{\underline{v}}\underline{u} - [\underline{u},\underline{v}]$$

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} - C^a{}_{bc}$$

derivation which

mmutes with contractions

ibnitzian

educes to d on fns.

$$C^a{}_{bc} = \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$$

are the structure constants of the basis.

$$\ldots \underline{v}) = \nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} - [\underline{u}, \underline{v}]$$

$$(u^a \nabla_a)$$

$$\ldots c = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a{}_{bc}$$

Defn The connection 1-forms / spin connection are

$$\underset{\sim}{\theta}^{a}{}_{c} = \Gamma^{a}{}_{bc}\,\underset{\sim}{\omega}^{b}$$

Then $\underset{\sim}{\theta}_{ac} + \underset{\sim}{\theta}_{ca} = \underset{\sim}{d}\,\underbrace{g_{ac}}_{\text{symm}}$ for metric connection

$$\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\nabla}\, g_{ab} = \underset{\sim}{\nabla}\left(\langle \underset{\sim}{g} \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle\right)$$

scalars

$$= \langle \underset{\sim}{g} \,|\, \nabla \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle + \langle \underset{\sim}{g} \,|\, \underset{\sim}{e}_a, \nabla \underset{\sim}{e}_b \rangle$$

$$= \underset{\sim}{\omega}^d \langle \underset{\sim}{g} \,|\, \Gamma^c_{da}\, \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle + \underset{\sim}{\theta}^c{}_b \langle \underset{\sim}{g} \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_c \rangle$$

$$= \underset{\sim}{\theta}^c{}_a\, g_{cb} + \underset{\sim}{\theta}^c{}_b\, g_{ac}$$

$$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$$

$_a, e_b \rangle \big)$

$g | e_a \nabla e_b \rangle$

$\rangle + \theta^c{}_b \langle g | e_a, e_c \rangle$

$^c{}_b \, g_{ac}$

Commonly use an o/n basis: $g_{ab} = \eta_{ab}$

So $d g_{ab} = 0$

$\Rightarrow \theta_{ab}$ is antisymm.

for metric connection $\qquad \overline{\text{symm}}$

Cartan's 1st eqn

Define torsion 2-form

$$\underset{\sim}{T}^a = \frac{1}{2} T^a{}_{bc}\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c = \frac{1}{2}\left(\Gamma^a{}_{bc} - \Gamma^a{}_{cb} - C^a{}_{bc}\right) \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$= \underbrace{\Gamma^a{}_{bc}\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c}_{\underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c} - \frac{1}{2} \langle \underset{\sim}{\omega}^a | [\underset{\sim}{e}_b, \underset{\sim}{e}_c] \rangle\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c .$$

...hat $\underline{\Theta}_{ab}$

Now use:

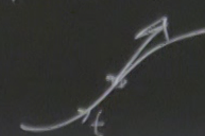
$$\langle d\underline{\omega}^a \,|\, \underline{e}_b, \underline{e}_c\rangle = \underline{e}_b\big(\cancel{\underbrace{\langle \underline{\omega}^a|\underline{e}_c\rangle}_{\text{by defn } \delta^a_c}}\big) - \underline{e}_c\big(\cancel{\langle\underline{\omega}^a|\underline{e}_b\rangle}\big) - \langle \underline{\omega}^a \,|\, [\underline{e}_b,\underline{e}_c]\rangle$$

components $(d\underline{\omega}^a)_{bc}$

$$\Rightarrow \underline{T}^a = \underline{\Theta}^a{}_c \wedge \underline{\omega}^c + \frac{1}{2}\boxed{\langle d\underline{\omega}^a \,|\, \underline{e}_b,\underline{e}_c\rangle}\,\underline{\omega}^b\wedge\underline{\omega}^c$$

$$= d\underline{\omega}^a + \underline{\Theta}^a{}_c\wedge\underline{\omega}^c$$

1st Structural eqn



$$I f = \frac{df}{dt}$$

Curvature

Defined as a commutator of derivs

$$R(\underline{u},\underline{v})\,\underline{w} = \left(\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right)\underline{w}$$

Tri linear map $T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$

cpts

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}.$$

of derivs

$\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]})\underline{W}$

$\times T_p(M) \to T_p(M)$

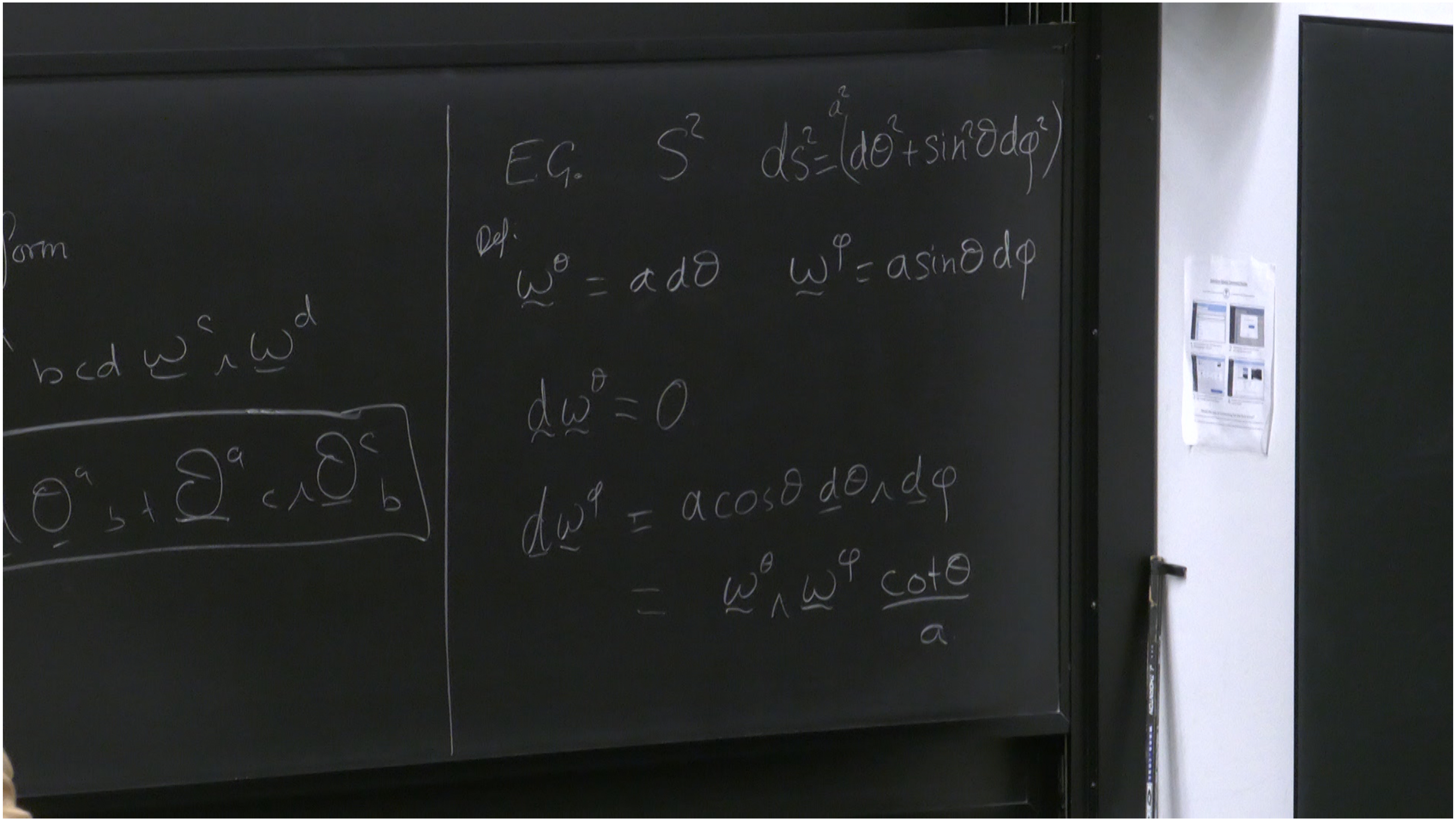
$\Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}.$

Cartan's 2nd eqn

Define curvature 2-form

$$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$

2nd eqn. $\boxed{\underline{R}^a{}_b = \underline{d}\,\underline{\Theta}^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b}$


form
$bcd\ \underset{\sim}{\omega}^c \wedge \underset{\sim}{\omega}^d$
$\Theta^a{}_b + \Theta^a{}_c \wedge \Theta^c{}_b$
E.G. S^2 $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
Def: $\underset{\sim}{\omega}^\theta = a\, d\theta$ $\underset{\sim}{\omega}^\varphi = a \sin\theta\, d\varphi$
$d\underset{\sim}{\omega}^\theta = 0$
$d\underset{\sim}{\omega}^\varphi = a \cos\theta\, d\theta \wedge d\varphi$
$= \underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^\varphi \frac{\cot\theta}{a}$

Get $\underset{\sim}{\Theta}^{\varphi}{}_{\theta} = \frac{1}{a}\cot\theta\, \underline{\omega}^{\varphi} = \cos\theta\, d\varphi$

$\underline{R}^{\varphi}{}_{\theta} = \underline{d}\,\underline{\Theta}^{\varphi}{}_{\theta} = -\sin\theta\, d\theta \wedge d\varphi$

$= -\underline{\omega}^{\theta} \wedge \underline{\omega}^{\varphi} / a^2$

$\Rightarrow R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{a^2}$ (in o/n basis)

only 1 indep cpt.

$\ldots s\theta d\varphi$

$d\varphi$

a^2

For coord basis

$$R^{\mu}{}_{\nu\lambda\sigma} = e_a{}^{\mu}\,\omega_\nu{}^b\,\omega_\lambda{}^c\,\omega_\sigma{}^d\,R^a{}_{bcd}$$

$$\rightarrow R^{\varphi}{}_{\theta\varphi\theta} = 1$$

$$R_{abcd} = \frac{1}{a^2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

(Const curv)

$R^{a}{}_{bcd}$

EG2 Static, sph symm

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

① o/n basis:

basis for subspace

$$\underset{\sim}{\omega}^{t} = A\,dt \;;\; \underset{\sim}{\omega}^{r} = B\,dr \;;\; \underset{\sim}{\omega}^{\hat{\alpha}} = C\,\underset{\sim}{\omega}_0{}^{\alpha}$$

$$d\underset{\sim}{\omega}^{t} = A'\,dr \wedge dt = -\frac{A'}{AB}\,\underset{\sim}{\omega}^{\hat{t}} \wedge \underset{\sim}{\omega}^{\hat{r}}$$

$$d\omega^{r} = 0$$

$$\underset{\sim}{d}\,\underset{\sim}{\omega}^{\alpha} = C'\,\underset{\sim}{d}r \wedge \underset{\sim}{\omega}_0^{\alpha} + C\,\underset{\sim}{d}\underset{\sim}{\omega}_0^{\alpha}$$

$$= -\frac{C'}{CB}\,\underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{r} - C\,\underset{\sim}{\Theta}_0{}^{\alpha}{}_{\beta} \wedge \underset{\sim}{\omega}_0^{\beta}.$$

Extract: $\underset{\sim}{\Theta}^{t}{}_{r} = \dfrac{A'}{AB}\,\underset{\sim}{\omega}^{t}$ $\quad \underset{\sim}{\Theta}^{\alpha}{}_{r} = \dfrac{C'}{CB}\,\underset{\sim}{\omega}^{\alpha}$

$$\underset{\sim}{\Theta}^{\alpha}{}_{\beta} = \underset{\sim}{\Theta}_0{}^{\alpha}{}_{\beta}.$$

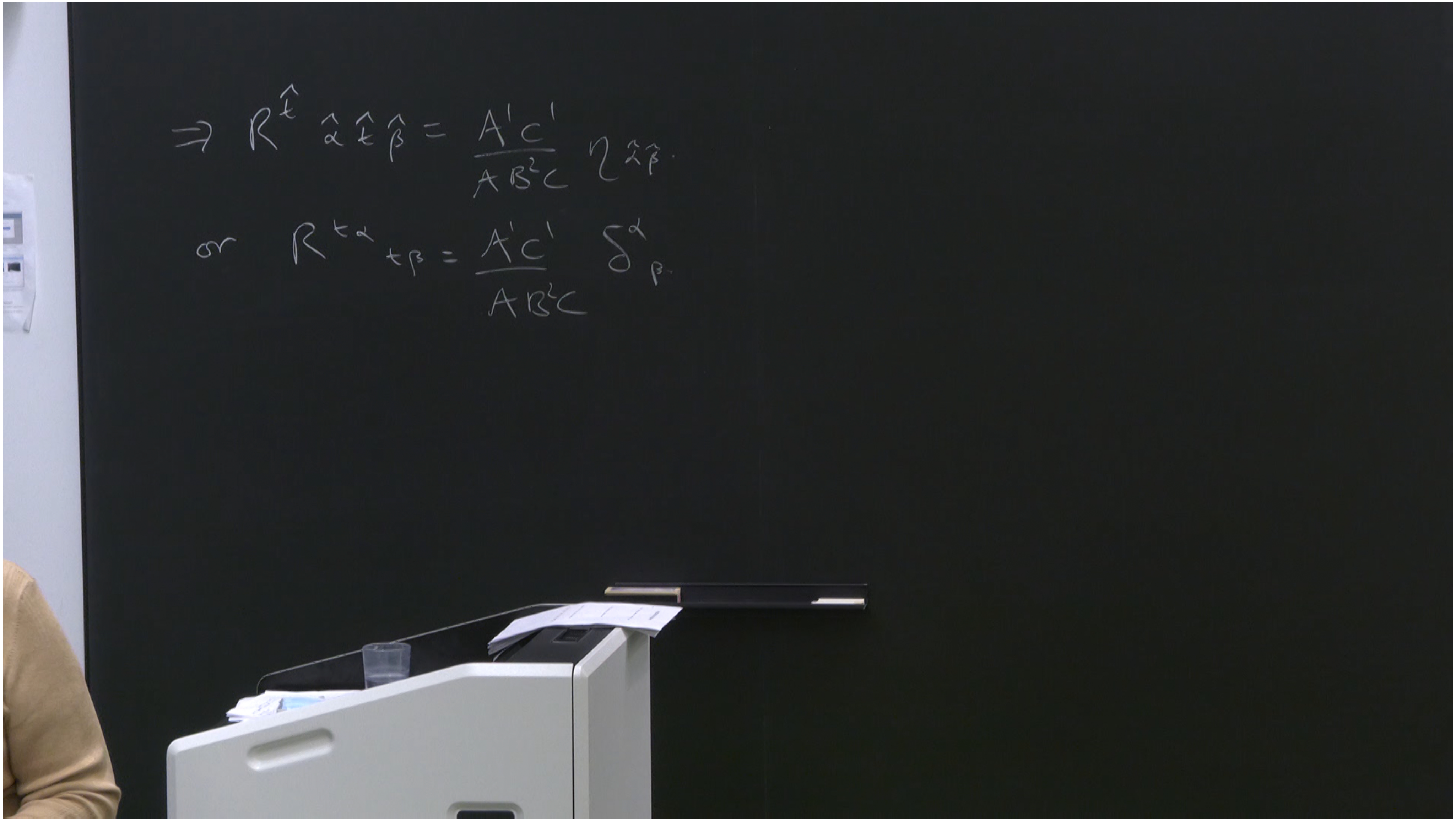
② Curv

$$\underline{R}^t{}_r = d\,\underline{\theta}^t{}_r = d\left(\frac{A'}{B}\,dt\right) = \left(\frac{A'}{B}\right)' \underline{dr}\wedge\underline{dt} \rightarrow R^t{}_{\hat r \hat r \hat t} = \frac{1}{AB}\left(\frac{A'}{B}\right)'$$

or $R^t{}_{rrt} = \frac{B}{A}\left(\frac{A'}{B}\right)'$ coord

$$\underline{R}^t{}_\alpha = \underline{\theta}^t{}_r \wedge \underline{\theta}^r{}_\alpha = -\frac{A'C'}{AB^2C}\,\underline{\omega}^t\wedge\underline{\omega}^\alpha$$

$$\underline{\theta}^r{}_\alpha = -\eta^{rr}\eta_{\alpha\beta}\,\theta^\beta{}_r = -\frac{C'}{CB}\,\underline{\omega}^\alpha$$


$\Rightarrow R^{\hat{t}}{}_{\hat{\alpha}\hat{t}\hat{\beta}} = \frac{A'C'}{AB^2C}\eta_{\hat{\alpha}\hat{\beta}}.$
or $R^{t\alpha}{}_{t\beta} = \frac{A'C'}{AB^2C}\delta^{\alpha}{}_{\beta}.$

for metric connection symm

$$\underline{R}^{\alpha}{}_{\beta} = \underline{d}\,\underline{\Theta}^{\alpha}{}_{\beta} + \underline{\Theta}^{\alpha}{}_{\gamma} \wedge \underline{\Theta}^{\gamma}{}_{\beta} + \underline{\Theta}^{\alpha}{}_{\gamma} \wedge \underline{\Theta}^{\gamma}{}_{\beta}$$

$$= \underline{d}\,\underline{\Theta}_{0}{}^{\alpha}{}_{\beta} - \frac{c'^2}{c^2 B^2}\,\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \qquad \underline{\Theta}_{0}{}^{\alpha}{}_{\gamma} \wedge \underline{\Theta}_{0}{}^{\gamma}{}_{\beta}.$$

$$= \underline{R}_{0}{}^{\alpha}{}_{\beta} - \frac{c'^2}{c^2 B^2}\,\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta}$$

$\Theta_{ba} + \Theta_{ab}$

$$R^{\alpha\beta}{}_{\gamma\delta} = \left(\frac{C'^2}{C^2 B^2} - \frac{1}{C^2}\right)\left(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma\right)$$

In 4D.

$$R^t{}_t = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + \overset{D-2}{2}\frac{A'C'}{AC}\right)$$

$$R^r{}_r = \frac{1}{B^2}\left(\frac{A''}{A} + 2\frac{C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + \overset{D-2}{2}\frac{C'}{C}\right)\right)$$

$$R^\alpha{}_\beta = \left[\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \overset{(D-3)}{}\frac{C'^2}{C^2}\right) - \frac{(D-3)}{C^2}\right]\delta^\alpha_\beta$$