Title: Gravitational Physics Lecture

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Collection: Gravitational Physics

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$$\underline{\underline{\epsilon}} = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$$

$$= \frac{1}{n!}\det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\mu_1\cdots\mu_n}\, dx'^{\mu_1}\wedge\cdots dx^{\mu_n}$$

weight

HODGE DUAL. $*: \Lambda^p \to \Lambda^{n-p}$ $\underline{A} \to *\underline{A} \leftarrow \frac{1}{p!}\epsilon_{a_1\cdots a_{n-p}}{}^{b_1\cdots b_p} A_{b_1\cdots b_p}$

$\underline{d}: \Lambda^p \to \Lambda^{p+1}$ s.t. reduces to df on fns, Leibnitz $d A^{(p)}\wedge B^q = dA\wedge B$ $(-)^p A\wedge dB$

AND $d^2 = 0$ $\delta = *d*$

$$\langle \underline{d\omega} \,|\, \underline{u}, \underline{v}\rangle = \underline{u}(\langle \underline{\omega}|\underline{v}\rangle) - \underline{v}(\langle \underline{\omega}|\underline{u}\rangle) - \langle \underline{\omega} \,|\, [\underline{u},\underline{v}]\rangle$$

$$\underline{\epsilon} = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$$

$$= \frac{1}{n!}\det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\mu_1\cdots\mu_n}\, dx'^{\mu_1}\wedge\cdots dx'^{\mu_n}$$

(weight)

$$\underline{\mathcal{E}} = \sqrt{g}\,\underline{\epsilon}$$

HODGE DUAL. $*: \Lambda^p \to \Lambda^{n-p}$ $\quad \underline{A} \to *\underline{A} \leftarrow \frac{1}{p!}\epsilon_{a_1\cdots a_{n-p}}{}^{b_1\cdots b_p} A_{b_1\cdots b_p}$

$\underline{d}: \Lambda^p \to \Lambda^{p+1}$ s.t. reduces to df on fns, Leibnitz $d A^{(p)}\wedge B^q = dA\wedge B + (-)^p A\wedge dB$

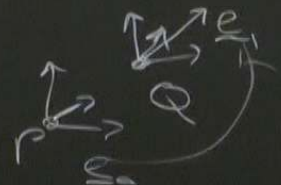
AND $\underline{d}^2 = 0$ $\quad \delta = *d*$

$$\longrightarrow \langle \underline{dw} \,|\, \underline{u},\underline{v}\rangle = \underline{u}\,(\langle \underline{w}|\underline{v}\rangle) - \underline{v}\,(\langle \underline{w}|\underline{u}\rangle) - \langle \underline{w} \,|\, [\underline{u},\underline{v}]\rangle$$

Lecture 3 Cartan's structural eqns

Recall $\nabla \underline{e}_b = \Gamma^c_{ab}\, \underline{e}_c \otimes \underline{\omega}^a$

deriv · basis · scalars · co-tgt basis



In GR we take a metric connection $\nabla g = 0$.

also usually take a torsion-free connection.

∇ is a derivation which

i) commutes with contractions

ii) Leibnizian

iii) reduces to d on fns.

torsion.

$$T(u, v) = \nabla_u v - \nabla_v u - [u, v]$$

$(u^a \nabla_a)$

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$$

derivation which

mutes with contraction

ibnitzian

duces to d on fns.

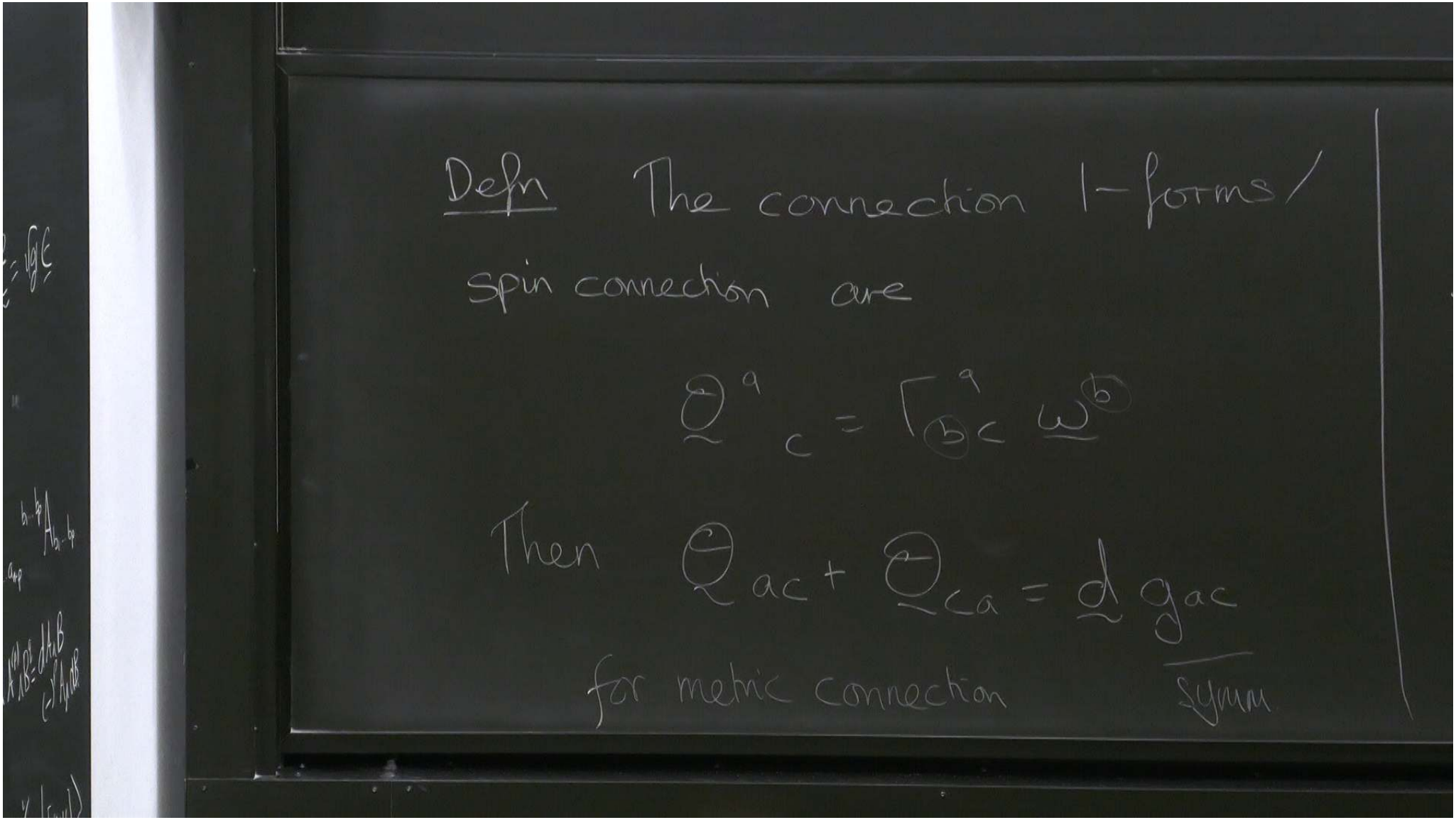
$$C^a_{bc} = \langle \underline{\omega}^a \,|\, [\underline{e}_b, \underline{e}_c] \rangle$$

are the structure constants of the basis.

$$\underline{V}) = \nabla_{\underline{u}} \underline{V} - \nabla_{\underline{v}} \underline{U} - [\underline{U}, \underline{V}]$$

$(u^a \nabla_a)$

$$= \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$$


Defn The connection 1-forms/
spin connection are
$\underset{\sim}{\Theta}{}^{a}{}_{c} = \Gamma^{a}_{bc}\,\underset{\sim}{\omega}{}^{b}$
Then $\underset{\sim}{\Theta}_{ac} + \underset{\sim}{\Theta}_{ca} = \underset{\sim}{d}\, g_{ac}$
for metric connection
symm

$$\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\nabla} g_{ab} = \underset{\sim}{\nabla}\left(\langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle\right)$$

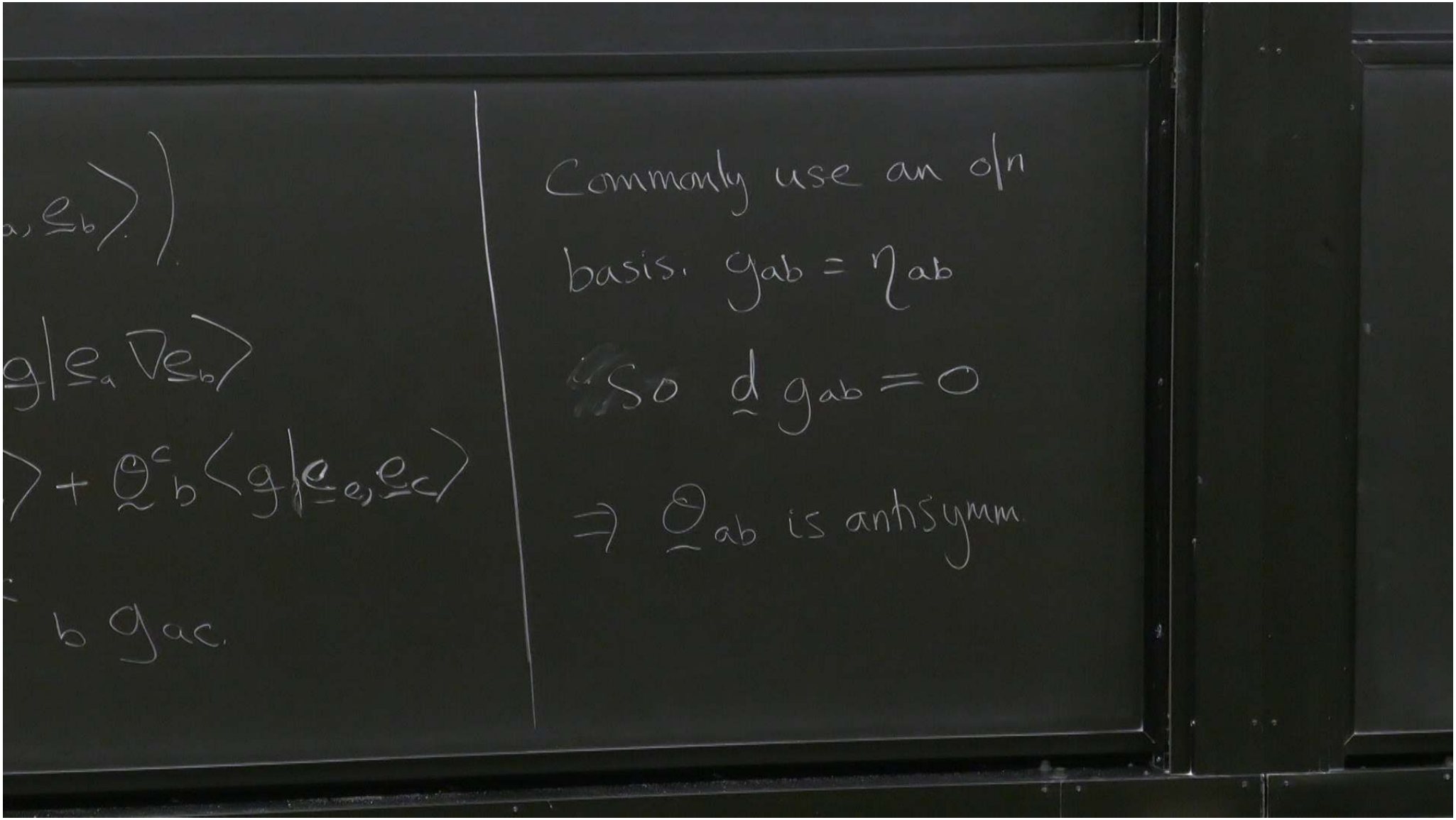
scalars

$$= \langle g | \nabla_{\underset{\sim}{e}_a} \underset{\sim}{e}_b \rangle + \langle g | \underset{\sim}{e}_a \nabla_{\underset{\sim}{e}_b} \rangle$$

$$= \underset{\sim}{\omega}^d \langle g | \Gamma^c_{da}\, \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle + \underset{\sim}{\theta}^c{}_b \langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_c \rangle$$

$$= \underset{\sim}{\theta}^c{}_a\, g_{cb} + \underset{\sim}{\theta}^c{}_b\, g_{ac}$$

$$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$$


Commonly use an o/n
basis. $g_{ab} = \eta_{ab}$
So $\underset{\sim}{d}\, g_{ab} = 0$
$\Rightarrow \underset{\sim}{\theta}_{ab}$ is antisymm

for metric connection $\nabla_{(a}\nabla_{b)} = \frac{\partial}{\partial}$ Symm $= 0$

Cartan's 1st eqn

Define torsion 2-form

$$T^a = \frac{1}{2} T^a_{bc}\, \omega^b \wedge \omega^c = \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}\right) \omega^b \wedge \omega^c$$

$$= \underbrace{\Gamma^a_{(bc)}\, \omega^b \wedge \omega^c}_{\Theta^a{}_c \wedge \omega^c} - \frac{1}{2} \langle \omega^a | [e_b, e_c] \rangle\, \omega^b \wedge \omega^c$$

$\omega^a{}_b \to \Theta_{ab}$

Now use:

$$\langle d\omega^a | e_b, e_c \rangle = e_b(\langle \omega^a | e_c \rangle) - e_c(\langle \omega^a | e_b \rangle) - \langle \omega^a | [e_b, e_c] \rangle$$

by defn $\delta^a{}_c$

components $(d\omega^a)_{bc}$

$$\Rightarrow T^a = \Theta^a{}_c \wedge \omega^c + \frac{1}{2} \langle d\omega^a | e_b, e_c \rangle \, \omega^b \wedge \omega^c$$

$$= d\omega^a + \Theta^a{}_c \wedge \omega^c$$

1st Structural eqn

$I f = \frac{df}{dt}$

Curvature

Defined as a commutator of derivs

$$R(\underline{u},\underline{v})\,\underline{w} = \left(\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right)\underline{w}$$

Tri linear map $T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$

cpks

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}.$$

of derivs

$\nabla_{\underline{v}} - \nabla_{[\underline{v},\underline{w}]})\underline{W}$

$\times T_p(M) \to T_p(M)$

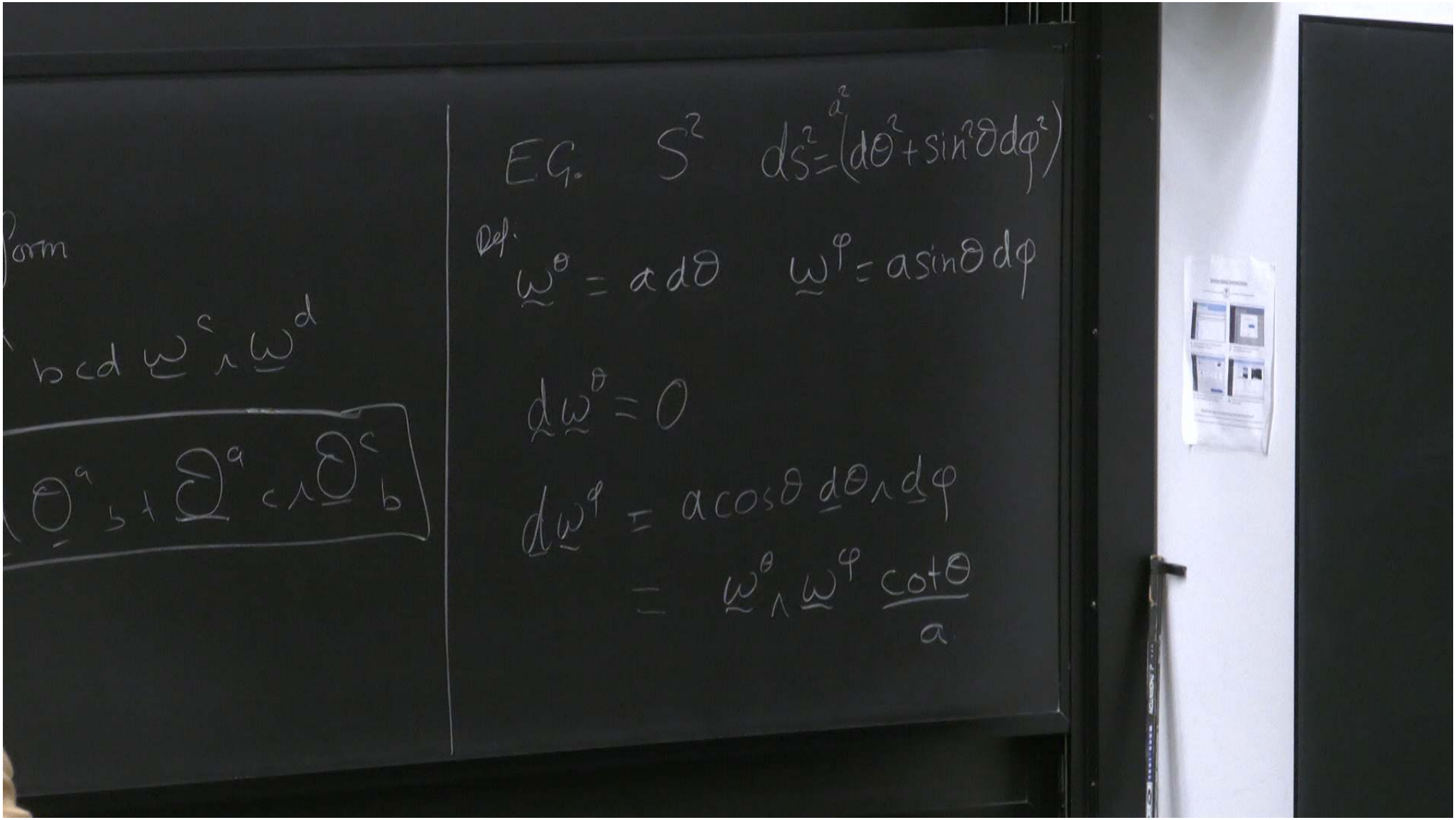
$\Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}.$

Cartan's 2nd eqn

Define curvature 2-form

$$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$

2nd eqn. $\boxed{\underline{R}^a{}_b = \underline{d}\underline{\theta}^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b}$


form
$\omega^c \wedge \omega^d$
$\Omega^a{}_b + \Omega^a{}_c \wedge \Omega^c{}_b$
EG. S^2 $ds^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2)$
Def:
$\underset{\sim}{\omega}^\theta = a\, d\theta$ $\underset{\sim}{\omega}^\varphi = a \sin\theta\, d\varphi$
$d\underset{\sim}{\omega}^\theta = 0$
$d\underset{\sim}{\omega}^\varphi = a\cos\theta\, d\theta \wedge d\varphi$
$= \underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^\varphi \frac{\cot\theta}{a}$

Get $\underset{\sim}{\Theta}{}^{\varphi}{}_{\theta} = \frac{1}{a}\cot\theta\,\underline{\omega}^{\varphi} = \cos\theta\, d\varphi$

$\underline{R}^{\varphi}{}_{\theta} = \underline{d}\,\underline{\Theta}^{\varphi}{}_{\theta} = -\sin\theta\, d\theta \wedge d\varphi$

$= -\underline{\omega}^{\theta} \wedge \underline{\omega}^{\varphi} / a^2$

$\Rightarrow R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{a^2}$ (in o/n basis)

only 1 ind dep cpt.

$\ldots \theta d\varphi$

$d\varphi$

a^2

For coord basis

$$R^{\mu}{}_{\nu\lambda\sigma} = e_a^{\mu}\, \omega_{\nu}^{\ b}\, \omega_{\lambda}^{\ c}\, \omega_{\sigma}^{\ d}\, R^{a}{}_{bcd}$$

$$\rightarrow R^{\theta}{}_{\varphi\theta\varphi} = 1$$

$$R_{abcd} = \frac{1}{a^2}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

(Const curv)

$R^a{}_{bcd}$

Eg 2 Static, sph symm

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r)\left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

① o/n basis:

basis for subspace

$$\underset{\sim}{\omega}^t = A\,dt \;;\; \underset{\sim}{\omega}^r = B\,dr \;;\; \underset{\sim}{\omega}^{\hat{\alpha}} = C\,\underset{\sim}{\omega}_o{}^{\alpha}$$

$$d\underset{\sim}{\omega}^t = A'\,dr \wedge dt = -\frac{A'}{AB}\,\underset{\sim}{\omega}^{\hat{t}} \wedge \underset{\sim}{\omega}^{\hat{r}}$$

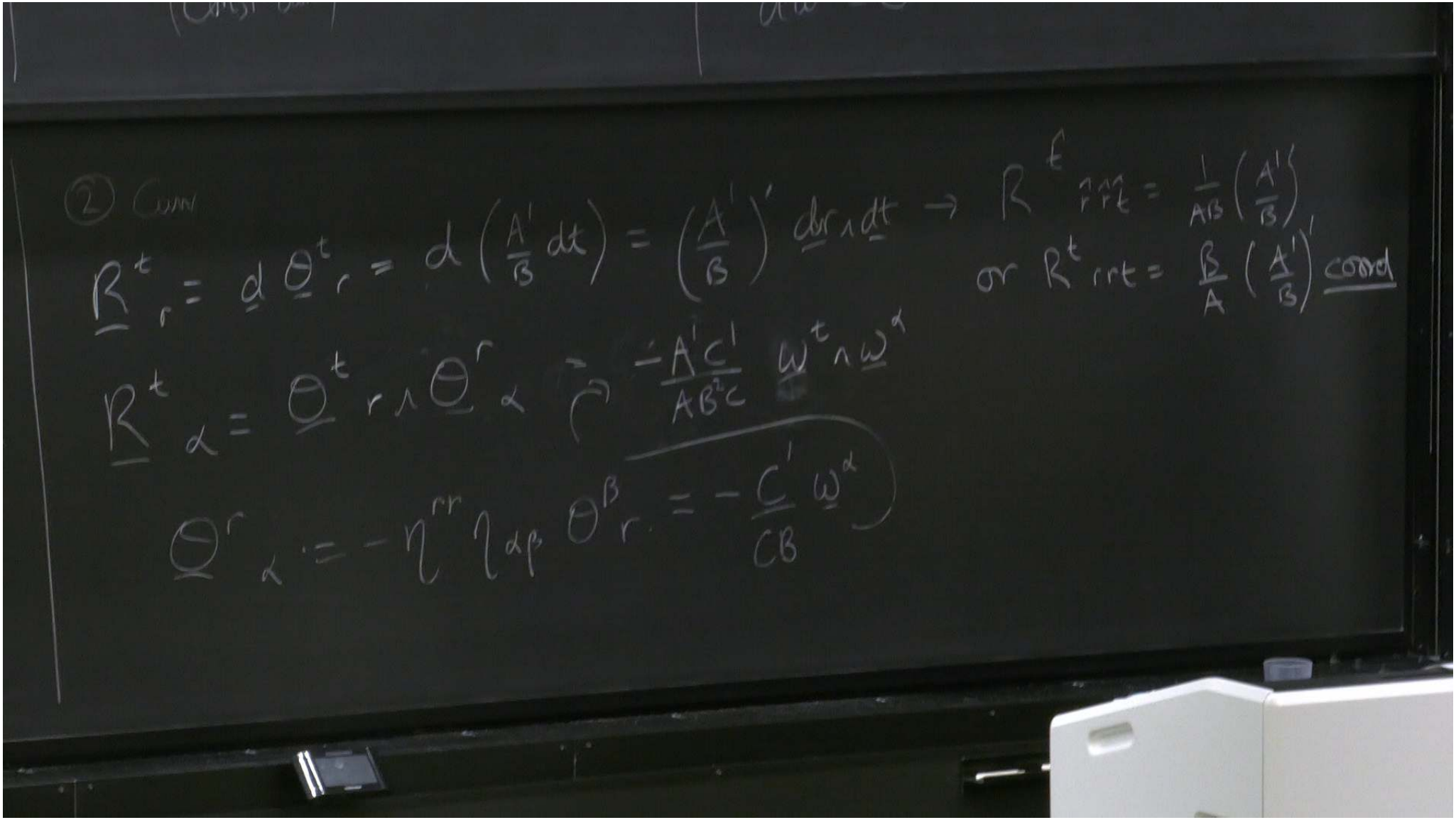
$$d\omega^r = 0$$

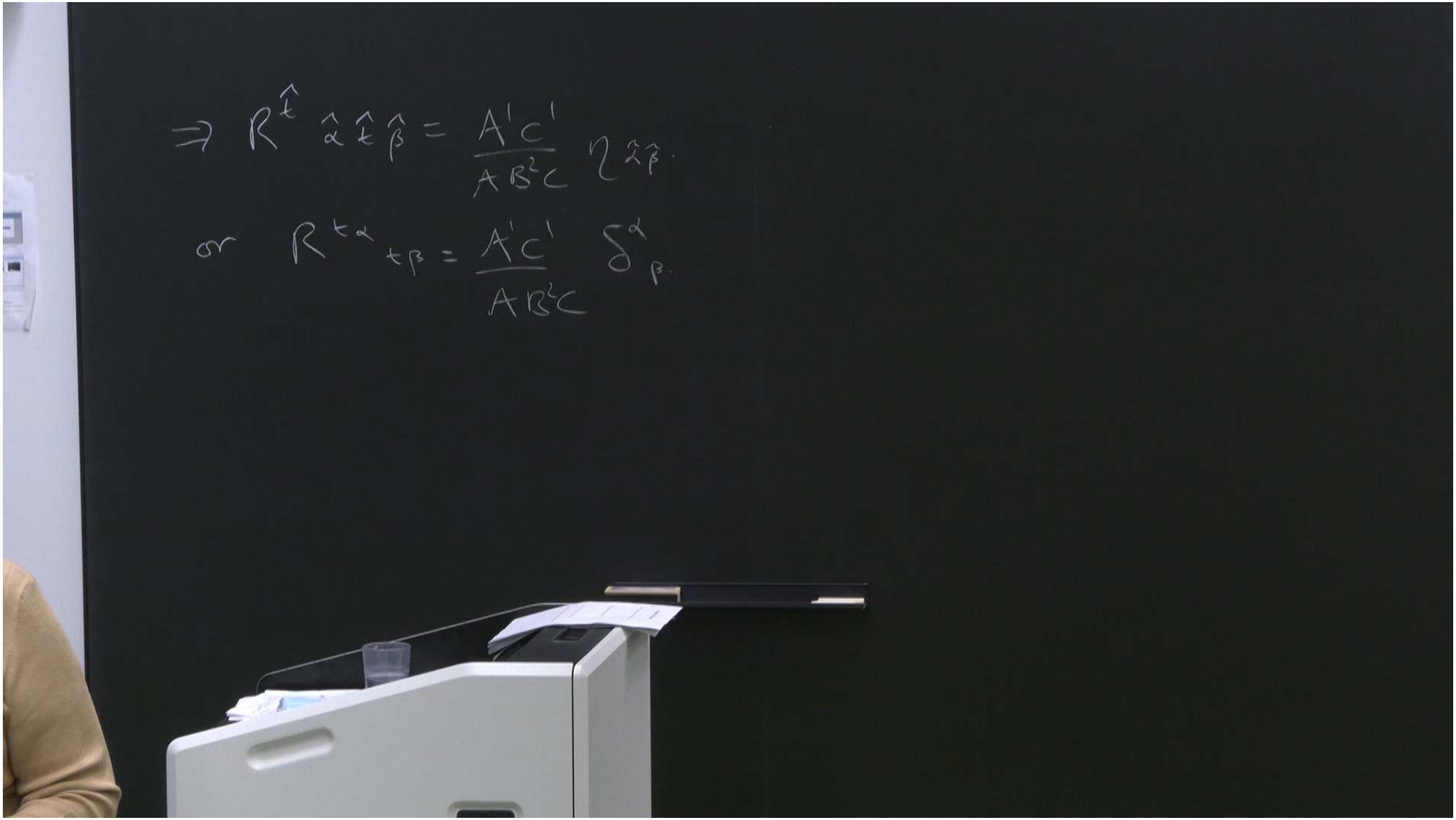
$$\underset{\sim}{d}\,\underset{\sim}{\omega}^{\alpha} = C'\,\underset{\sim}{dr}\wedge\underset{\sim}{\omega}_0^{\alpha} + C\,\underset{\sim}{d}\underset{\sim}{\omega}_0^{\alpha}$$

$$= -\frac{C'}{CB}\,\underset{\sim}{\omega}^{\alpha}\wedge\underset{\sim}{\omega}^{r} - C\,\underset{\sim}{\Theta}_0{}^{\alpha}{}_{\beta}\wedge\underset{\sim}{\omega}_0^{\beta}$$

Extract: $\underset{\sim}{\Theta}{}^{t}{}_{r} = \frac{A'}{AB}\,\underset{\sim}{\omega}^{t}$ $\quad \underset{\sim}{\Theta}{}^{\alpha}{}_{r} = \frac{C'}{CB}\,\underset{\sim}{\omega}^{\alpha}$

$$\underset{\sim}{\Theta}{}^{\alpha}{}_{\beta} = \underset{\sim}{\Theta}_0{}^{\alpha}{}_{\beta}$$


② Conn
$\underline{R}^t{}_r = \underline{d}\,\underline{\theta}^t{}_r = d\left(\frac{A'}{B}dt\right) = \left(\frac{A'}{B}\right)' \underline{dr} \wedge \underline{dt} \rightarrow R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = \frac{1}{AB}\left(\frac{A'}{B}\right)'$
or $R^t{}_{rrt} = \frac{B}{A}\left(\frac{A'}{B}\right)'$ coord
$\underline{R}^t{}_\alpha = \underline{\theta}^t{}_r \wedge \underline{\theta}^r{}_\alpha \rightarrow -\frac{A'C'}{AB^2C}\,\underline{\omega}^t \wedge \underline{\omega}^\alpha$
$\underline{\theta}^r{}_\alpha = -\eta^{rr}\eta_{\alpha\beta}\theta^\beta{}_r = -\frac{C'}{CB}\underline{\omega}^\alpha$


$\Rightarrow R^{\hat{t}}{}_{\hat{\alpha}\hat{t}\hat{\beta}} = \frac{A'C'}{AB^2C}\eta_{\hat{\alpha}\hat{\beta}}$
or $R^{t\alpha}{}_{t\beta} = \frac{A'C'}{AB^2C}\delta^{\alpha}_{\beta}$

for metric connection symm

$$\underline{R}^{\alpha}{}_{\beta} = \underline{d}\,\underline{\Theta}^{\alpha}{}_{\beta} + \underline{\Theta}^{\alpha}{}_{\gamma} \wedge \underline{\Theta}^{\gamma}{}_{\beta} + \underline{\Theta}^{\alpha}{}_{\gamma} \wedge \underline{\Theta}^{\gamma}{}_{\beta}$$

$$= \underline{d}\,\underline{\Theta}_{0}{}^{\alpha}{}_{\beta} - \frac{c'^{2}}{c^{2}B^{2}}\,\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} \qquad \underline{\Theta}_{0}{}^{\alpha}{}_{\gamma} \wedge \underline{\Theta}_{0}{}^{\gamma}{}_{\beta}$$

$$= \underline{R}_{0}{}^{\alpha}{}_{\beta} - \frac{c'^{2}}{c^{2}B^{2}}\,\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta}$$

$\Theta_{bat}\Theta_{ab}$

$$R^{\alpha\beta}{}_{\gamma\delta} = \left(\frac{C'^2}{C^2 B^2} - \frac{1}{C^2}\right)\left(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma\right)$$

In 4D.

$$R^t_t = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + \overset{D-2}{(2)}\frac{A'C'}{AC}\right)$$

$$R^r_r = \frac{1}{B^2}\left(\frac{A''}{A} + 2\frac{C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + \overset{D-2}{(2)}\frac{C'}{C}\right)\right)$$

$$R^\alpha_\beta = \left[\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \overset{D-3}{(D-3)}\frac{C'^2}{C^2}\right) - \frac{\overset{D-3}{(D-3)}}{C^2}\right]\delta^\alpha_\beta$$