Title: Gravitational Physics Lecture

Speakers: Ruth Gregory

Collection: Gravitational Physics

Date: January 10, 2024 - 9:00 AM

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Lecture 2: Lie (& maybe d)

Recall

$$\text{"}\frac{dQ}{dt}\text{"} = \lim_{\delta t \to 0} \frac{Q_{t+\delta t} - \boxed{Q}_t}{\boxed{\delta t}}.$$

1)

2)

Need to "connect" $T_p(\mathcal{M})$ & $T_{p'}(\mathcal{M})$

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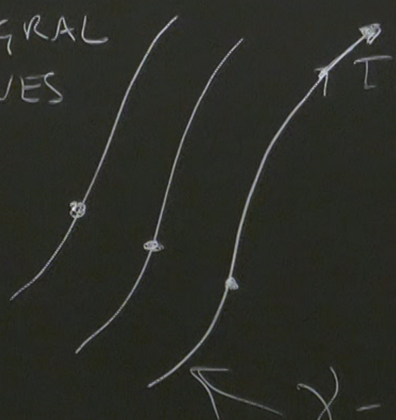
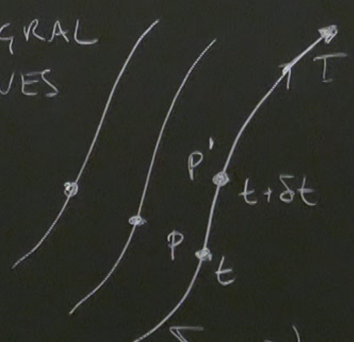
INTEGRAL CURVES



γ - set of curves which have I as their tangent

Need to "connect" $T_p(\mathcal{M})$ & $T_{p'}(\mathcal{M})$

INTEGRAL CURVES

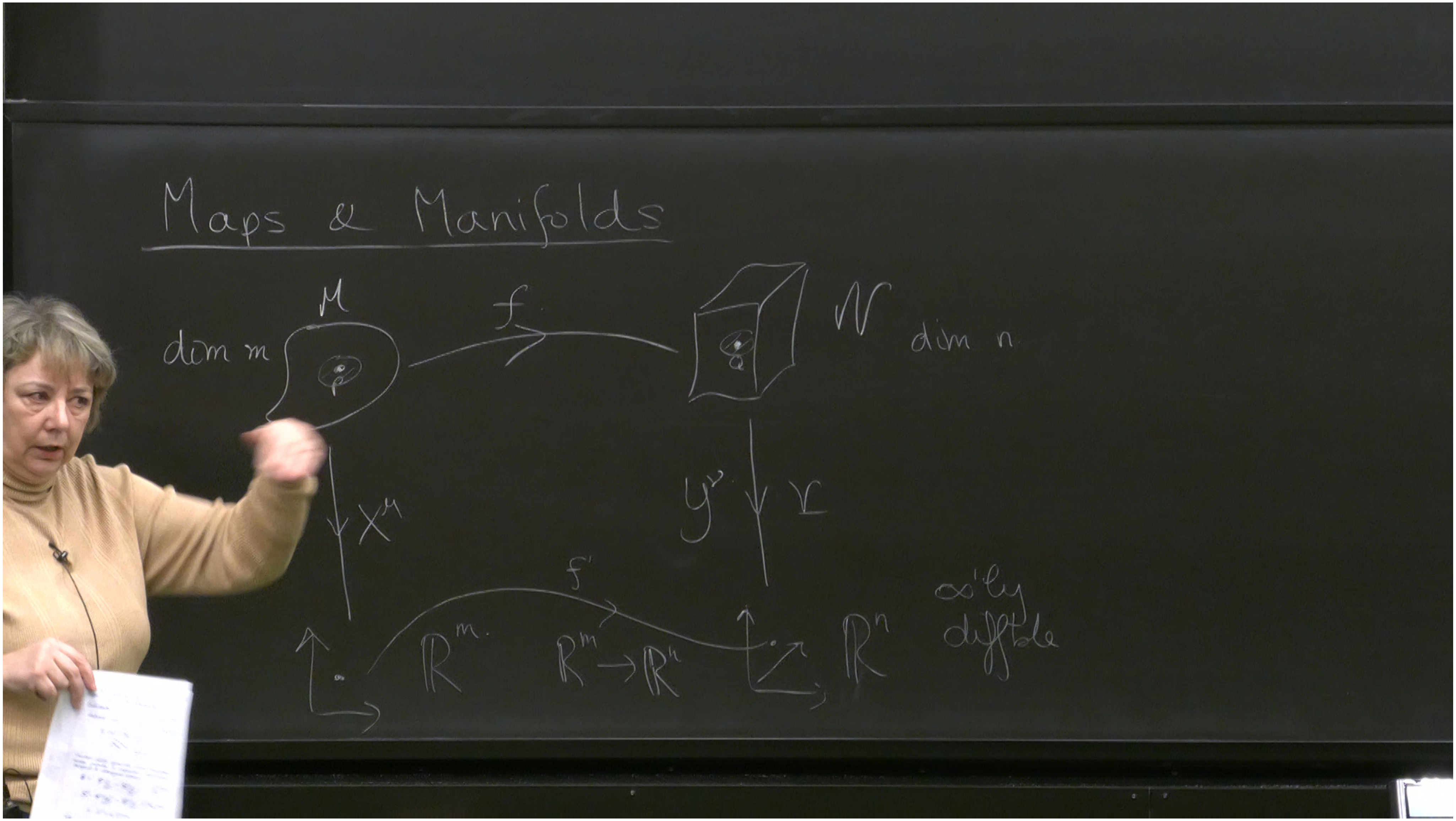


γ - set of curves which have T as their tangent

$$P \leftrightarrow \underset{\text{local chart}}{x^\mu} \longrightarrow \underbrace{x^\mu + \delta x^\mu}_{\text{Can think of as coord transfm}} \leftrightarrow P'$$

$\rightarrow x'^{\mu}$

$$\delta x^\mu = \epsilon T^\mu + \dots$$



Maps & Manifolds
M
dim m
f
N
dim n
p
q
x^μ
y^ν
$\mathbb{R}^m$
$\mathbb{R}^m \to \mathbb{R}^n$
$\mathbb{R}^n$
f'
∞'ly
diff'ble

Here, interested in $n = m$, $\mathcal{N} = \mathcal{M}$

If f is 1-1, onto, C^∞, then

it is a <u>diffeomorphism</u> & we say

$\mathcal{N}$ & $\mathcal{M}$ are the same.

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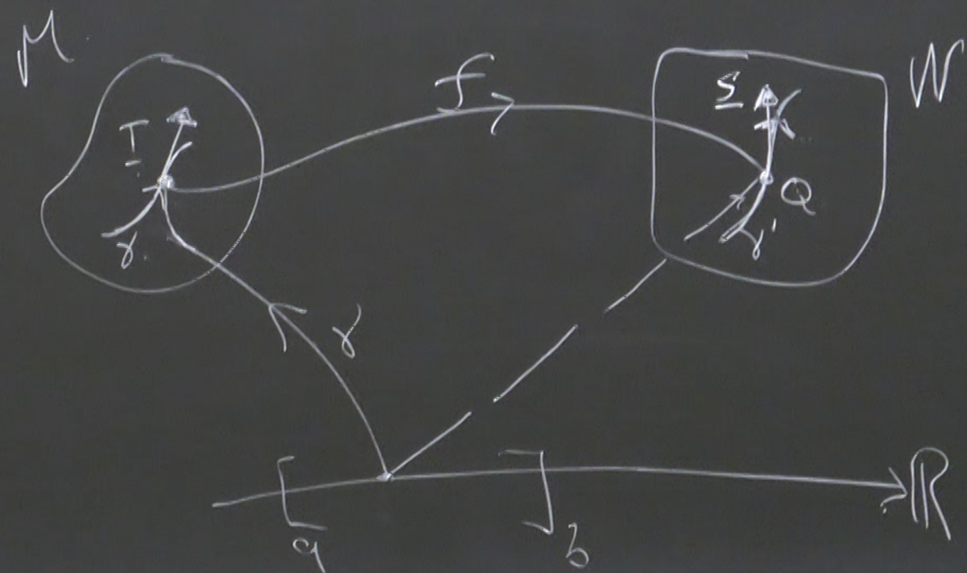
same

<u>Push forward</u> $f_* \; T_P(\mathcal{M}) \to T_Q(\mathcal{N})$



$\mathbb{R}^m$. $\quad \mathbb{R}^m \to \mathbb{R}^n \quad \mathbb{R}^n$ diff'ble

Can identify $\underset{\sim}{S}$ as the vector in $\mathcal{N}$
corr. to I in $\mathcal{M}$

$$\begin{array}{ccc} f_*(I) & = & \underset{\sim}{S} \\ \downarrow & & \downarrow \\ T_p(\mathcal{M}) & & T_Q(\mathcal{N}) \end{array}$$

ly
diff'ble

<u>Pull-back</u> $f^*: T_Q^*(N) \to T_p^*(M)$

defined via $\langle f^*(\underline{\omega}) | \underline{I} \rangle = \langle \underline{\omega} | f_*(\underline{I}) \rangle$

$$
\begin{tikzcd}
T_p(M) \arrow[r, "f_*"] \arrow[dr, dashed] & T_Q(N) \arrow[d, "\underline{\omega}"] \\
& \mathbb{R}
\end{tikzcd}
$$

If

a b

$$T^*_Q(N) \to T^*_p(M)$$

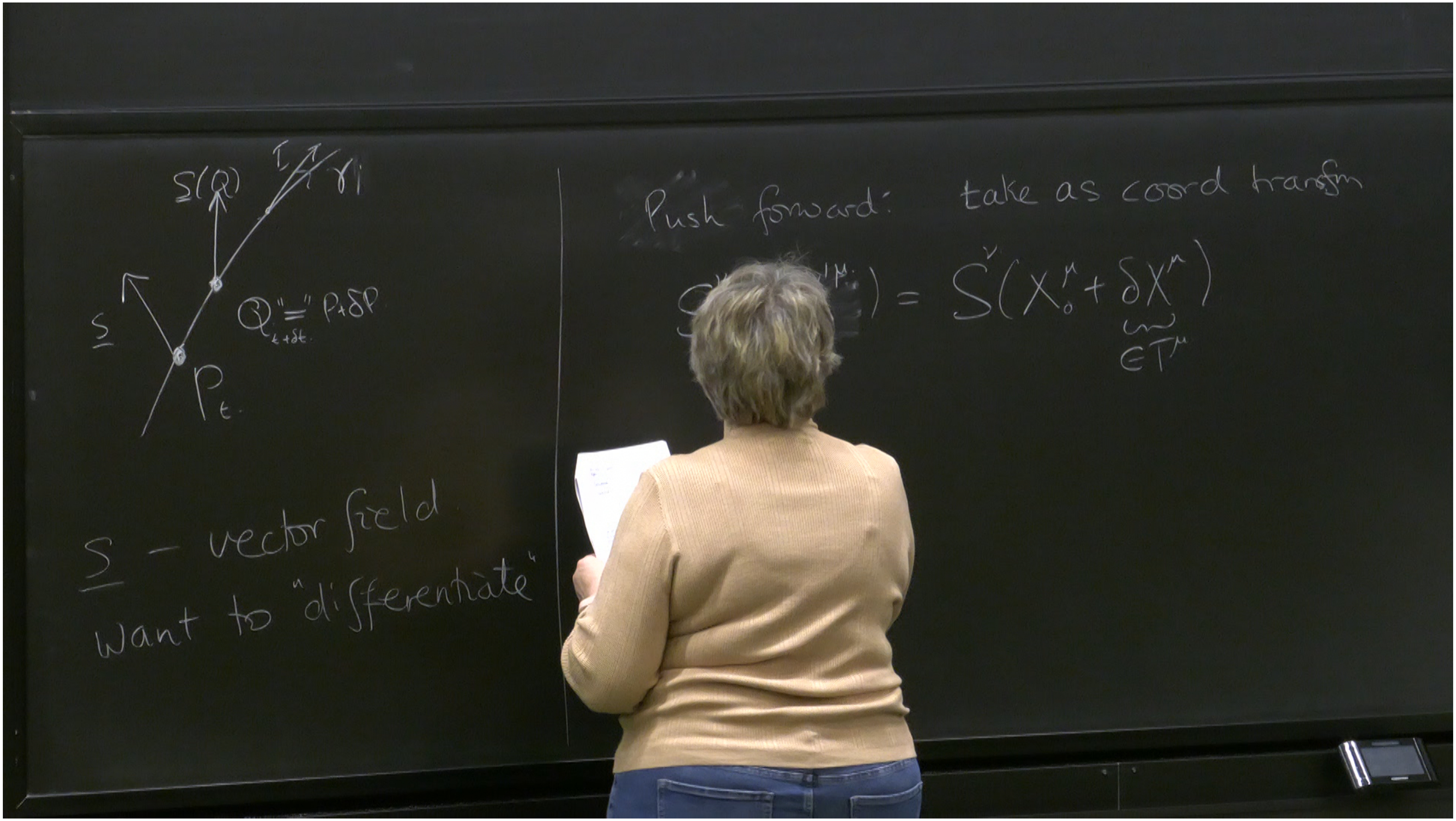
$$\langle f^*(\underline{\omega}) | \underline{I} \rangle = \langle \underline{\omega} | f_*(\underline{I}) \rangle$$

$$\xrightarrow{f_*} T_Q(N)$$

$$\downarrow \underline{\omega}$$

$$\mathbb{R}$$

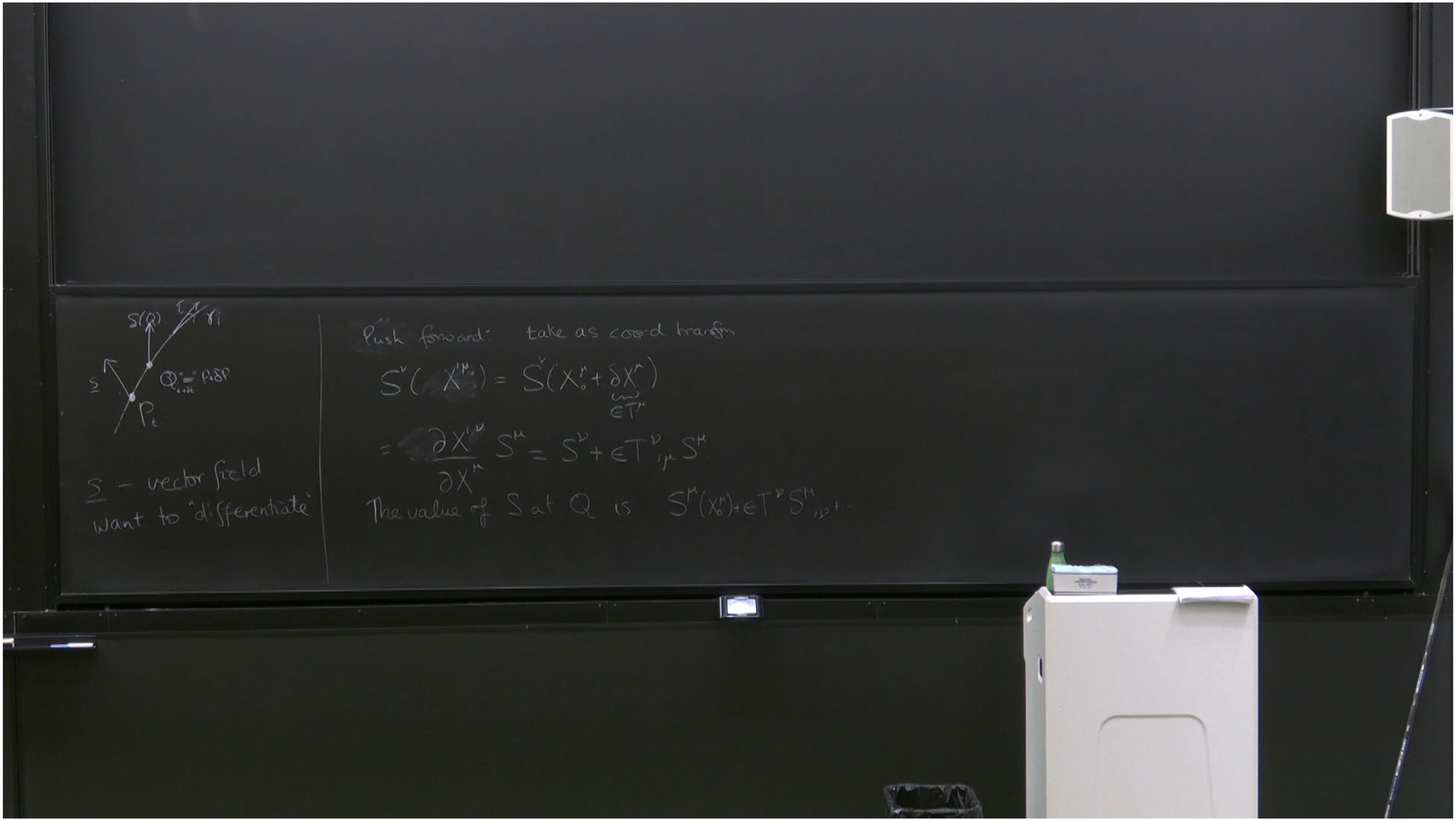
If f a diffeo, can push forward & pull back in both directions.


S(Q)
γ
S
Q "=" P+δP
P
S - vector field
want to "differentiate"
Push forward: take as coord transfm
ET

Push forward: take as coord transfm

$$S^\nu(X'^\mu) = S^\nu(X_0^\mu + \underbrace{\delta X^\mu}_{\epsilon T^\mu})$$

$$= \frac{\partial X'^\nu}{\partial X^\mu} S^\mu = S^\nu + \epsilon T^\nu{}_{,\mu} S^\mu$$


S(Q)
T
γ
S
Q "=" P + δP
t+dt
P_t
S - vector field
want to "differentiate"
Push forward: take as coord transfm
$S^\nu(X'^\mu) = S^\nu(X_0^\mu + \underbrace{\delta X^\mu}_{\epsilon T^\mu})$
$= \frac{\partial X'^\nu}{\partial X^\mu} S^\mu = S^\nu + \epsilon T^\nu{}_{,\mu} S^\mu$
The value of S at Q is $S^\mu(X_0^\mu) + \epsilon T^\nu S^\mu{}_{,\nu} + \dots$

Push forward: take as coord transfm

$$S^{\nu}(X'^{\mu}) = S^{\nu}(X_0^{\mu} + \underbrace{\delta X^{\mu}}_{\epsilon T^{\mu}})$$

$$= \frac{\partial X'^{\nu}}{\partial X^{\mu}} S^{\mu} = S^{\nu} + \epsilon T^{\nu}{}_{,\mu} S^{\mu}$$

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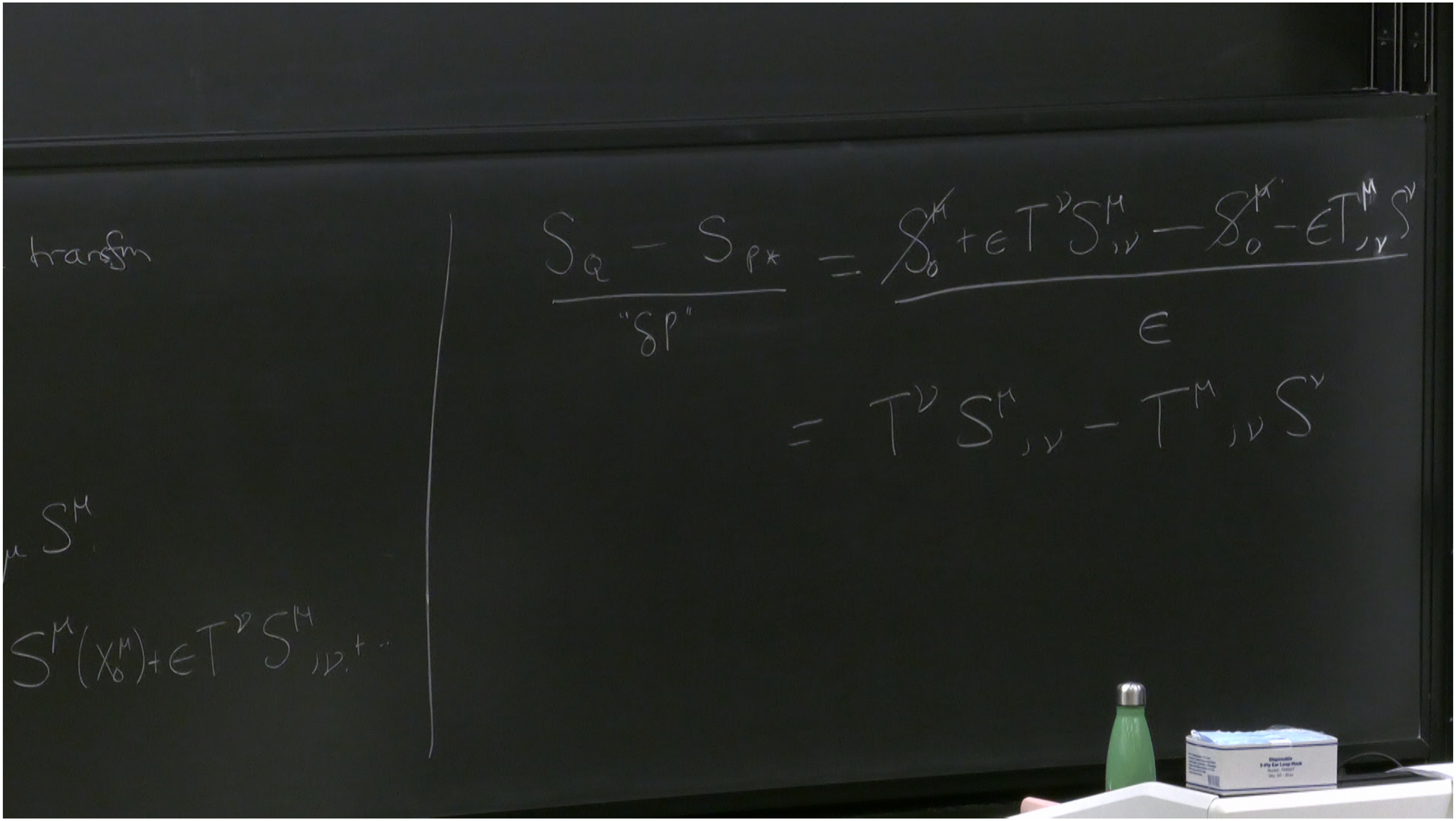
transfm
$\frac{S_Q - S_{P*}}{\text{"}\delta P\text{"}} = \frac{\cancel{S_0^\mu} + \epsilon T^\nu S^\mu_{,\nu} - \cancel{S_0^\mu} - \epsilon T^\mu_{,\nu} S^\nu}{\epsilon}$
$= T^\nu S^\mu_{,\nu} - T^\mu_{,\nu} S^\nu$
$_\mu S^\mu$
$S^\mu(x_0^\mu) + \epsilon T^\nu S^\mu_{,\nu} + \dots$

transfm

$_\mu S^\mu$

$S^\mu(X_0^\mu) + \epsilon T^\nu S^\mu_{,\nu} + \cdots$

$$\frac{S_Q - S_{P*}}{\text{“}\delta P\text{”}} = \frac{\cancel{S_0^\mu} + \epsilon T^\nu S^\mu_{,\nu} - \cancel{S_0^\mu} - \epsilon T^\mu_{,\nu} S^\nu}{\epsilon}$$

$$= T^\nu S^\mu_{,\nu} - T^\mu_{,\nu} S^\nu$$

$$= [T, S]^\mu$$

transfm

μS^μ

$S^\mu(X_0^\mu) + \epsilon T^\nu S^\mu{}_{,\nu} + \cdots$

$$\frac{S_Q - S_{P*}}{\text{"}\delta P\text{"}} = \frac{\cancel{S_0^\mu} + \epsilon T^\nu S^\mu{}_{,\nu} - \cancel{S_0^\mu} - \epsilon T^\mu{}_{,\nu} S^\nu}{\epsilon}$$

$$= T^\nu S^\mu{}_{,\nu} - T^\mu{}_{,\nu} S^\nu$$

$$= [T, S]^\mu$$

$\boxed{\mathcal{L} = \text{Lie deriv.}}$

ex, use pull-back to define

$$(\mathcal{L}_T \omega)_\mu = T^\nu \omega_{\mu,\nu} + \omega_\nu T^\nu{}_{,\mu}$$

Geometrical Significance

The Lie bracket tells us
if transport along sets of integral
curves commutes.

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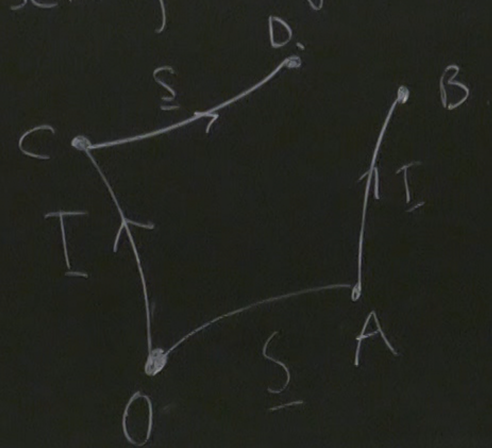
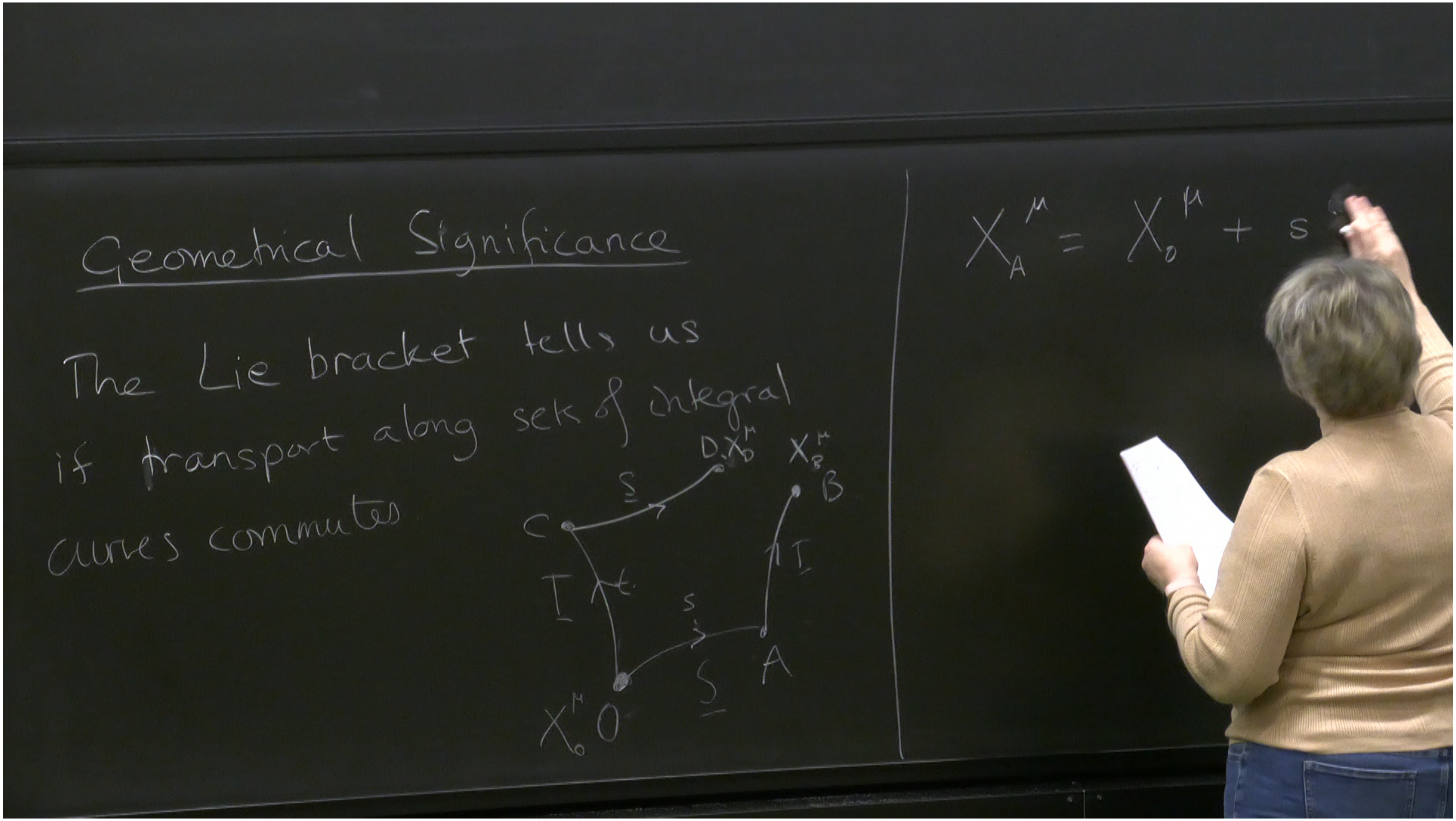



Geometrical Significance
The Lie bracket tells us
if transport along sets of integral
curves commutes
$X_A^\mu = X_0^\mu + s$
C
D X_D^μ
X_B^μ B
A
X_0^μ O

S at Q is $S^\mu(X_0^\mu) + \epsilon T^\nu S^\mu{}_{,\nu} + \dots$

$= [T, S]$

ex, use pull-back to deine

$(\mathcal{L}_T \omega)_\mu = T^\nu \omega_{\mu,\nu} + \omega_\nu T^\nu{}_{,\mu}$

$\mathcal{L}$ = Lie deriv

$$X_A^\mu = X_0^\mu + s S_0^\mu + \frac{1}{2} s^2 S_0^\nu S_0^\mu{}_{,\nu} + O(s^3)$$

$$\Rightarrow X_B^\mu = X_A^\mu + t T_A^\mu + \frac{1}{2} t^2 T_A^\nu T_A^\mu{}_{,\nu} + \dots$$

$$= X_0^\mu + s S_0^\mu + \frac{1}{2} s^2 S_0^\nu S_0^\mu{}_{,\nu} + t\left(T_0^\mu + s S_0^\nu T^\mu_{0,\nu}\right) + \frac{1}{2} t^2 T_0^\nu T^\mu_{0,\nu}$$

X_D^μ has $S_0 \leftrightarrow T_0$ & $s \leftrightarrow t$

$$X_D^\mu = X_0^\mu + t T_0^\mu + \frac{1}{2} t^2 T_0^\nu T_0^\mu{}_{,\nu} + s\left(S_0^\mu + t T_0^\nu S_0^\mu{}_{,\nu}\right) + \frac{1}{2} s^2 S_0^\nu S_0^\mu{}_{,\nu}$$

So $X_D^{\mu} - X_B^{\mu} = st\left(T_0^{\nu} S_0{}^{\mu}{}_{,\nu} - S_0^{\nu} T_0{}^{\mu}{}_{,\nu}\right)$

$$[T, S]^{\mu}$$

The Lie derivative also tells us about symmetries.

X_0^μ O $\underset{-}{S}$ A

$X_D^\nu = X_0^\nu + t\,T_0 + \ldots$

So $$X_D^\mu - X_B^\mu = st\left(T_0^\nu S_0^\mu{}_{,\nu} - S_0^\nu T_0^\mu{}_{,\nu}\right)$$

$$[T, S]^\mu$$

The Lie derivative also tells us about symmetries.

Defn: A Killing vector

X_0^μ O S A

$X_D^\nu = X_0^\nu + t\, T_0 +$

So $X_D^\mu - X_B^\mu = st\left(T_0^\nu S_0^\mu{}_{,\nu} - S_0^\nu T_0^\mu{}_{,\nu}\right)$

$$[T, S]^\mu$$

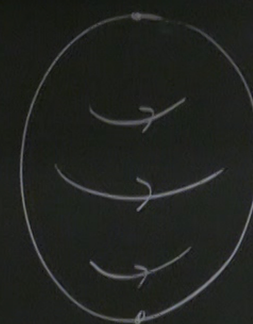
The Lie derivative also tells us about symmetries.

DEFN: A Killing vector is a vector field along which the metric is Lie invariant. $\mathcal{L}_K g = 0$

$$+\, t\, T_0^{\mu} + \frac{1}{2} t^2 T_0^{\nu} T_0^{\mu}{}_{,\nu} + s\left(S_0^{\mu} + t\, T_0^{\nu} S_0^{\mu}{}_{,\nu}\right) + \frac{1}{2} s^2 S_0^{\nu} S_0^{\mu}{}_{,\nu}.$$

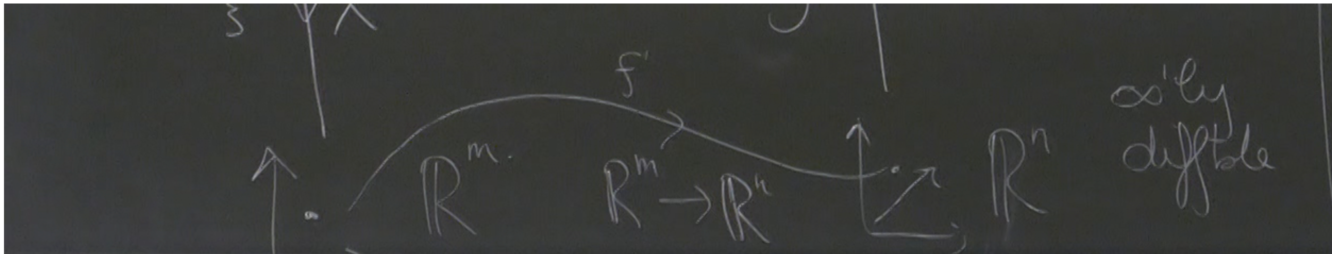
es.

e.g. S^2 : $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$



$$\underline{k} = \frac{\partial}{\partial \varphi} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(\mathcal{L}_k g)_{\mu\nu} = \underbrace{k^{\sigma} g_{\mu\nu,\sigma}}_{\frac{\partial}{\partial \phi} g_{\mu\nu} = 0} + \cancel{k^{\sigma}{}_{,\mu}} g_{\sigma\nu} + \cancel{k^{\sigma}{}_{,\nu}} g_{\sigma\mu}$$



Exterior deriv $\underline{d}$

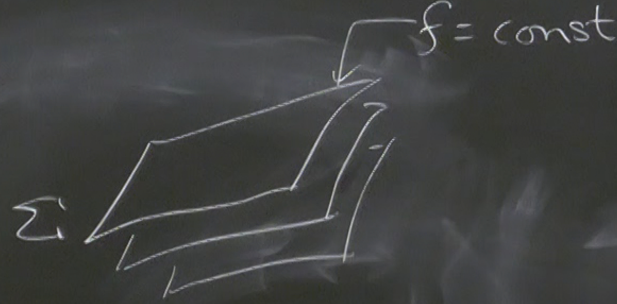
The exterior derivative is a purely anti-symmetric deriv.

Acting on a fn, $\underline{d}$ maps fn to a covector, $\underline{d}f$

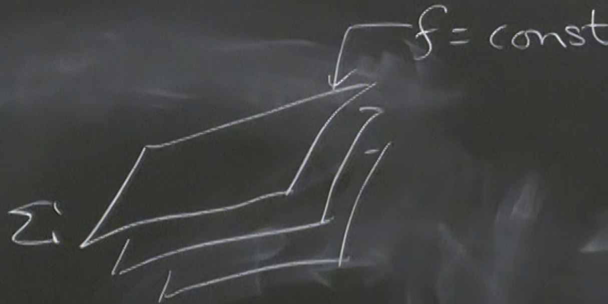
$$\underline{d}: C^\infty(\mathcal{M}) \to T^*(\mathcal{M})$$

$$\langle \underline{d}f \,|\, \underline{T}\rangle = \underline{T}f \quad \forall\, \underline{T} \in T_p(\mathcal{M})$$

$p \in \mathcal{M}$.



Can think of $\underline{d}f$ as "normal" to surfaces of const f.



Can think of $\underline{d}f$ as "normal" to surfaces of const f.

A **p-form** is an anti-symm covariant tensor. Construct in a similar way to tensor product.

$$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A}$$

Both defining "wedge", $\wedge$, & the 2-form

In components

$$\left(\underline{A}^{(p)} \wedge \underline{B}^{(q)}\right)_{a_1 \ldots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}$$

$$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq}\, \underline{B} \wedge \underline{A}$$

$p \leftrightarrow$ rank of form.

m.

The p-forms lie in $\Lambda^p(\mathcal{M})$

Note $p \leq n$. $\dim(\mathcal{M})$

The n-form is unique up to a factor.

$\epsilon_{abcd} = \pm 1$ (sign of perm of a, b, c, d)