Title: Mathematical Physics Core Lecture

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ex. of principal bundle: frame bundle $F(M)$ of manifold M

$F = GL(n,\mathbb{R})$ $\qquad$ $G = GL(n,\mathbb{R})$

trivialisations $\quad (U,(x^1,\dots,x^n))$, $(V,(y^1,\dots,y^n))$ charts on M

trivialisation

$\phi : (e, A^\alpha_1 \frac{\partial}{\partial x^\alpha}\big|_e, A^\alpha_2 \frac{\partial}{\partial x^\alpha}\big|_e, \dots, A^\alpha_n \frac{\partial}{\partial x^\alpha}\big|_e) \in \pi^{-1}$

$\psi : (e, B^\alpha_1 \frac{\partial}{\partial y^\alpha}\big|_e, \dots, B^\alpha_n \frac{\partial}{\partial y^\alpha}\big|_e) \mapsto$

$F($

$\pi :$

manifold M

$$F(M) = \bigcup_{e \in M} \{e\} \times F_e M$$

$$F_e M = \left\{ (v_1, \dots, v_n) \in (T_e M)^n \;\middle|\; v_1, \dots, v_n \text{ linearly indep} \right\}$$

$n = \dim(M)$

$$\pi : (e, v_1, \dots, v_n) \in F(M) \mapsto e \in M$$

$$\left(\dots \frac{\partial}{\partial x^\alpha}\Big|_e, \dots, A^\alpha_n \frac{\partial}{\partial x^\alpha}\Big|_e \right) \in \pi^{-1}(U) \mapsto (e, A) \in U \times GL(n, \mathbb{R})$$

$$\left(\dots B^\alpha_n \frac{\partial}{\partial y^\alpha}\Big|_e \right) \longmapsto (e, B)$$

$$\phi \circ \psi^{-1} : (a, A) \in (U \cap V) \times GL(n,\mathbb{R}) \mapsto \phi\left(a, \frac{\partial x}{\partial y}(a)\, A\right) \in (U \cap V) \times GL(n,\mathbb{R})$$

$$\left(\frac{\partial x}{\partial y}\right)^{\alpha}_{\ \beta} = \frac{\partial x^{\alpha}}{\partial y^{\beta}}$$

$$\psi^{-1}(a, A) = \left(a, A^{\alpha}_{1} \frac{\partial}{\partial y^{\alpha}}\Big|_{a}, \ldots, A^{\alpha}_{n} \frac{\partial}{\partial y^{\alpha}}\Big|_{a}\right)$$

$$= \left(a, A^{\alpha}_{1} \frac{\partial x^{\beta}}{\partial y^{\alpha}}(a) \frac{\partial}{\partial x^{\beta}}\Big|_{a}, \ldots, A^{\alpha}_{n} \frac{\partial x^{\beta}}{\partial y^{\alpha}}(a) \frac{\partial}{\partial x^{\beta}}\Big|_{a}\right)$$

$(n,\mathbb{R})$

$\frac{\alpha}{\beta}$

$t_{UV}: U\cap V \to GL(n,\mathbb{R})$ smooth

$a \mapsto \frac{\partial x}{\partial y}(a)$

action of $GL(n,\mathbb{R})$ on $F(M)$

$(a, v_1, \dots, v_n) \triangleleft g = (a, (v_1, v_2 \dots v_n)\, g)$

Gauge theory $\rightarrow$ need a connection on P

Lie(G)

Def: $\pi: P \to M$ principal G-bundle. A connection on P is a $\mathfrak{g}$-valued

such that:

- if $R_g: p \in P \mapsto p \triangleleft g \in P$, then $R_g^* \omega = g^{-1} \omega g$
- if $\xi \in \mathfrak{g}$, $X_\xi : P \to TP$ such that $(X_\xi f)(p) = \frac{d}{dt}\left(f(p \triangleleft e^{t\xi})\right)\Big|_{t=0}$

$, B_h^\alpha \frac{\partial}{\partial y^\alpha}\Big|_e \Big) \longmapsto (a, B)$

ω is a matrix of 1-forms

Lie(G)

n P is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$

$\omega : P \to T^*P \otimes \mathfrak{g}$

$\omega = g^{-1} \omega g$

$)(p) = \frac{d}{dt}\left(f(p \cdot e^{t\xi}) \right)\Big|_{t=0}$ then $\omega(X_\xi) = \xi$

Sections of bundles

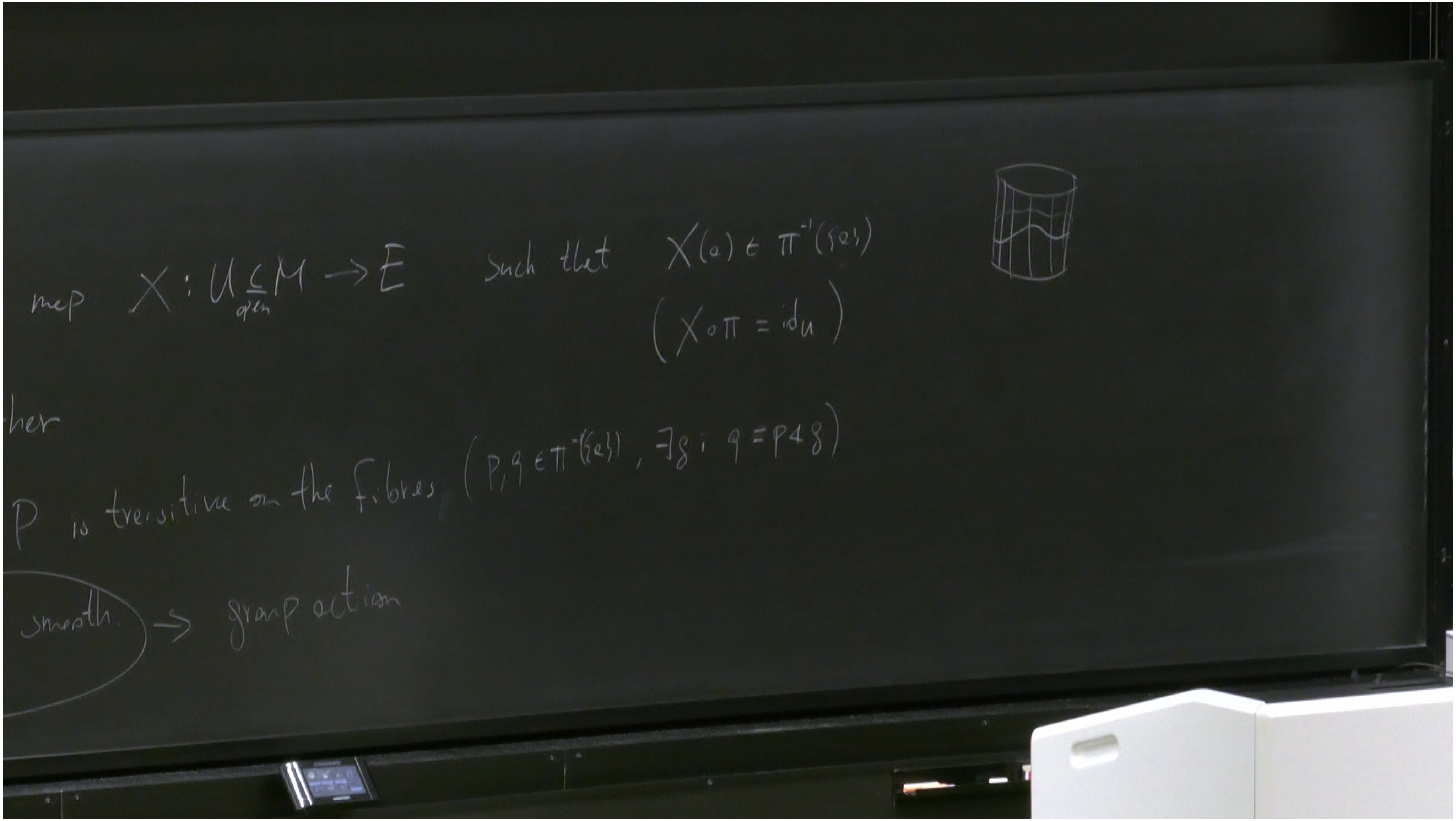
Def: a section of a bundle $\pi : E \to M$ is a smooth map $X : U \underset{open}{\subseteq} M \to E$

Principal bundles

we can transform sections into each other

if $s : U \to P$, $s' : U \to P$ since the action of G on P is transitive on the fibres

$$s'(u) = s(u) \triangleleft g(u)$$ for $g : U \to G$ smooth. $\to$ group action


map $X : U \underset{open}{\subseteq} M \to E$ such that $X(a) \in \pi^{-1}(\{a\})$
$(X \circ \pi = id_U)$
her
P is transitive on the fibres $(p, q \in \pi^{-1}(\{a\}), \exists g : q = p \triangleleft g)$
smooth -> group action

... such that $(X_\xi f)(p) = \frac{d}{dt}\left(f(p \triangleleft e^{t\xi})\right)\Big|_{t=0}$

Let $\{(U_i, \phi_i)\}_{i \in I}$ maximal G-atlas of P

Define $s_i : u \in U_i \mapsto \phi_i^{-1}(u, e) \in P$

on $U_i \cap U_j$ $\quad s_j : u \in U_i \cap U_j \mapsto \phi_j^{-1}(u, e) =$

↳ all possible sections!

→ consequence: P has a global trivialisation iff

$\partial t \quad \Big|_{t=0}$

$t_i \mapsto \phi_i^{-1}(u,e) \in P$

$s_j : u \in U_i \cap U_j \mapsto \phi_j^{-1}(u,e) = \phi_i^{-1}(\phi_{ij}(u,e)) = \phi_i^{-1}(u, t_{ij}(u)) = \phi_i^{-1}(u,e) \triangleleft t_{ij}(u)$

$= s_i(u) \triangleleft t_{ij}(u)$

ctions!

l trivialisation iff P has a global section

$s'(u) = s(u) \triangleleft g(u)$ (for $g: U \to G$ smooth.) $\Rightarrow$ group action

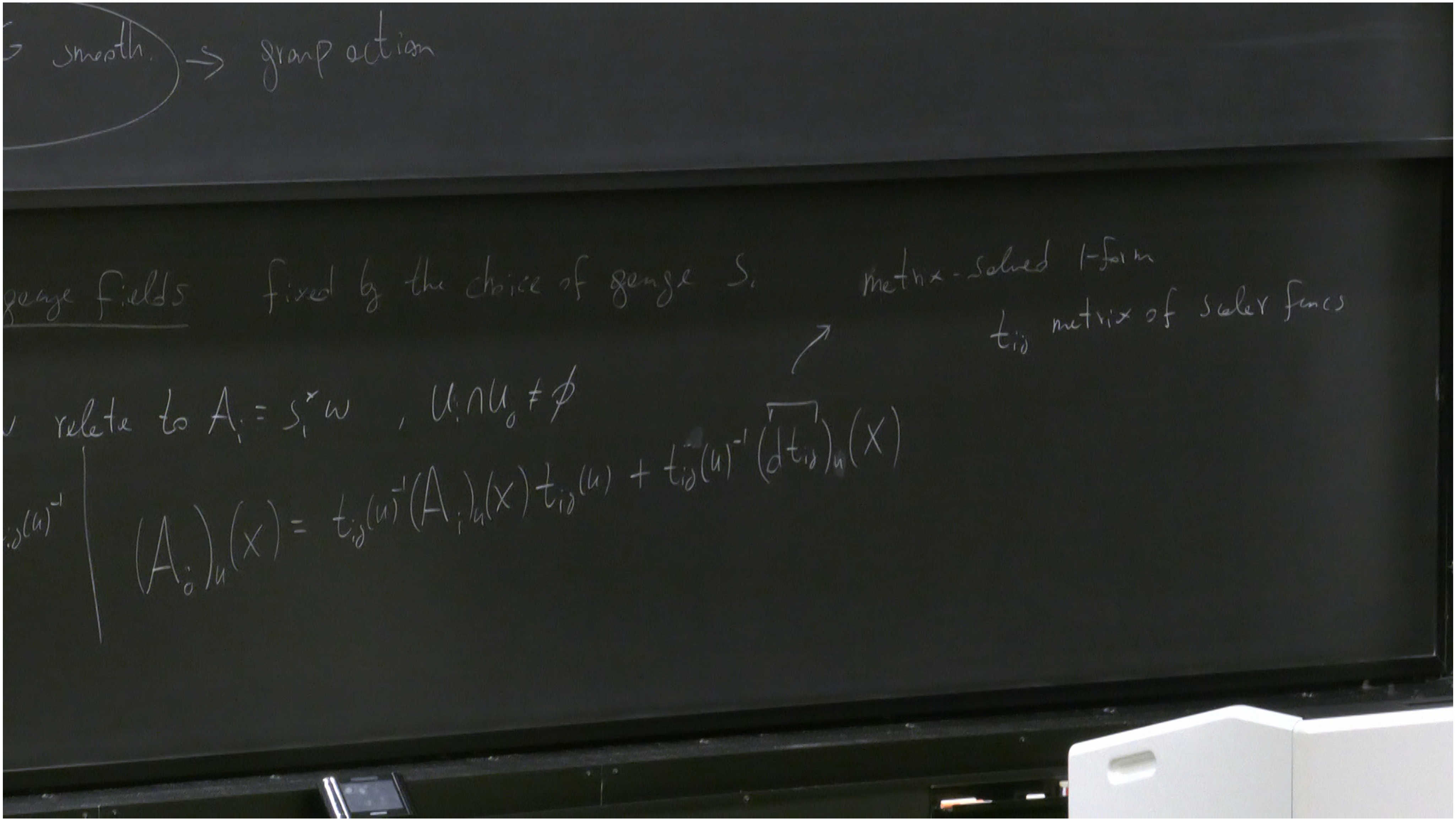
Gauge fields $A_i = s_i^* \omega \in \Omega^1(U_i \subseteq M, \mathfrak{g})$ local gauge fields fixed by the

gauge dependent. how does it transform? how is $A_j = s_j^* \omega$ relate to $A_i = s_i^* \omega$, U_i

$$\boxed{A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} d t_{ij}}$$

$t_{ij}^{-1} : u \mapsto t_{ij}(u)^{-1}$

$(A_j)_u(x) = t_{ij}(u)^{-1}(A_i$


smooth. → group action
gauge fields fixed by the choice of gauge s_i
matrix-valued 1-form
t_{ij} matrix of scalar fncs
relate to $A_i = s_i^* \omega$, $U_i \cap U_j \neq \emptyset$
$(A_j)_u(X) = t_{ij}(u)^{-1}(A_i)_u(X)\, t_{ij}(u) + t_{ij}(u)^{-1}(dt_{ij})_u(X)$

Prop: $g : U \subseteq M \to G$ smooth. Then $g^{-1}dg \in \Omega^1(U, \mathfrak{g})$

Let $X : U \to TM$ be a vector field

$$(g^{-1}dg(X))(u) = g(u)^{-1}\, dg_u(X_u) = g(u)^{-1} X_u g$$

$$= g(u)^{-1}\dot\gamma(0) g = g(u)^{-1}(g\circ\gamma)'(0) = \frac{d}{dt}\left(g(u)^{-1} g(\gamma(t))\right)\Big|_{t=0}$$

Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be the unique int

$\dot\gamma(0) f = (f\circ\gamma)'(0)$

G matrix Lie gro

$\mathfrak{g} = \{\dot\gamma(0) \mid \gamma : (-\varepsilon,$

↳ component-

$\in \Omega^1(U, \mathfrak{g})$

G matrix Lie group

$\mathfrak{g} = \{ \dot{\gamma}(0) \mid \gamma : (-\varepsilon, \varepsilon) \to G \text{ smooth with } \gamma(0) = \mathbb{1} \}$

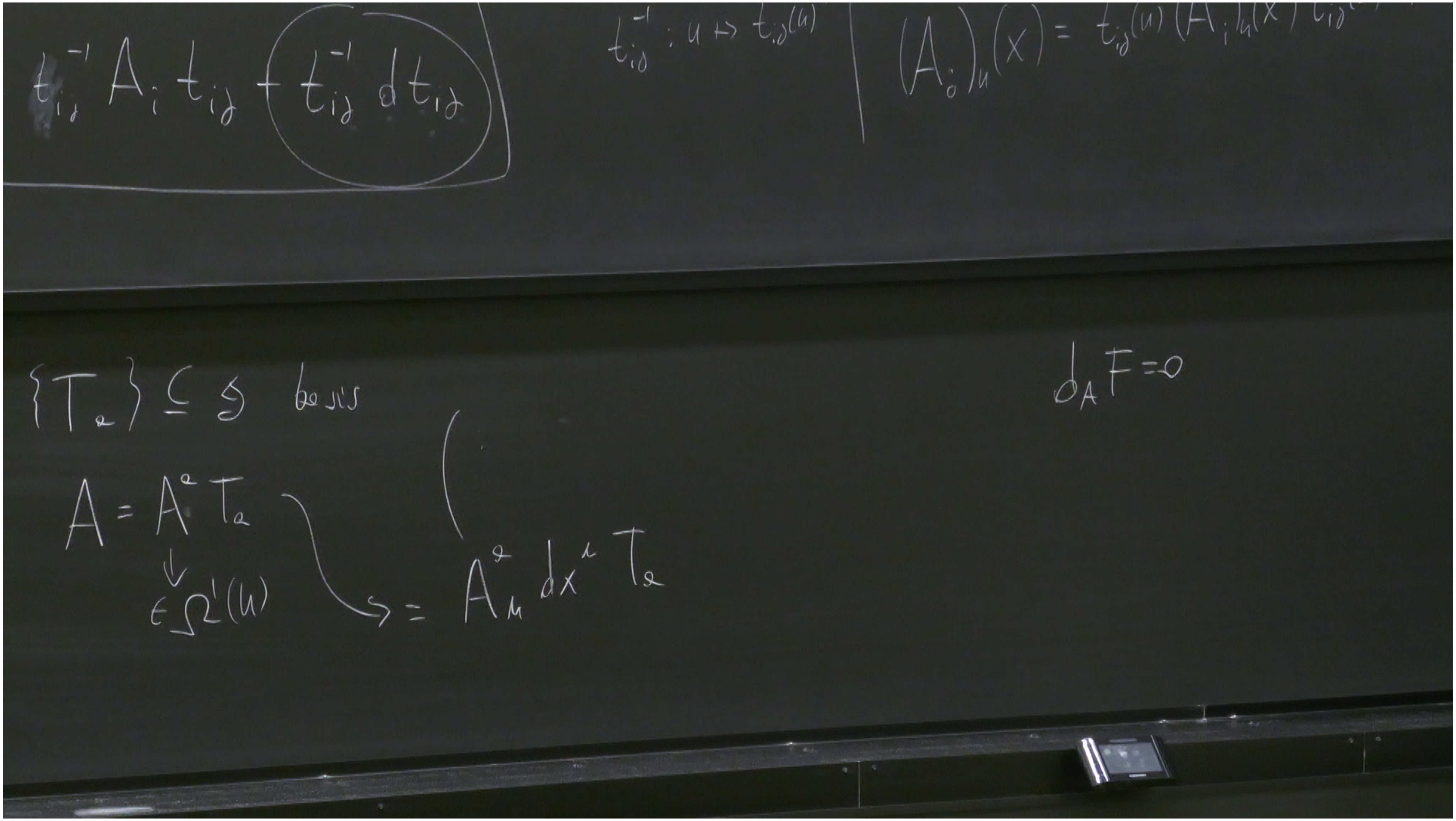
$\hookrightarrow$ component-wise derivation

$$\begin{cases} \dot{\gamma}(t) = X|_{\gamma(t)} \\ \gamma(0) = u \end{cases}$$

Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be the unique integral curve of X with $\gamma(0) = u$

$)) \Big|_{t=0}$

$\dot{\gamma}(0) f = (f \circ \gamma)'(0)$



$t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} d t_{ij}$
$t_{ij}^{-1} : u \mapsto t_{ij}(u)$
$(A_j)_\mu(x) = t_{ij}(u)(A_i)_\mu(x)\, t_{ij}$
$\{T_a\} \subseteq \mathfrak{g}$ basis
$A = A^a T_a$
$\in \Omega^1(U)$
$= A^a_\mu dx^\mu T_a$
$d_A F = 0$