Title: Mathematical Physics Core Lecture

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Alternate def. of vector bundle

don't put vector space struc

$\pi : E \to M$ bundle with fibre $F = \mathbb{R}^n$; E is a real vector bundle of ran

if $\{(U_i, \phi_i)\}$ trivialisations, then the transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$

ϕ_{ij} preserves the fibre: $\phi_{ij}(a, v) = (a, \dots)$ $\Big|$ $\phi_{ij}\big|_{\{a\}\times\mathbb{R}^n} : \{a\}\times\mathbb{R}^n \to \{a\}\times\mathbb{R}^n$

ctorspace structure on F_e

ector bundle of rank n if

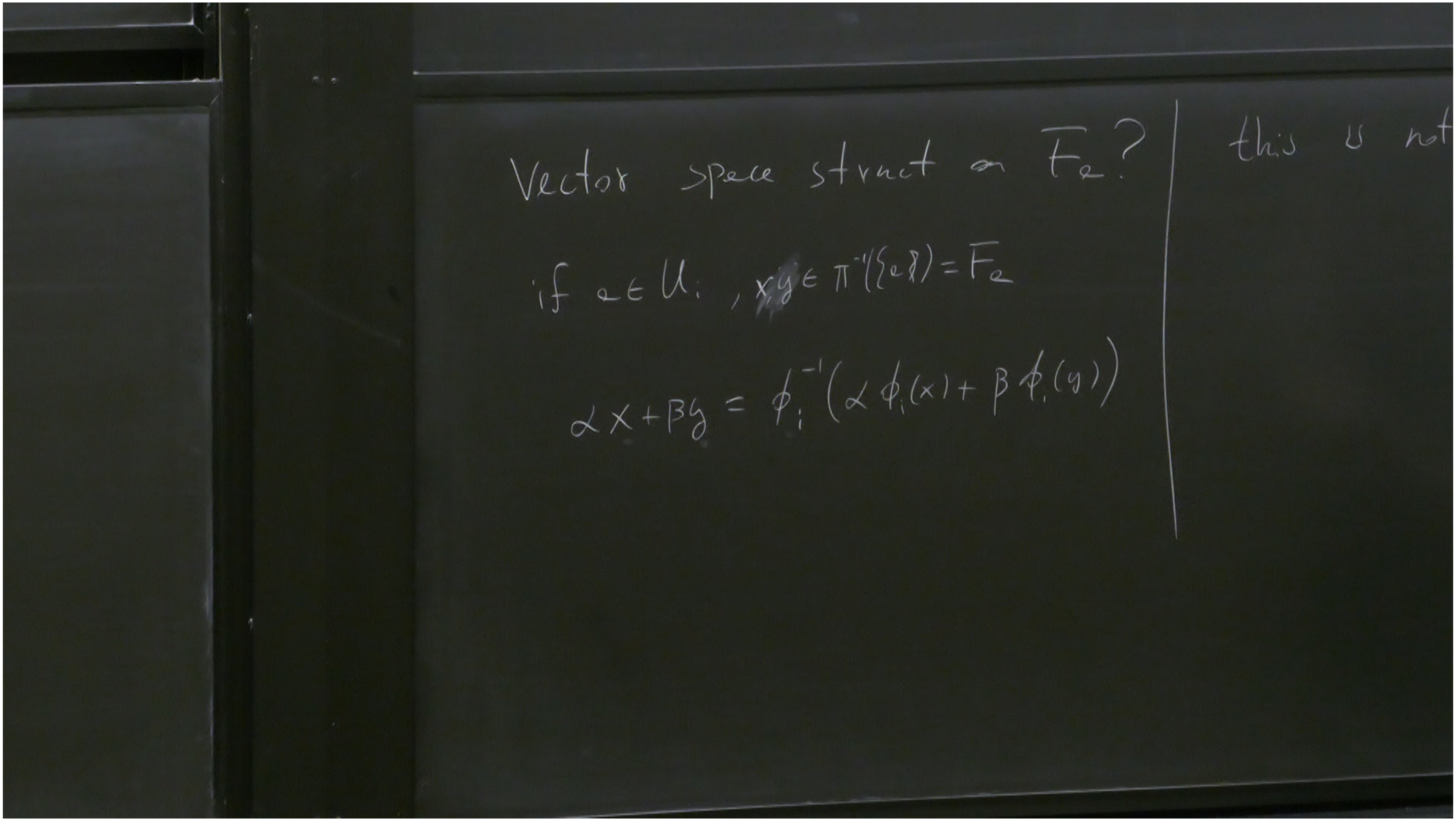
s $\quad \phi_{ij} = \phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ are linear

$\Downarrow$

V. space structure on $U \times \mathbb{R}^n$: $\alpha(e, v) + \beta(e, w) = (e, \alpha v + \beta w)$

$\mathbb{R}^n$: $\{e\} \times \mathbb{R}^n \to \{e\} \times \mathbb{R}^n$

$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$


Vector space struct on F_e?
if $e \in U_i$, $x, y \in \pi^{-1}(\{e\}) = F_e$
$\alpha x + \beta y = \phi_i^{-1}(\alpha \phi_i(x) + \beta \phi_i(y))$
this is not

this is not dependent on trivialisation

suppose $e \in U_i \cap U_j$

$$\phi_j^{-1}\left(\alpha \phi_j(x) + \beta \phi_j(y)\right) = \phi_j^{-1}\left(\alpha \phi_{ji}(\phi_i(x)) + \beta \phi_{ji}(\phi_i(y))\right)$$

$$= \phi_j^{-1}\left(\phi_{ji}\left(\alpha \phi_i(x) + \beta \phi_i(y)\right)\right) = \phi_i^{-1}\left(\alpha \phi_i(x) + \beta \phi_i(y)\right)$$

Def: Let $\pi : E \to M$ be a F.B. with fibre F. Assume there is a (l

$$G \times F \to F \qquad \Big| \qquad e \triangleright f = f$$
$$(g, f) \mapsto g \triangleright f \qquad \Big| \qquad g \triangleright (h \triangleright f) = (gh) \triangleright f$$

A G-atlas for E is a collection of trivialisations $\{(U_i, \phi_i)\}_{i \in I}$ such th

and $\phi_{ij}(u, f) = (u, t_{ij}(u) \triangleright f)$ where $t_{ij} : U_i \cap U_j \to G$ is a smooth dutch

F. Assume there is a (left) group action of G on F Lie group

atlas $\{(U_i, \phi_i)\}_{i\in I}$ such that: $\bigcup_i U_i = M$

$U_i \cap U_j \to G$ is a smooth clutching function

<u>Note:</u> ϕ_{ij} is linear is equivalent to $\phi_{ij}(u,v) = ($

<u>Def.</u> $\pi : E \to M$ fibre bundle with fibre F, we say that

<u>structure group</u> G, if G acts on F and there is

uivalent to $\quad \phi_{ij}(u,v) = (u, t_{ij}(u)\, v) \qquad t_{ij}(u) \in GL(n,\mathbb{R})$

$t_{ij} : U_i \cap U_j \to GL(n,\mathbb{R})$ smooth

with fibre F, we say that E has

G acts on F and there is a maximal G-atlas

nothing can be added to it

$\mathbb{R})$

$L(n,\mathbb{R})$ smooth

it

Vector bundles are fibre bundles with fibre $\mathbb{R}^n$ and structure group $GL(n,\mathbb{R})$

$g \in GL(n,\mathbb{R})$ $\quad v \in \mathbb{R}^n$

$$g \triangleright v = g v$$

Properties of clutching funcs

- $t_{ii}(u) = e \quad \forall u \in U_i$
- $t_{ij}(u)^{-1} = t_{ji}(u)$
- <u>Cocycle property</u>

  if $u \in U_i \cap U_j \cap U_k$

  $t_{ij}(u) = t_{ik}(u)\, t_{kj}(u)$

Principal bundles → gauge theory

A principal G-bundle is a fibre bundle $\pi : P \to M$ with fibre $F = G$ ✓

Lie group

fibre $F = G$ and structure group G

natural action of G on G

$g \triangleright h = gh$

Principal bundles → gauge theory

A principal G-bundle is a fibre bundle $\pi : P \to M$ with fibre $F = G$

extra structure: there is a right action of G on P

→ gauge theory

Lie group

is a fibre bundle $\pi : P \to M$ with fibre $F = G$ and structure group G

s a right action of G on P

Lie group

$=G$ and structure group G

natural action of G on G

$g \triangleright h = gh$

trivial. $P = M \times G$

$\pi : P \to M$ with fibre $F = G$ and structure group G

trivial. $P = M \times G$

...f G on P

action $p \in P$ $g \in G$

$$p \triangleleft g = \phi_i^{-1}\left(\phi_i(p) \triangleleft g\right)$$

$p \in \pi^{-1}(U_i)$

$\phi_i(p) \in U_i \times G$

if $(u,g) \in U_i \times G$: $h \triangleright (u,g) = (u, hg)$

$(u,g) \triangleleft h = (u, gh)$

$\in P \quad g \in G$

$\left(\phi_i(p) \triangleleft g \right)$

$\phi_i(p) \in U_i \times G$

if $(u,g) \in U_i \times G$: $h \triangleright (u,g) = (u, hg)$

$(u,g) \triangleleft h = (u, gh)$

$\downarrow$

$\downarrow \qquad x \in (U_i \cap U_j) \times G$

$\phi_{ij}(x) = t_{ij}(u) \triangleright x$

$x = (u,g)$

$\phi_{ij}(u,g) = (u, t_{ij}(u)\, g)$

$= t_{ij}(u) \triangleright (u,g)$

$p \in \pi^{-1}(U_i)$

ϕ_{ij}

$$\phi_j^{-1}(\phi_j(p) \triangleleft g) = \phi_j^{-1}(\phi_{ji}(\phi_i(p)) \triangleleft g)$$

$$= \phi_j^{-1}((t_{ji}(\pi(p)) \triangleright \phi_i(p)) \triangleleft g)$$

$$= \phi_i^{-1} \circ \phi_{ij}(\cdots\cdots) = \phi_i^{-1}((t_{ij}(\pi(p)) \triangleright t_{ji}(\pi(p)) \triangleright \phi_i(p)) \triangleleft g)$$

$\downarrow$

$\downarrow \qquad x \in (U_i \cap U_j) \times G$

$\phi_{ij}(x) = t_{ij}(u) \triangleright x$

$x = (u, g)$

$\phi_{ij}(u,g) = (u, t_{ij}(u)\, g)$

$= t_{ij}(u) \triangleright (u, g)$

t_{ij}^{-1}

$\uparrow$

$$\phi_1^{-1}\Big(\big(t_{ij}(\pi(p)) \triangleright t_{j1}(\pi(p)) \triangleright \phi_1(p)\big) \triangleleft g\Big) = \phi_1^{-1}\big(\phi_i(p) \triangleleft g\big)$$

perty

U_n

$_{ij}(u)$

Other properties of $p \triangleleft g$

- action is fibre-preserving : $p \in F_x$, $p \triangleleft g \in F_x$
- action is transitive on F_x : $p, q \in F_x$, $\exists g \in G$ s.t. $q = p \triangleleft g$
- action is free : if $p \triangleleft g = p$ for some $p \in P$, then $g = e$
- the trivialisations are G-equivariant

$$\phi_i(p \triangleleft g) = \phi_i(p) \triangleleft g$$