Title: Mathematical Physics Core Lecture

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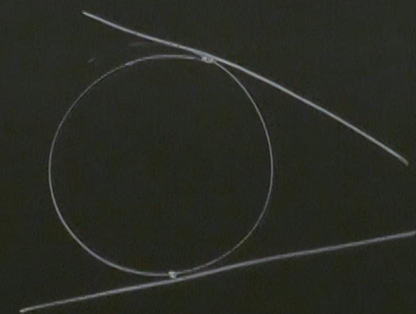
Fibre bundles

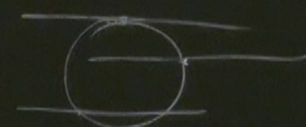
M manifold, space F want to

simplest way: $M \times F = \bigcup_{a \in M} \{a\} \times F$ $\rightarrow$ attach F to ea

F Want to attach a copy of F to each $a \in M$

attach F to each $a \in M$ in the <u>the same way</u>





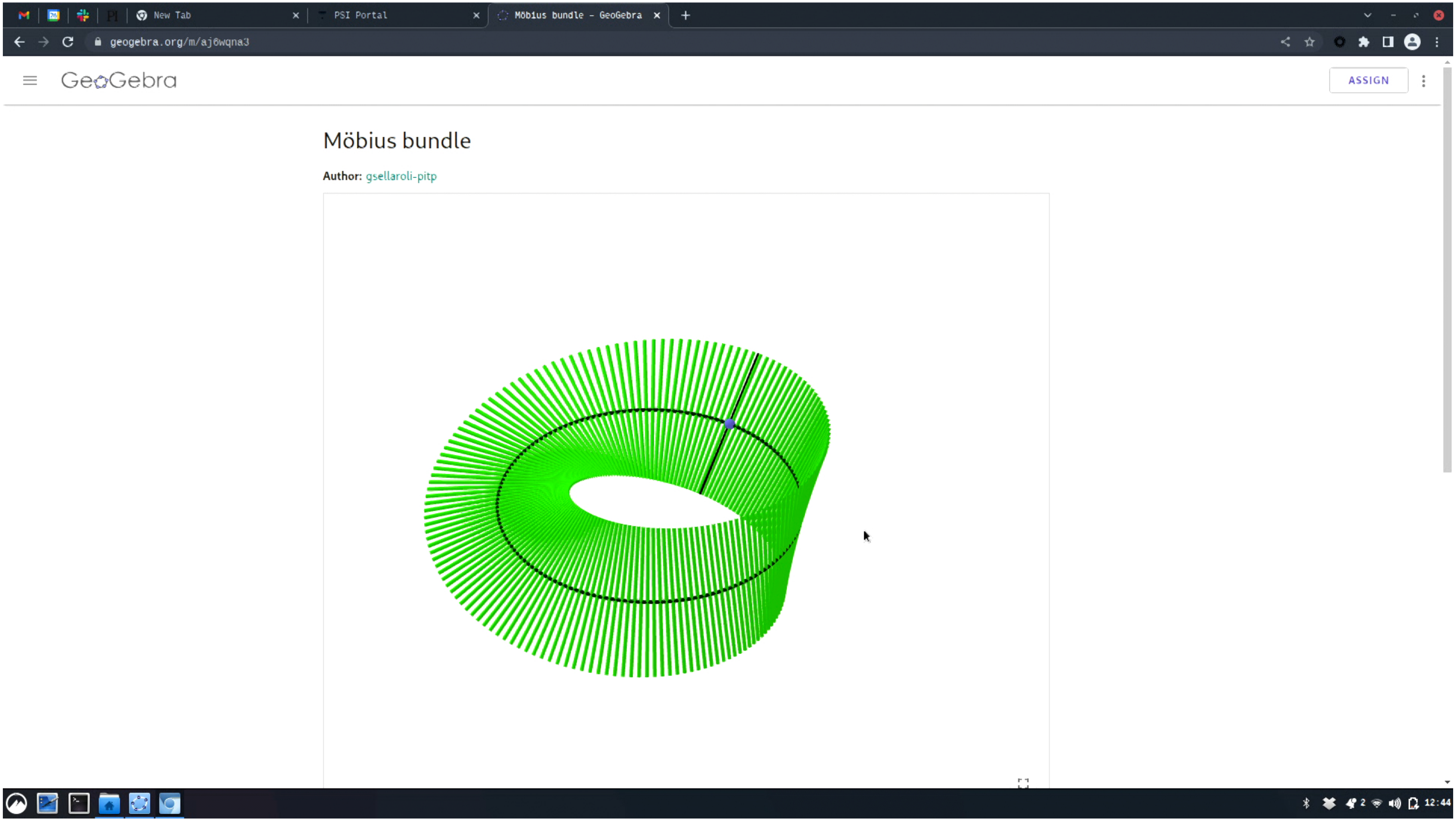
Fibre bundles

M manifold, space F want to

simplest way: $M \times F = \bigcup_{a \in M} \{a\} \times F$ → attach F to each

$\pi: M \times F \to M$ projection

$(a, f) \mapsto a$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Author: gsellaroli-pitp
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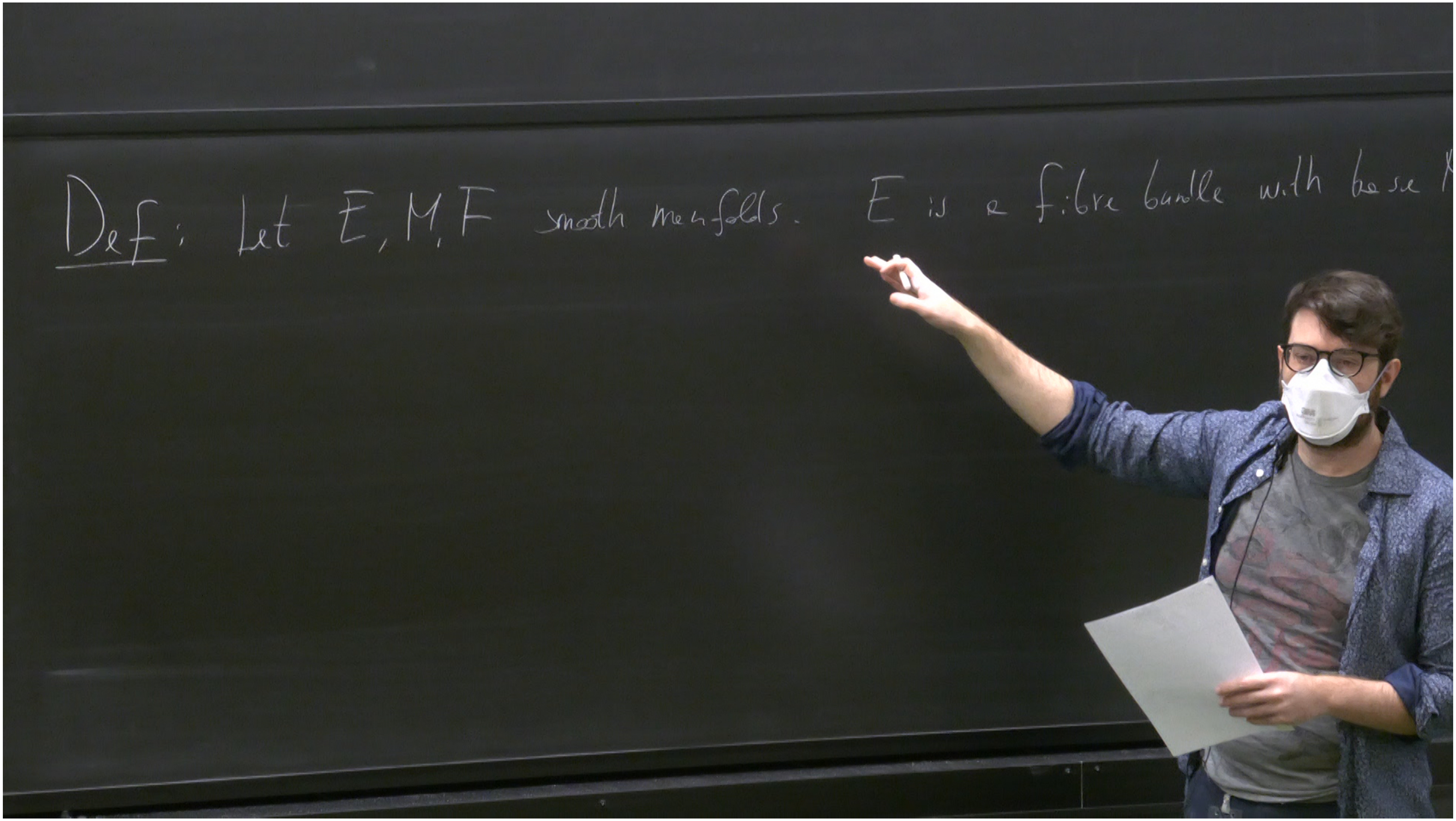
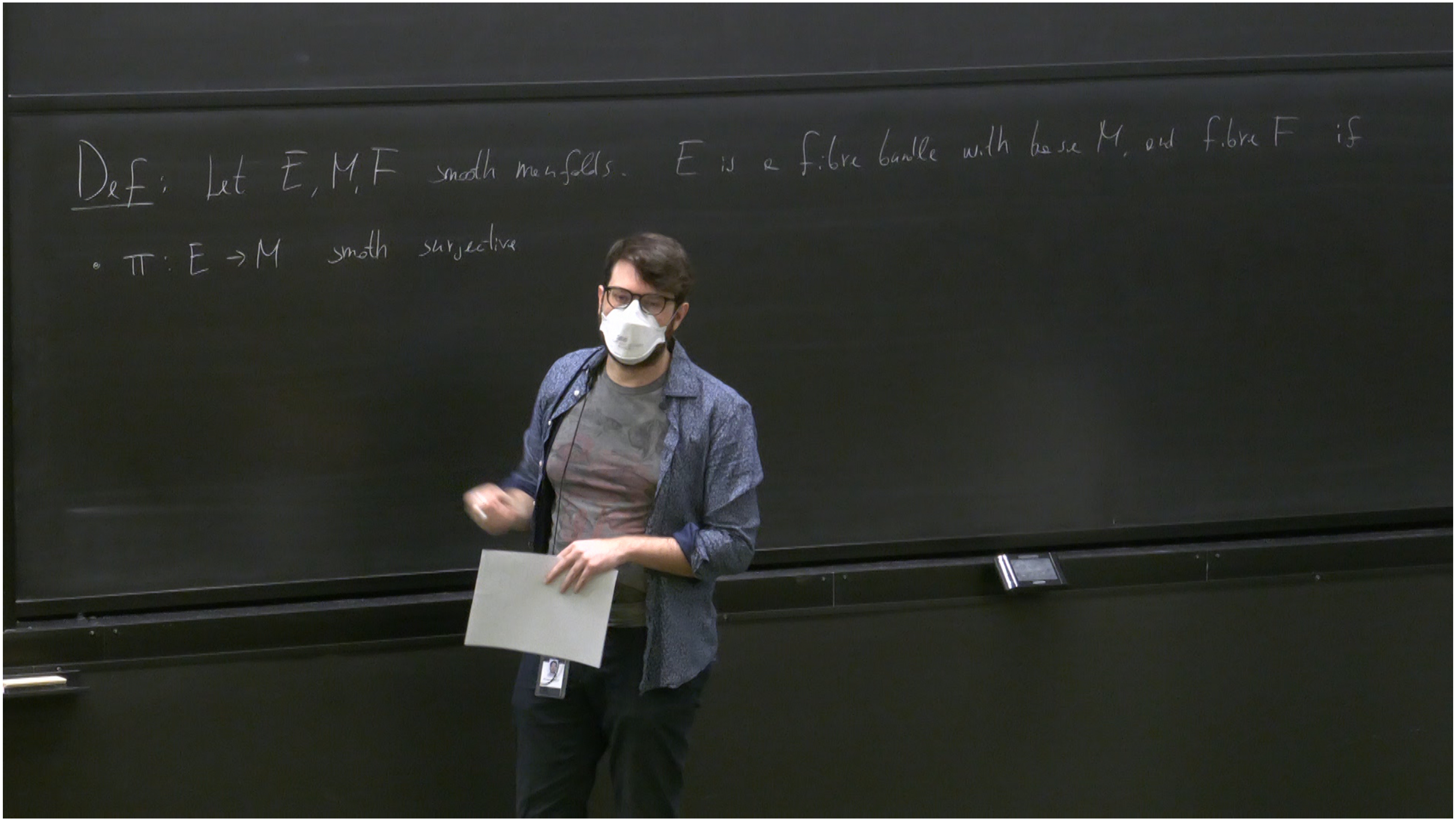
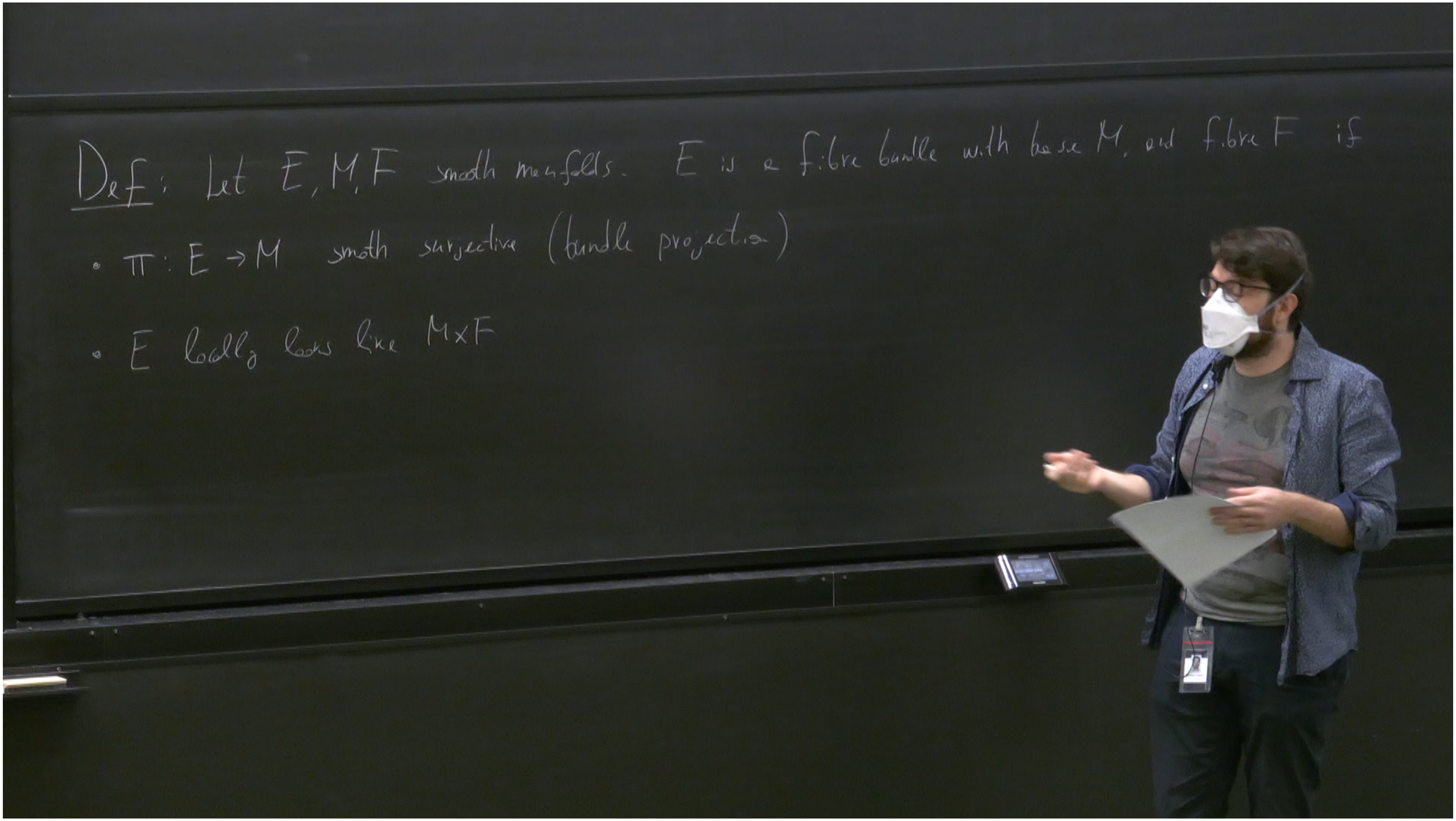
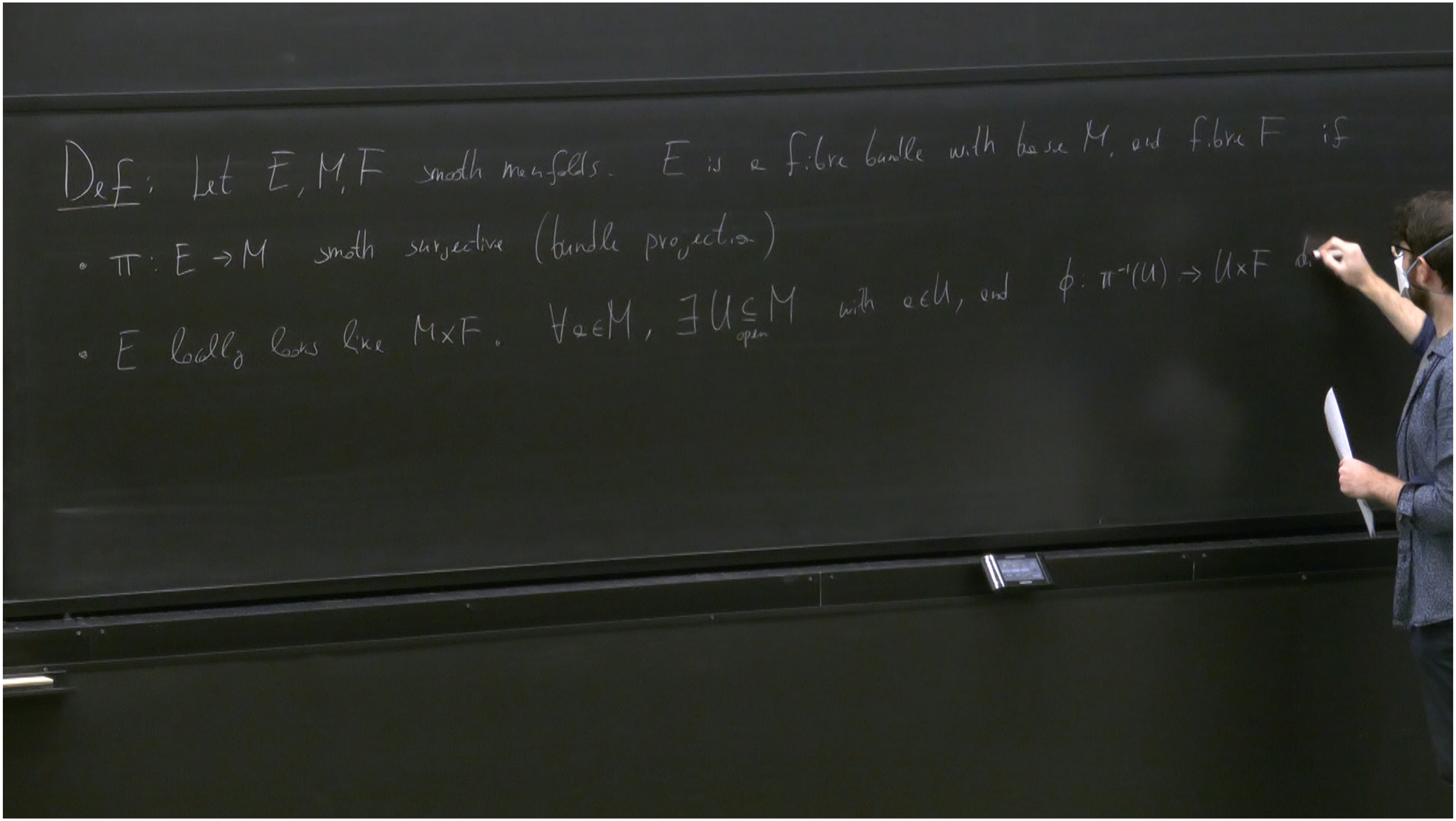
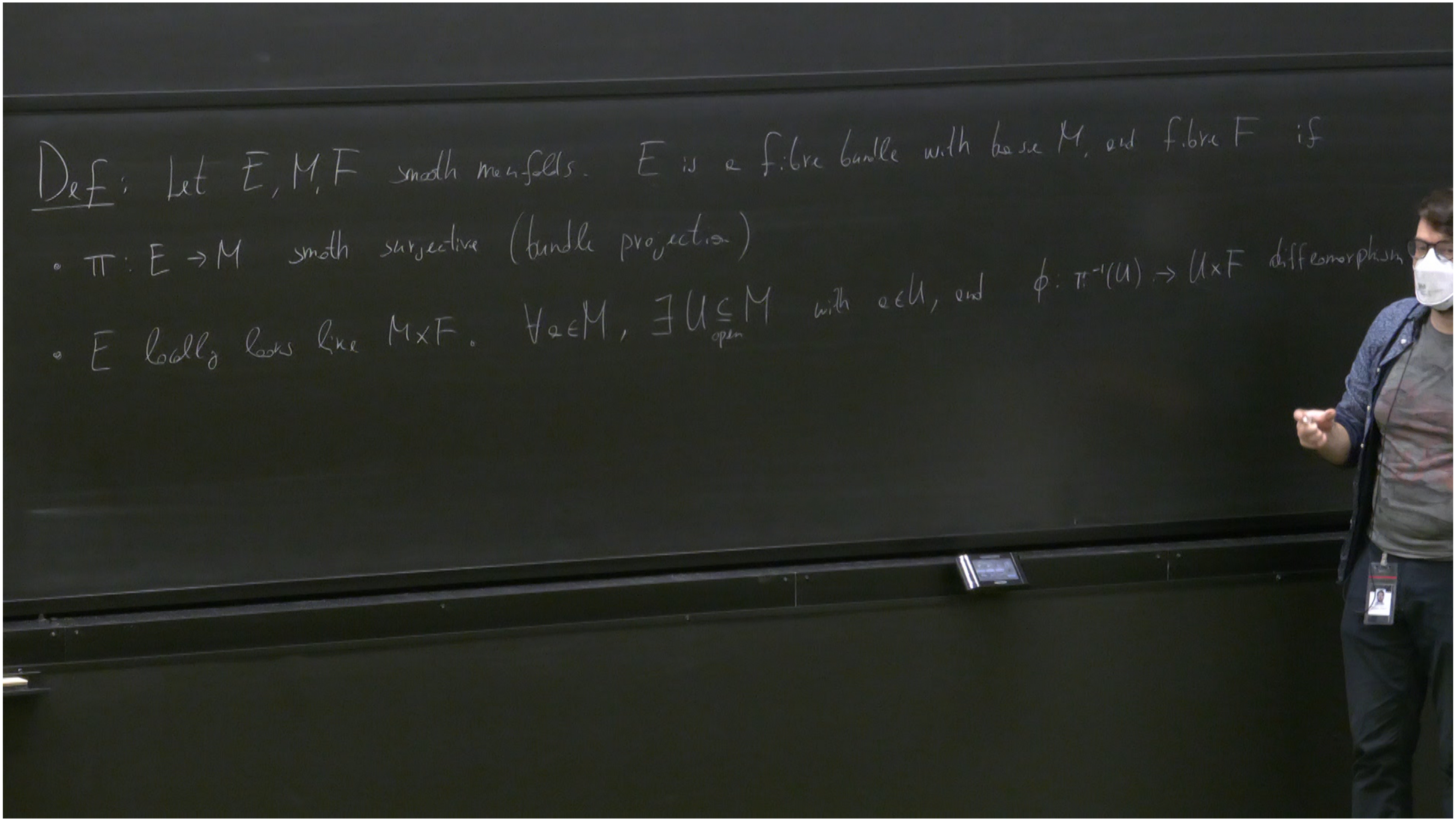
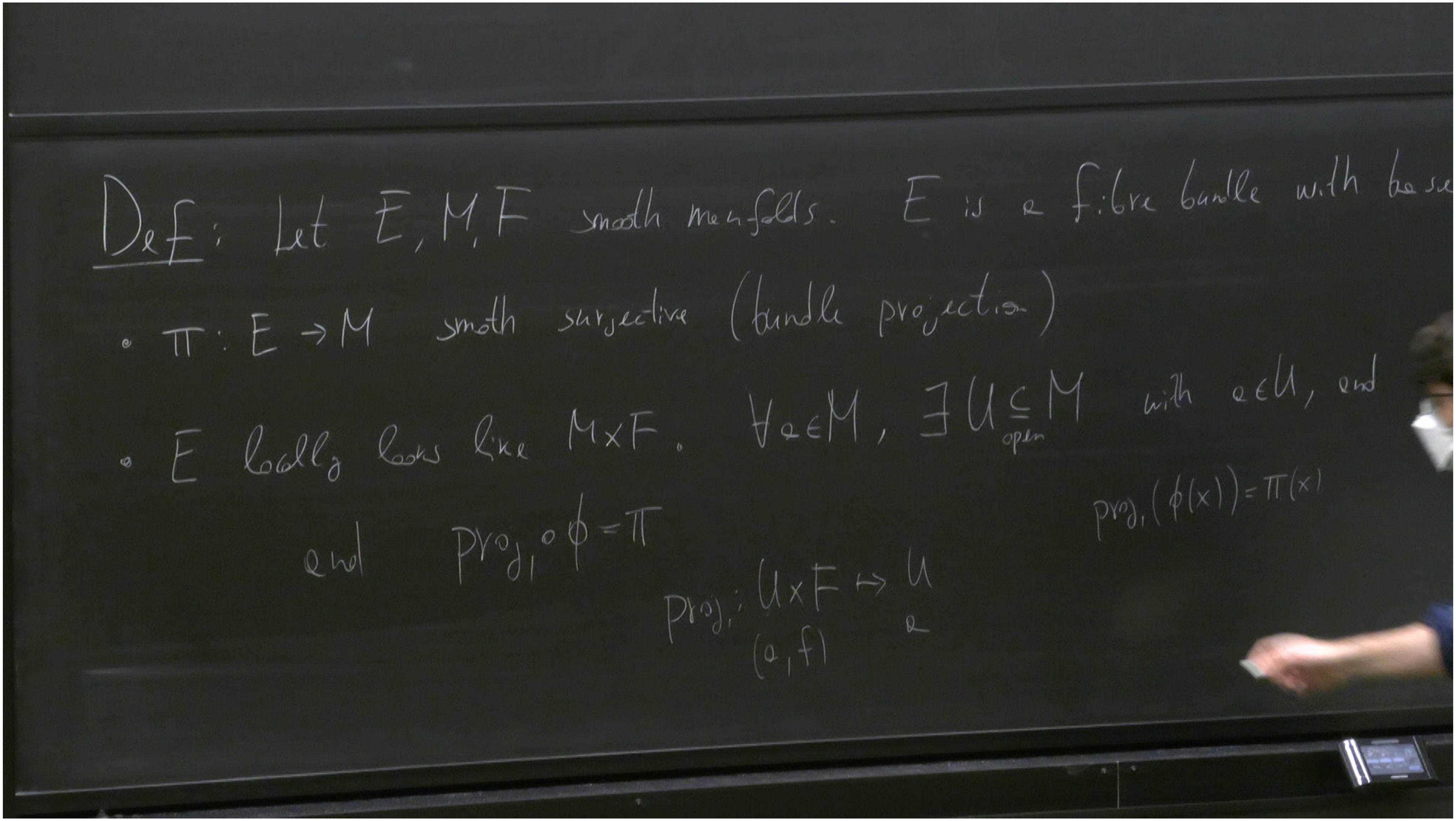

Def: Let E, M, F smooth manifolds. E is a fibre bundle with base


Def: Let E, M, F smooth manifolds. E is a fibre bundle with base M, and fibre F if
• π : E → M smooth surjective


Def: Let E, M, F smooth manifolds. E is a fibre bundle with base M, and fibre F if
• π : E → M smooth surjective (bundle projection)
• E locally looks like M×F


Def: Let E, M, F smooth manifolds. E is a fibre bundle with base M, and fibre F if
• $\pi : E \to M$ smooth surjective (bundle projection)
• E locally looks like $M \times F$. $\forall a \in M$, $\exists U \underset{\text{open}}{\subseteq} M$ with $a \in U$, and $\phi : \pi^{-1}(U) \to U \times F$


Def: Let E, M, F smooth manifolds. E is a fibre bundle with base M, and fibre F if
• π : E → M smooth surjective (bundle projection)
• E locally looks like M×F. ∀a∈M, ∃U ⊆ M with a∈U, and φ: π⁻¹(U) → U×F diffeomorphism
open


Def: Let E, M, F smooth manifolds. E is a fibre bundle with base
• $\pi : E \to M$ smooth surjective (bundle projection)
• E locally looks like $M \times F$. $\forall a \in M$, $\exists\, U \subseteq_{open} M$ with $a \in U$, and
and $proj_1 \circ \phi = \pi$
$proj_1(\phi(x)) = \pi(x)$
$proj_1 : U \times F \mapsto U$
$(a, f) \quad a$

mooth manifolds. E is a fibre bundle with base M, and fibre F if

rjective (bundle projection)

$\ldots \times F$. $\forall a \in M$, $\exists\, U \underset{\text{open}}{\subseteq} M$ with $a \in U$, and $\phi : \pi^{-1}(U) \to U \times F$ diffeomorphism

$= \pi$

$\mathrm{proj}_1(\phi(x)) = \pi(x)$

$\mathrm{proj}_1 : U \times F \mapsto U$

$(a, f) \qquad a$

if $x \in \pi^{-1}(\{a\})$ $\quad \mathrm{proj}_1(\phi(x)) = a$

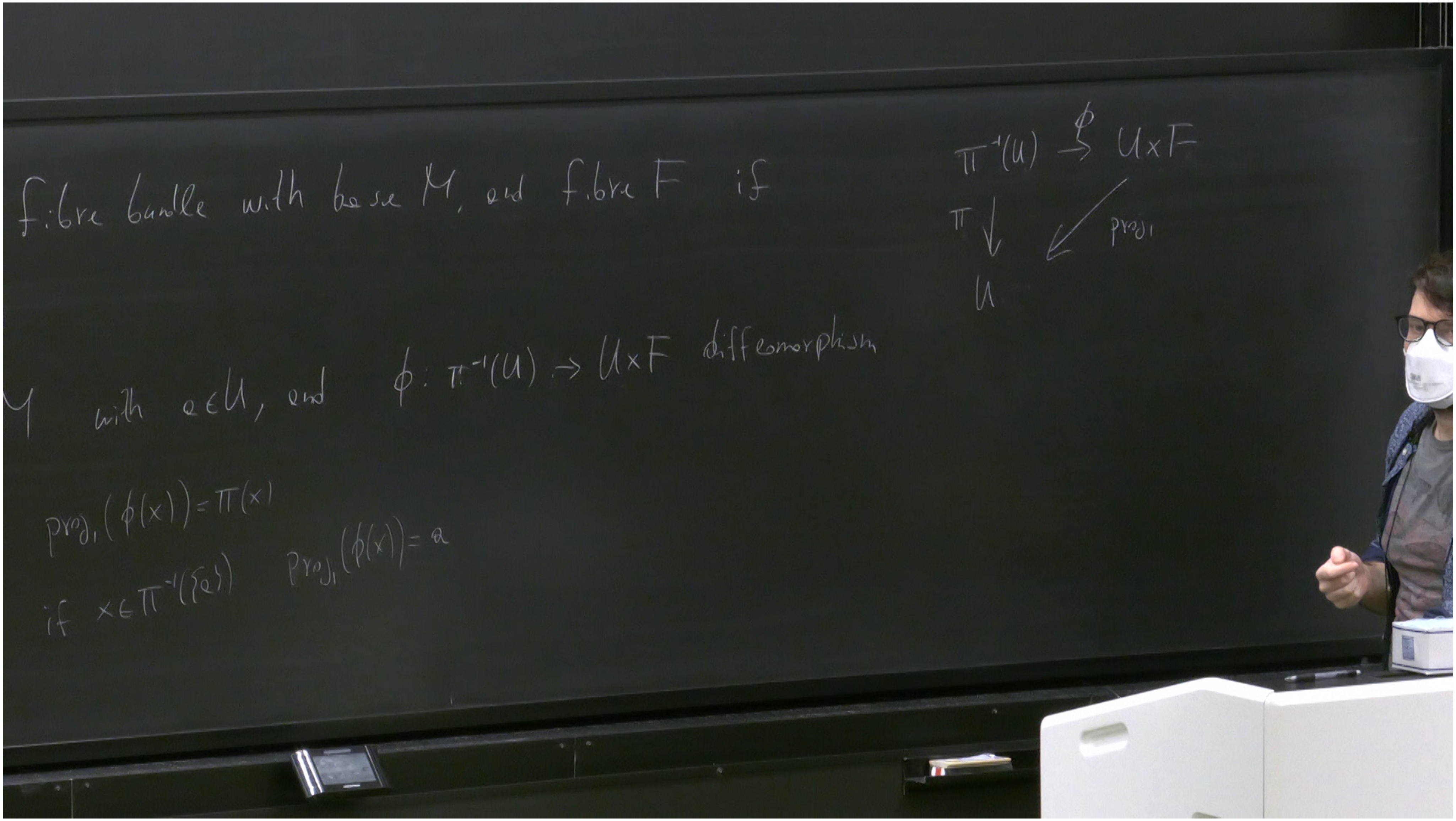
fibre bundle with base M, and fibre F if

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\ \pi \downarrow & & \\ U & & \end{array}$$

M with $a \in U$, and $\phi : \pi^{-1}(U) \to U \times F$ diffeomorphism

$proj_1(\phi(x)) = \pi(x)$

if $x \in \pi^{-1}(\{a\})$ $proj_1(\phi(x)) = a$


fibre bundle with base M, and fibre F if
$\pi^{-1}(U) \xrightarrow{\phi} U\times F$
π
proj$_1$
U
M with $a \in U$, and $\phi : \pi^{-1}(U) \to U\times F$ diffeomorphism
$\text{proj}_1(\phi(x)) = \pi(x)$
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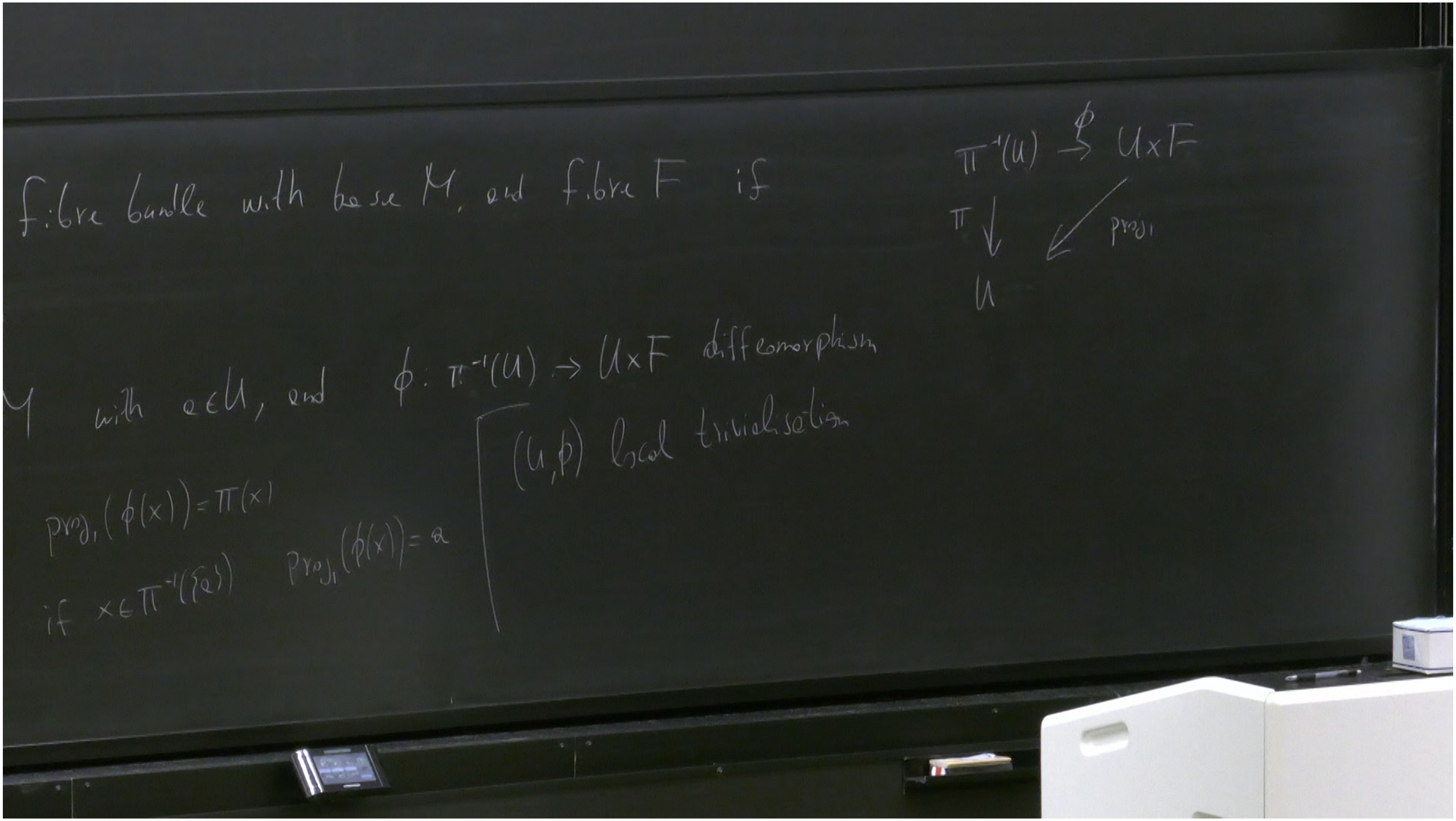
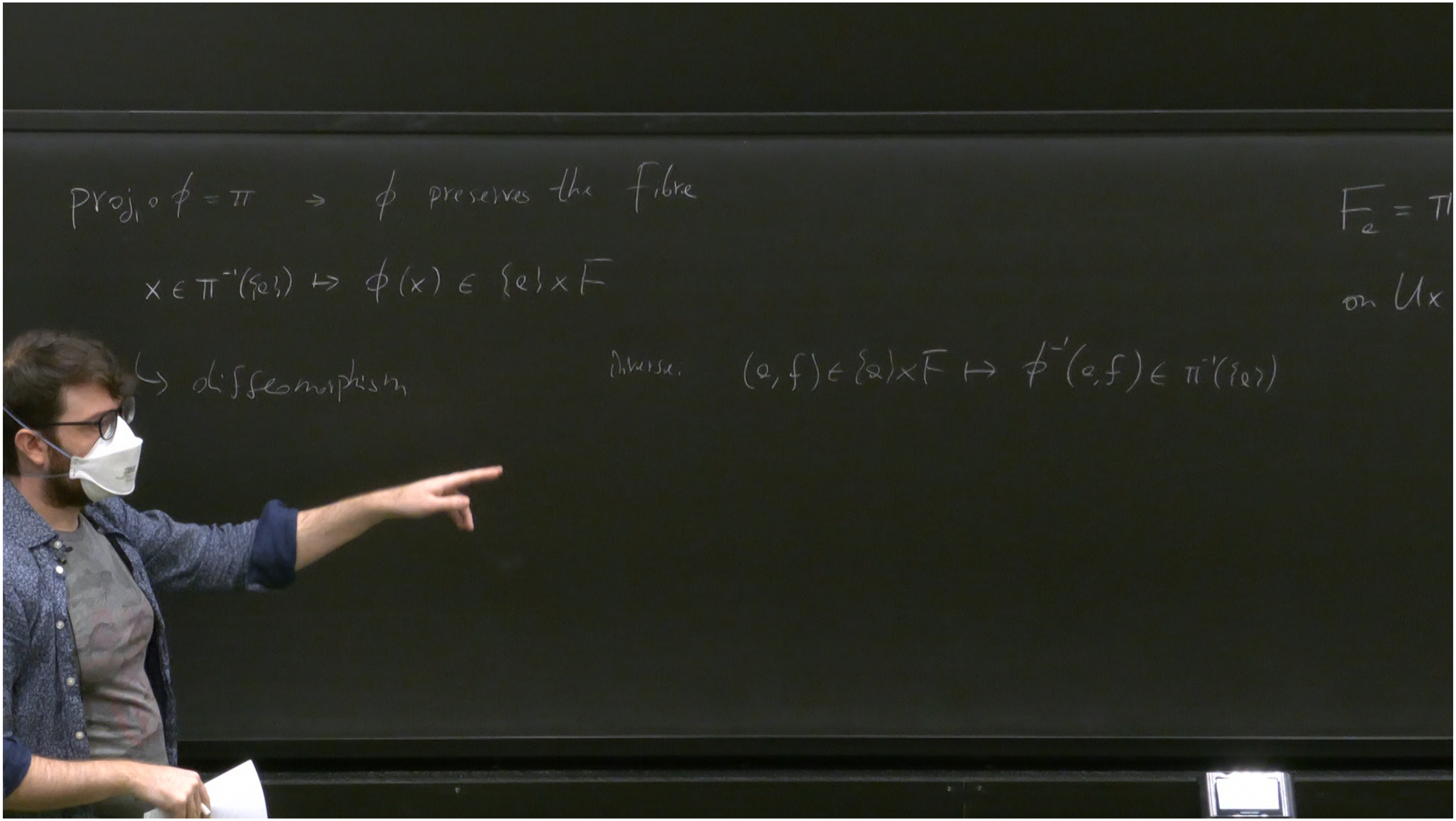

fibre bundle with base M, and fibre F if
$\pi^{-1}(U) \xrightarrow{\phi} U \times F$
π
proj$_1$
U
M with $a \in U$, and $\phi : \pi^{-1}(U) \to U \times F$ diffeomorphism
(U, ϕ) local trivialisation
proj$_1(\phi(x)) = \pi(x)$
if $x \in \pi^{-1}(\{a\})$ proj$_1(\phi(x)) = a$

$F_e = \pi^{-1}(\{e\})$ fibre at $e \in M$ $\rightarrow$ copy of F attached to $e \in M$

on $U \times F$, $\{e\} \times F$ is the copy of F attached to $e \in U$

- $\mathrm{proj}_1 \circ \phi = \pi \Rightarrow \phi$ preserves the fibre

$x \in \pi^{-1}(\{e\}) \mapsto$


$proj_1 \circ \phi = \pi$ → ϕ preserves the fibre
$x \in \pi^{-1}(\{e\}) \mapsto \phi(x) \in \{e\} \times F$
↳ diffeomorphism
Inverse: $(e,f) \in \{e\} \times F \mapsto \phi^{-1}(e,f) \in \pi^{-1}(\{e\})$
$F_e = \pi$
on U_x

$\mathrm{proj}_1 \circ \phi = \pi \;\rightarrow\; \phi$ preserves the fibre

$x \in \pi^{-1}(\{e\}) \mapsto \phi(x) \in \{e\} \times F$

$\hookrightarrow$ diffeomorphism

Inverse: $(e, f) \in \{e\} \times F \mapsto \phi^{-1}(e, f) \in \pi^{-1}(\{e\})$

$F_e \cong \{e\} \times F \cong F$

π surjective: $E = \bigcup_{e \in M} \pi^{-1}(e) = \bigcup_{e \in M} F_e$

$F_e = \pi$

on U_x

$F_e = \pi^{-1}(\{e\})$ fibre at $e \in M$ $\rightarrow$ copy of F attached to $e \in M$

on $U \times F$, $\{e\} \times F$ is the copy of F attached to $e \in U$

$\pi^{-1}(U) = \{ x \in E \mid \pi(x) \in U \}$ pre-image of U

$\downarrow$

subset of $\mathrm{Im}(\pi)$

π injective iff $\pi^{-1}(\{e\})$ has a single element

$F_e = \pi^{-1}(\{e\})$ fibre at $e \in M$ $\rightarrow$ copy of F attached to $e \in M$

on $U \times F$, $\{e\} \times F$ is the copy of F attached to $e \in U$

$e\})$

$$\pi^{-1}(U) = \{ x \in E \mid \pi(x) \in U \}$$ pre-image of U

$\downarrow$

subset of $\mathrm{Im}(\pi)$

π invertible iff $\pi^{-1}(\{e\})$ has a single element

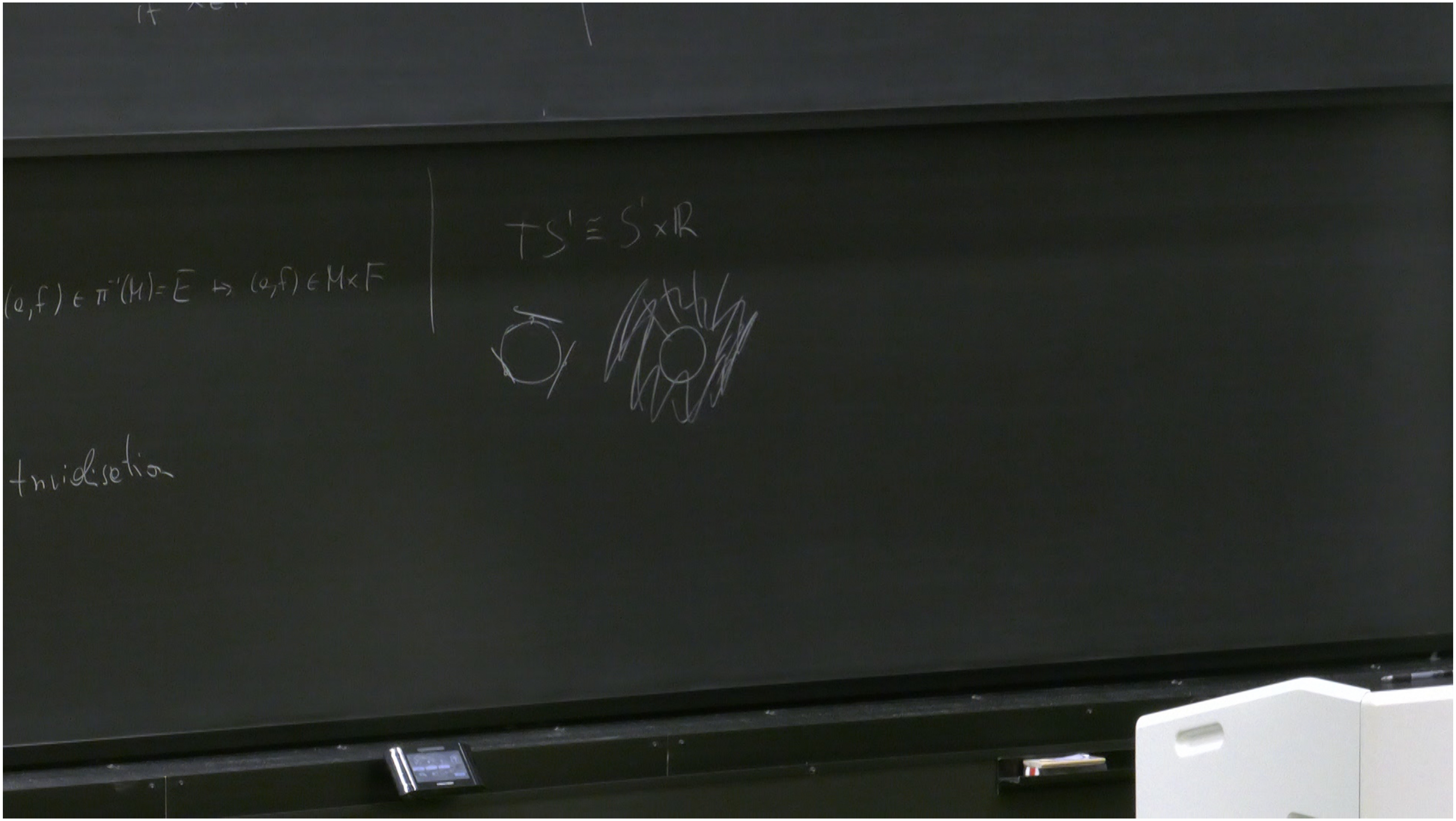
(e, f)

ex $\quad E = M \times F \quad \pi : (e, f) \in E \mapsto e \in M$

global trivialisation $\quad (U, \phi) = (M, id_{M \times F}) \quad \phi : (e, f) \in \pi^{-1}(M) = E \mapsto (e, f) \in M \times F$

$$F_e = \{e\} \times F$$

$\pi : E \to M$ is trivial if it admits a global trivialisation


$TS^1 \cong S^1 \times \mathbb{R}$
$(e,f) \in \pi^{-1}(M) = E \leftrightarrow (e,f) \in M \times F$
trivialisation

$a \in M$

Ex 2 M manifold, TM tangent bundle, base M, fibre $F = \mathbb{R}^{\dim(M)}$

$q \in M$

bre $F = \mathbb{R}^{\dim(M)}$

$$TM = \bigcup_{q \in M} \{q\} \times T_q M$$

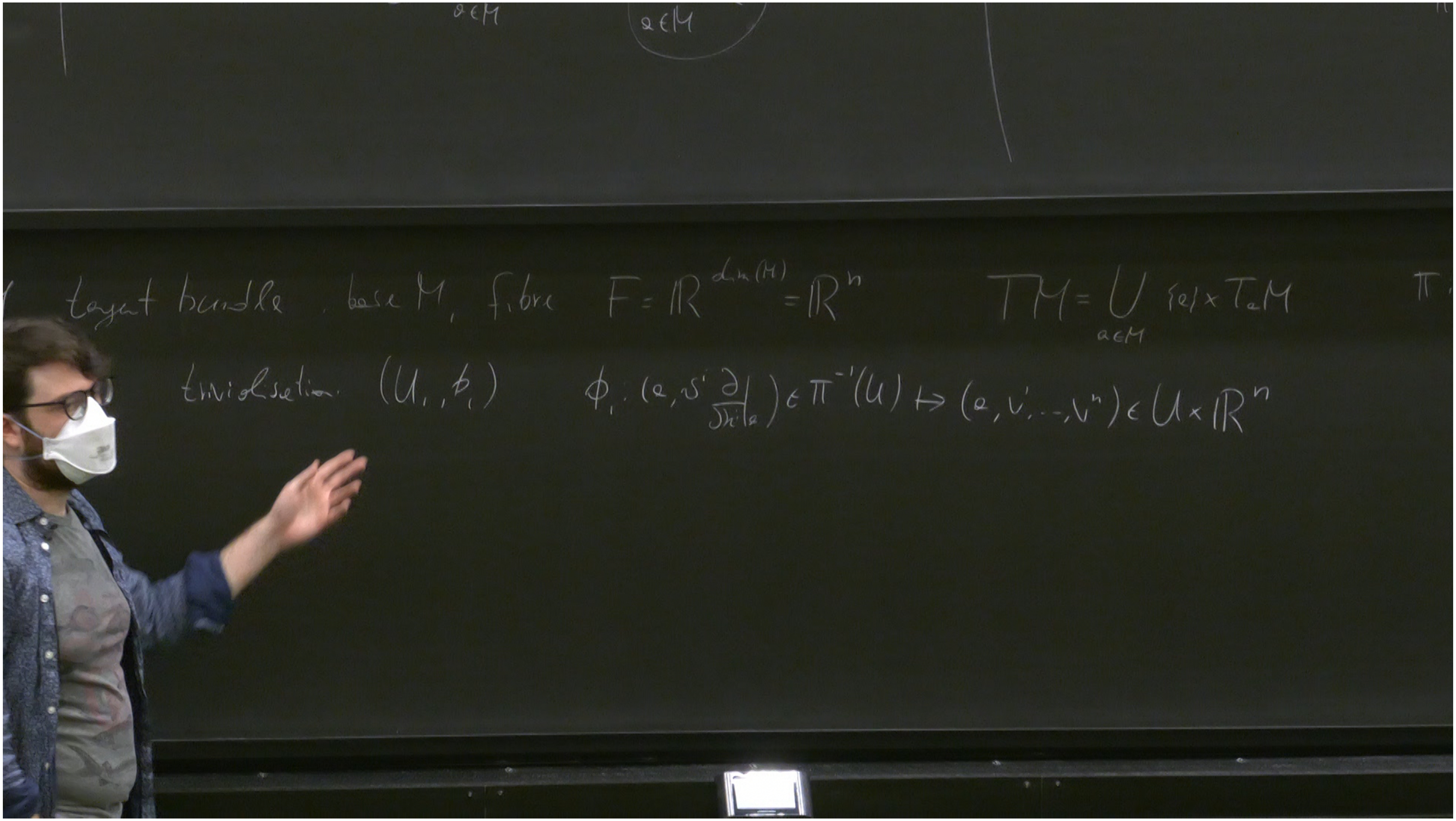
$\pi : (q, v) \in TM \mapsto q \in M$

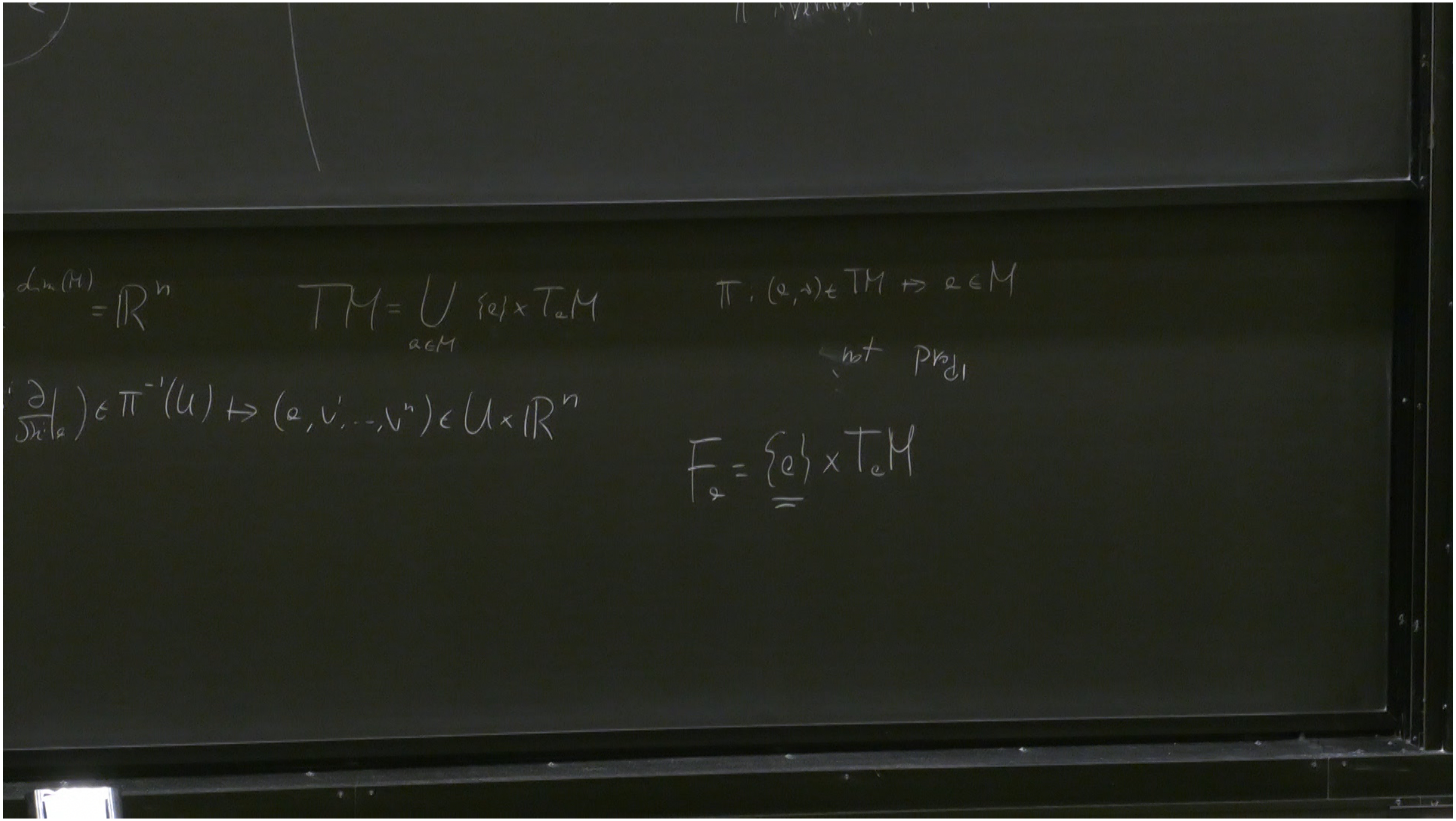
not prod

Ex 2

M manifold, TM tangent bundle, base M, fibre $F = \mathbb{R}^{\dim(M)}$

if $(U_i, (x^1, \dots, x^n)) \in A_M$ trivialisation (U_i, ϕ_i) $\phi_i : (e, v^i \frac{\partial}{\partial x^i}\big|_e) \in$


tangent bundle, base M, fibre $F = \mathbb{R}^{\dim(M)} = \mathbb{R}^n$
$TM = \bigcup_{a\in M} \{a\} \times T_aM$
trivialisation (U_i, ϕ_i)
$\phi_i : (a, v^i \frac{\partial}{\partial x^i}\big|_a) \in \pi^{-1}(U) \mapsto (a, v^1, \dots, v^n) \in U \times \mathbb{R}^n$


$\dim(M)$
$= \mathbb{R}^n$
$TM = \bigcup_{e \in M} \{e\} \times T_e M$
$\pi : (e, v) \in TM \mapsto e \in M$
not pr_1
$\frac{\partial}{\partial x^i}\Big|_e) \in \pi^{-1}(U) \mapsto (e, v^1, \ldots, v^n) \in U \times \mathbb{R}^n$
$F_e = \{e\} \times T_e M$

We can give more structure to F $\rightarrow$ e.g. F vector space

Def: a real vector bundle of rank n is a fiber bundle $\pi: E \rightarrow M$ with $F = \mathbb{R}^n$

re structure to F $\rightarrow$ e.g. F vector space

vector bundle of rank n is a fibre bundle $\pi : E \rightarrow M$ with $F = \mathbb{R}^n$, such that each $F_e = \pi$

F vector space

a fiber bundle $\pi: E \to M$ with $F = \mathbb{R}^n$, such that each $F_e = \pi^{-1}(\{e\})$ is a real vector space

We can give more structure to F $\rightarrow$ e.g. F vector space

Def: a real vector bundle of rank n is a fibre bundle $\pi : E \rightarrow M$ with $F = \mathbb{R}^n$,

and if (U, ϕ) local trivialisation, $x \in \pi^{-1}(\{e\}) \mapsto \phi(x) \in \{e\} \times F$

F $\rightarrow$ e.g. F vector space

rank n is a fibre bundle $\pi : E \to M$ with $F = \mathbb{R}^n$, such that each $F_e = \pi^{-1}(\{e\})$ is a real vec

ialisation, $x \in \pi^{-1}(\{e\}) \mapsto \phi(x) \in \underbrace{\{e\} \times \mathbb{R}^n}$ linear ($\Rightarrow$ vector space isomorphism)

$\downarrow$ $\alpha(e, v) + \beta(e, w) = (e, \alpha v + \beta w)$

$\dim(M) = \mathbb{R}^n$

$TM = \bigcup_{a \in M} \{a\} \times T_a M$

$\pi : (a, v) \in TM \mapsto a \in M$

not prod

$\frac{\partial}{\partial x^i}\Big|_a) \in \pi^{-1}(U) \mapsto (a, v^1, \dots, v^n) \in U \times \mathbb{R}^n$

$F_a = \{a\} \times T_a M \quad \alpha(a, v) = (s, \alpha v)$