Title: Mathematical Physics Core Lecture

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Collection: Mathematical Physics - Core 2023/24

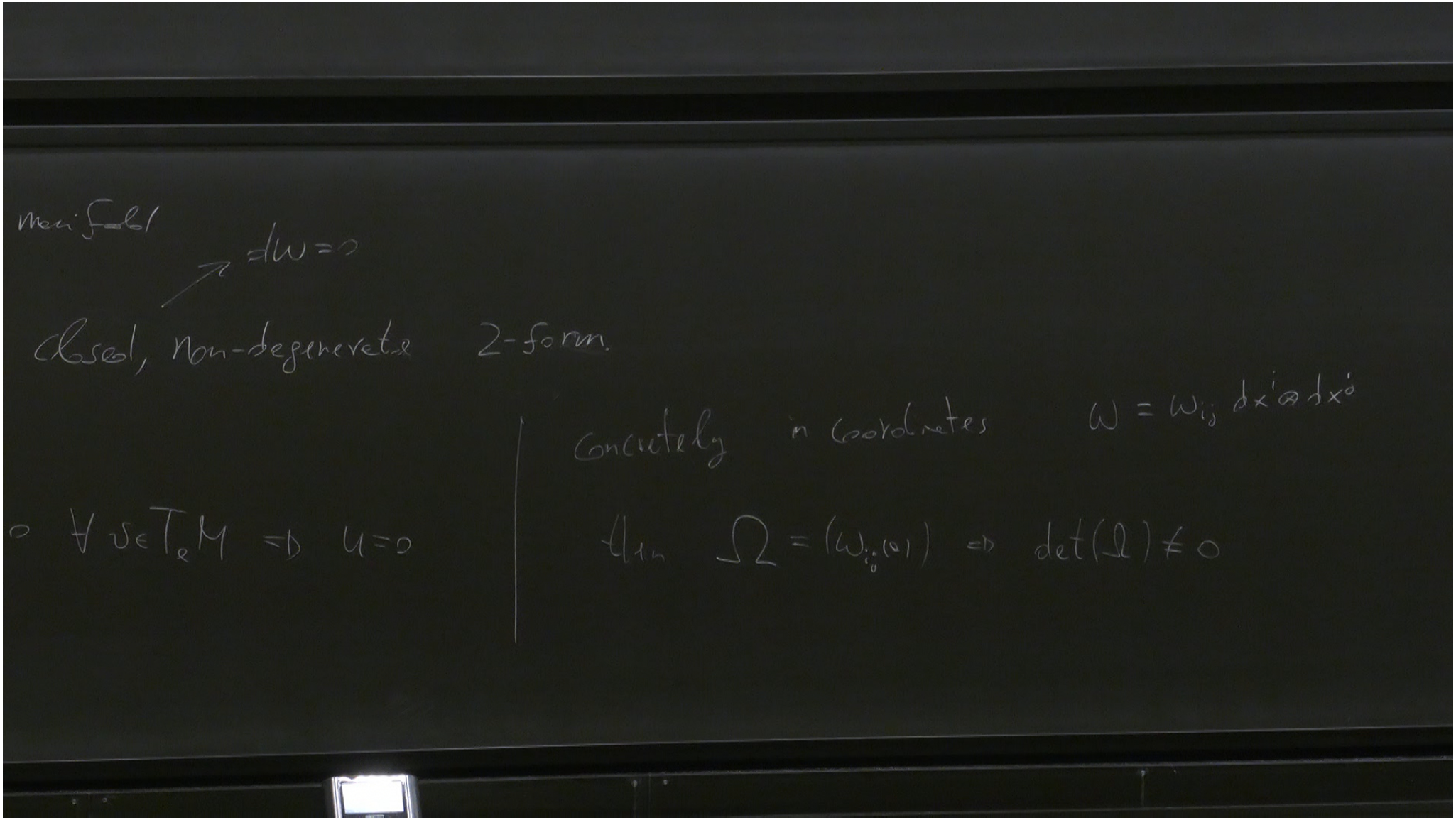
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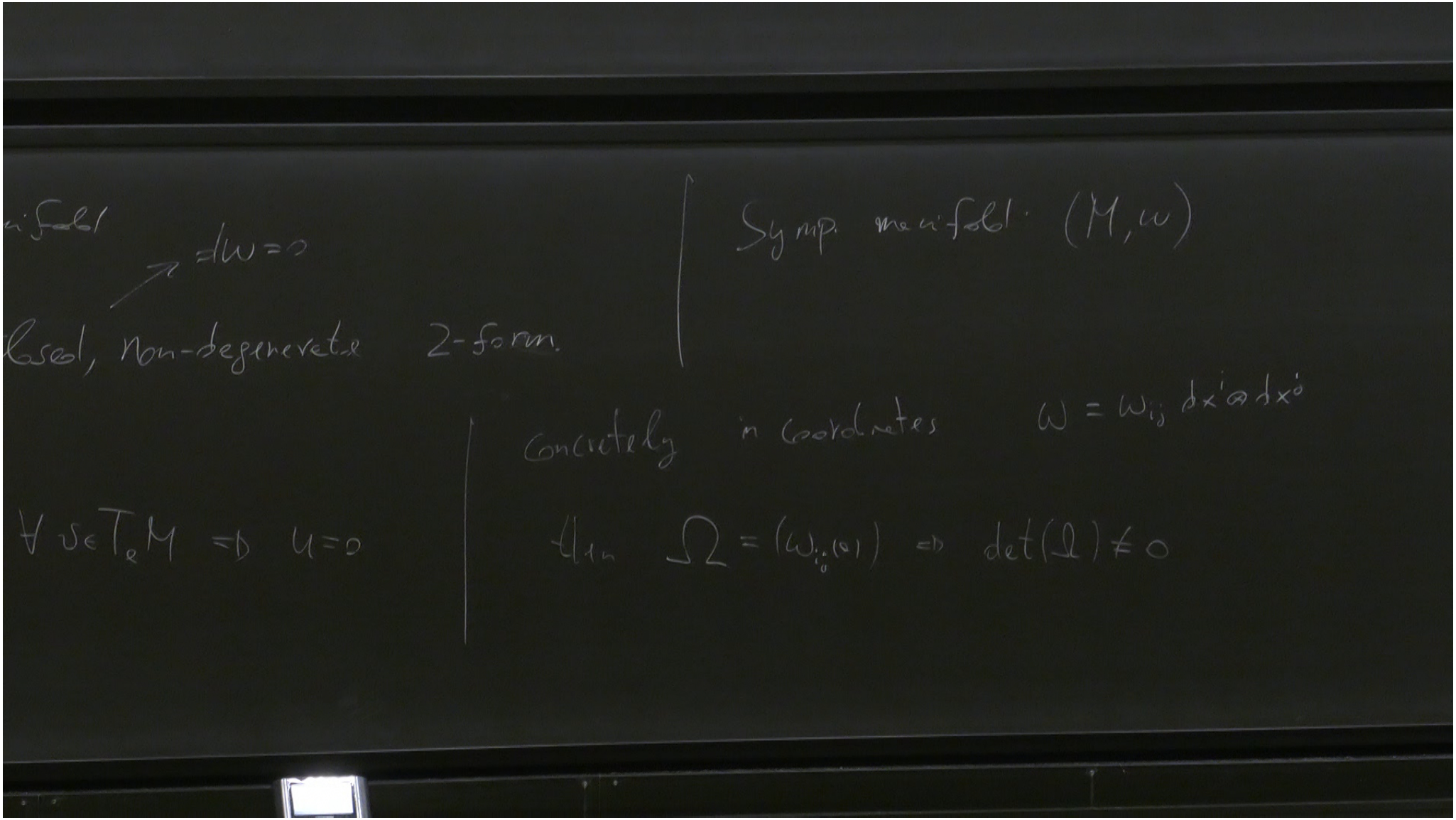
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Symplectic manifolds

Def a symplectic form ω on M (smooth manifold) is a closed ($d\omega = 0$), non-degenerate

non-degenerate: if $\omega(x) = (x, \omega_x)$ $\quad \omega_x(u, v) = 0 \ \forall v \in T_x M \Rightarrow u = 0$


manifold
$d\omega = 0$
closed, non-degenerate 2-form.
$\forall\, v \in T_e M \Rightarrow u = 0$
Concretely in coordinates $\omega = \omega_{ij}\, dx^i \otimes dx^j$
then $\Omega = (\omega_{ij}(0)) \Rightarrow \det(\Omega) \neq 0$


dω = 0
closed, non-degenerate 2-form.
Symp. manifold: (M, ω)
Concretely in coordinates ω = ω_ij dx^i ⊗ dx^j
∀ v ∈ T_e M ⇒ u = 0
then Ω = (ω_ij(0)) ⇒ det(Ω) ≠ 0

Very restrictive!

- $\dim(M)$ must be even

$$\det(\Omega) = \det(-\Omega^t)$$
$$= (-1)^{\dim(M)} \det(\Omega^t)$$
$$= (-1)^{\dim(M)} \det(\Omega)$$

Thm (Darboux theorem)

if (M, ω) is symplectic and $\dim(M) = 2n$

for each $e \in M$, there is a chart $(U, \varphi = (q^1, \dots, q^n, P_1, \dots, P_n))$

containing e s.t.

$$\omega|_U = dq^i \wedge dP_i$$

!

even

$(-\Omega^t)$

$\det(\Omega^t)$

$\det(\Omega)$

Thm (Darboux theorem)

if (M, ω) is symplectic and $\dim(M) = 2n$

for each $a \in M$, there is a chart $(U, \varphi = (q^1, \dots, q^n, p_1, \dots, p_n))$

containing a s.t.

$$\omega|_U = dq^i \wedge dp_i$$

→ locally all symplectic manifolds are the same → differences are global

(ex) R^{2n} $\omega = \Omega_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ $\Omega = \begin{pmatrix} 0_n & \\ -\mathbb{1}_n & \end{pmatrix}$

$$\begin{cases} q^i = x^i & i=1,\dots n \\ p_i = x^{i+n} & i=1,\dots n \end{cases}$$

$$\vec{x} = (\vec{q}, \vec{p})$$

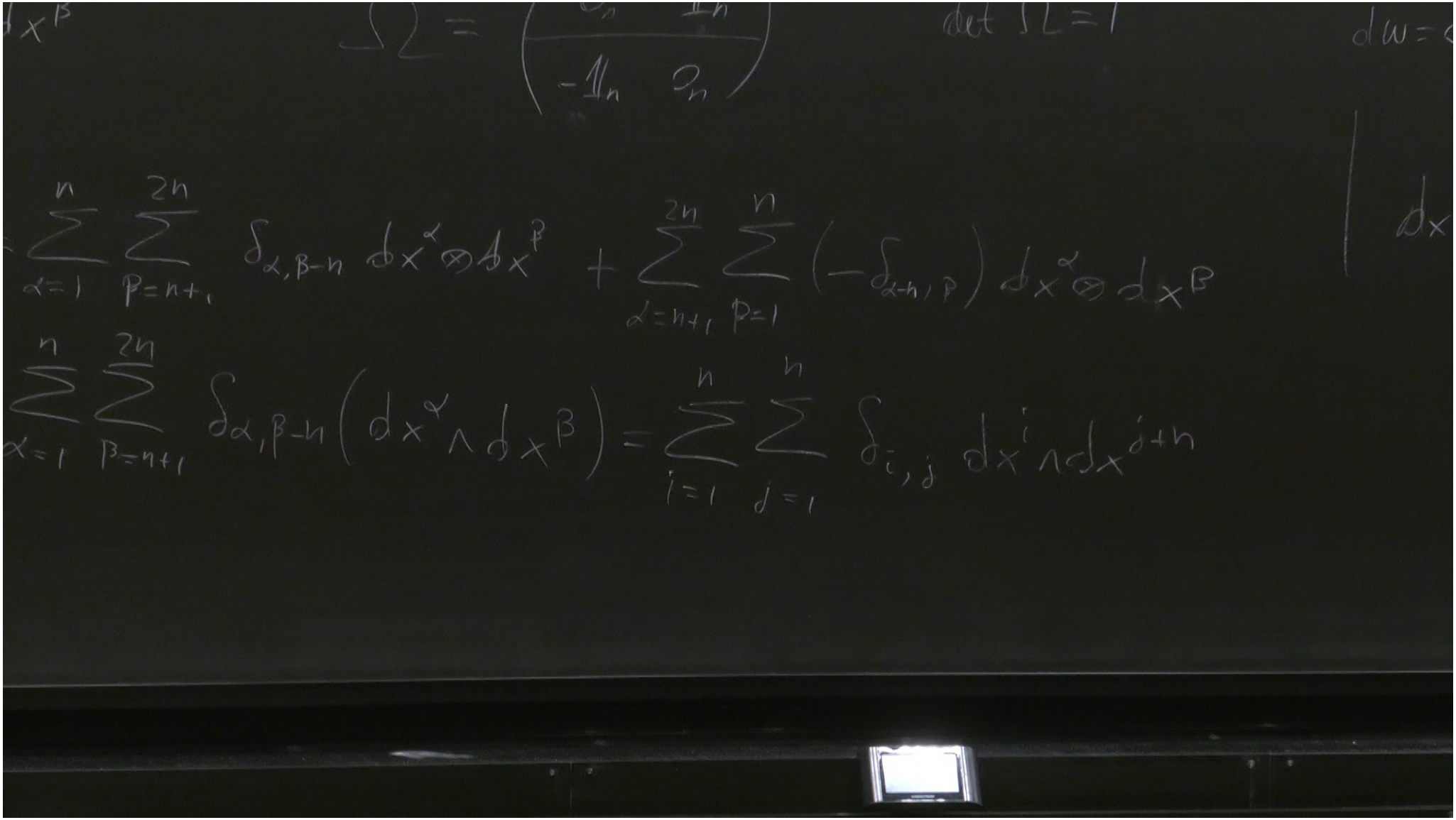
$$\omega = \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} \delta_{\alpha,\beta-n}\, dx^\alpha \otimes dx^\beta + \sum_{\alpha=1}^{}$$

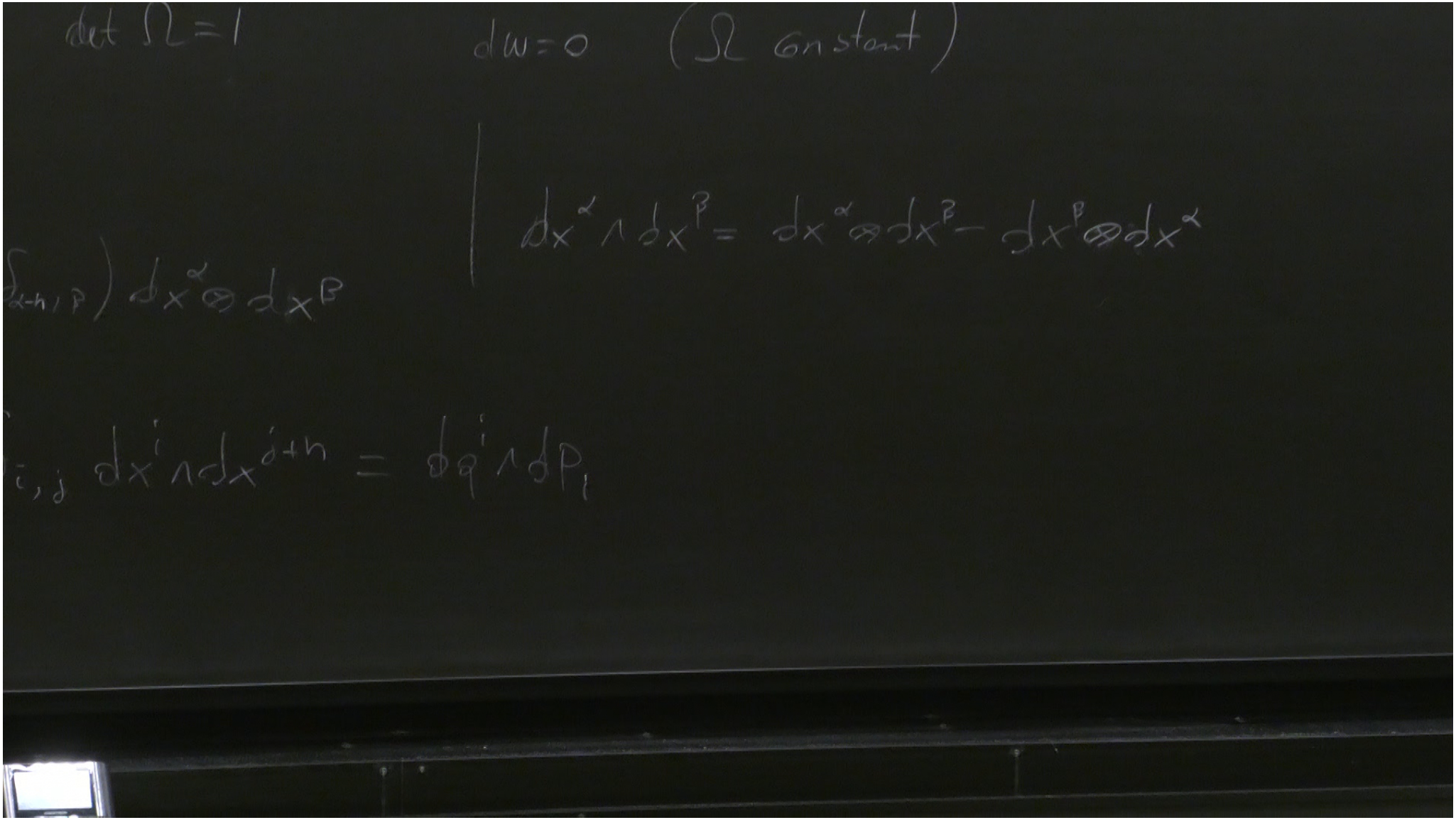
$$= \sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} \delta_{\alpha,\beta-n} \left(dx^\alpha \wedge dx^\beta \right)$$

$$\begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix} \qquad \det \Omega = 1 \qquad d\omega = 0 \quad (\Omega \text{ constant})$$

$$+ \sum_{\alpha=n+1}^{2n} \sum_{\beta=1}^{n} (-\delta_{\alpha-n,\beta})\, dx^\alpha \otimes dx^\beta$$

$$dx^\alpha \wedge dx^\beta = dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha$$


$\Omega = \begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix}$
$\det \Omega = 1$
$\sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} \delta_{\alpha,\beta-n}\, dx^\alpha \otimes dx^\beta + \sum_{\alpha=n+1}^{2n} \sum_{\beta=1}^{n} (-\delta_{\alpha-n,\beta})\, dx^\alpha \otimes dx^\beta$
$\sum_{\alpha=1}^{n} \sum_{\beta=n+1}^{2n} \delta_{\alpha,\beta-n} (dx^\alpha \wedge dx^\beta) = \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{i,j}\, dx^i \wedge dx^{j+n}$

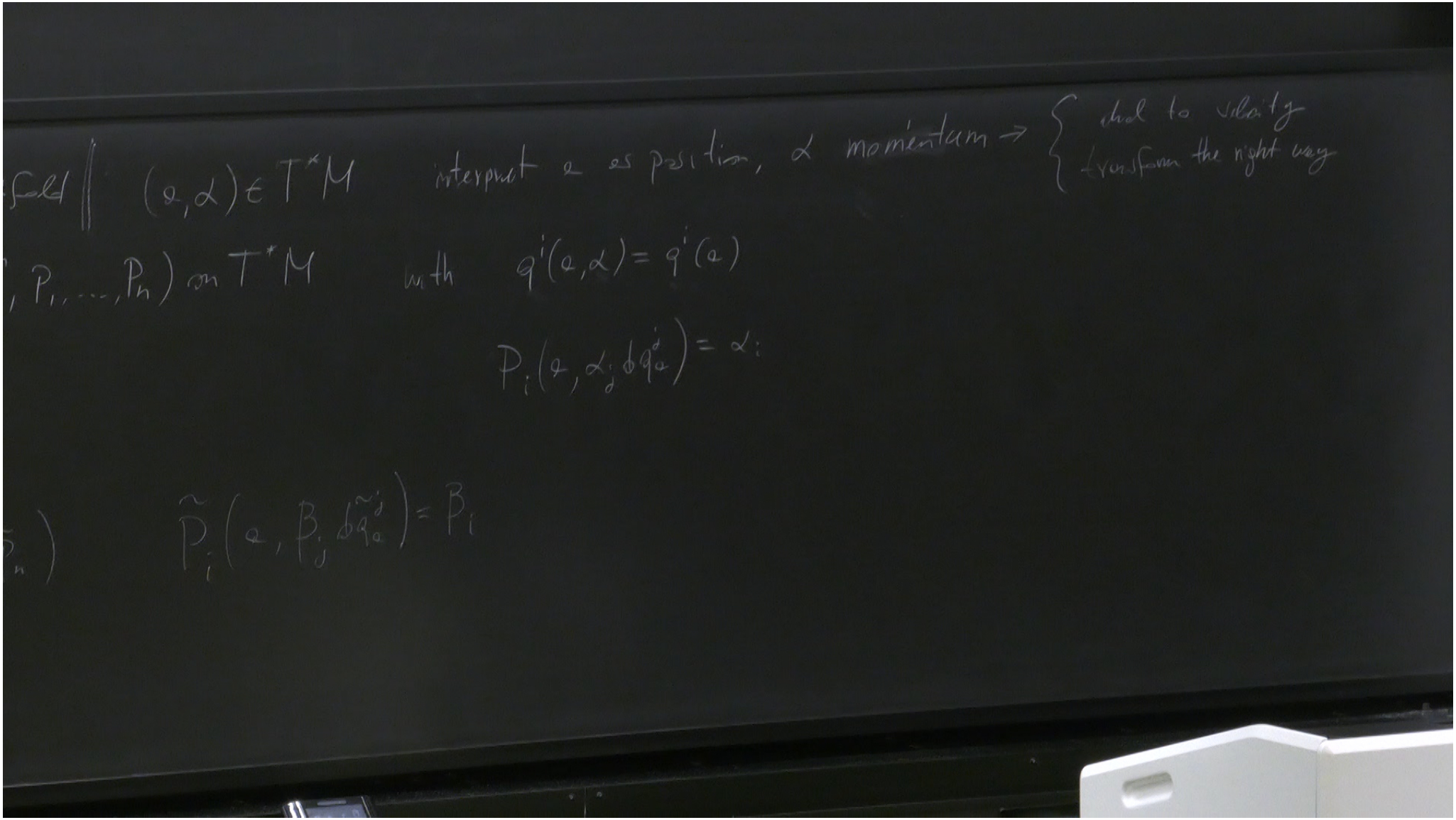

det Ω = 1
dω = 0 (Ω constant)
$dx^\alpha \wedge dx^\beta = dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha$
$\sum_{i,j} dx^i \wedge dx^{j+n} = dq^i \wedge dp_i$

(Ex) M smooth manifold, T^*M is (canonically) a symplectic manifold $\|$ $(q,\alpha) \in T$

Coordinates $(q^1, \dots, q^n)$ on M → extend to $(q^1, \dots, q^n, P_1, \dots, P_n)$ on T^*M

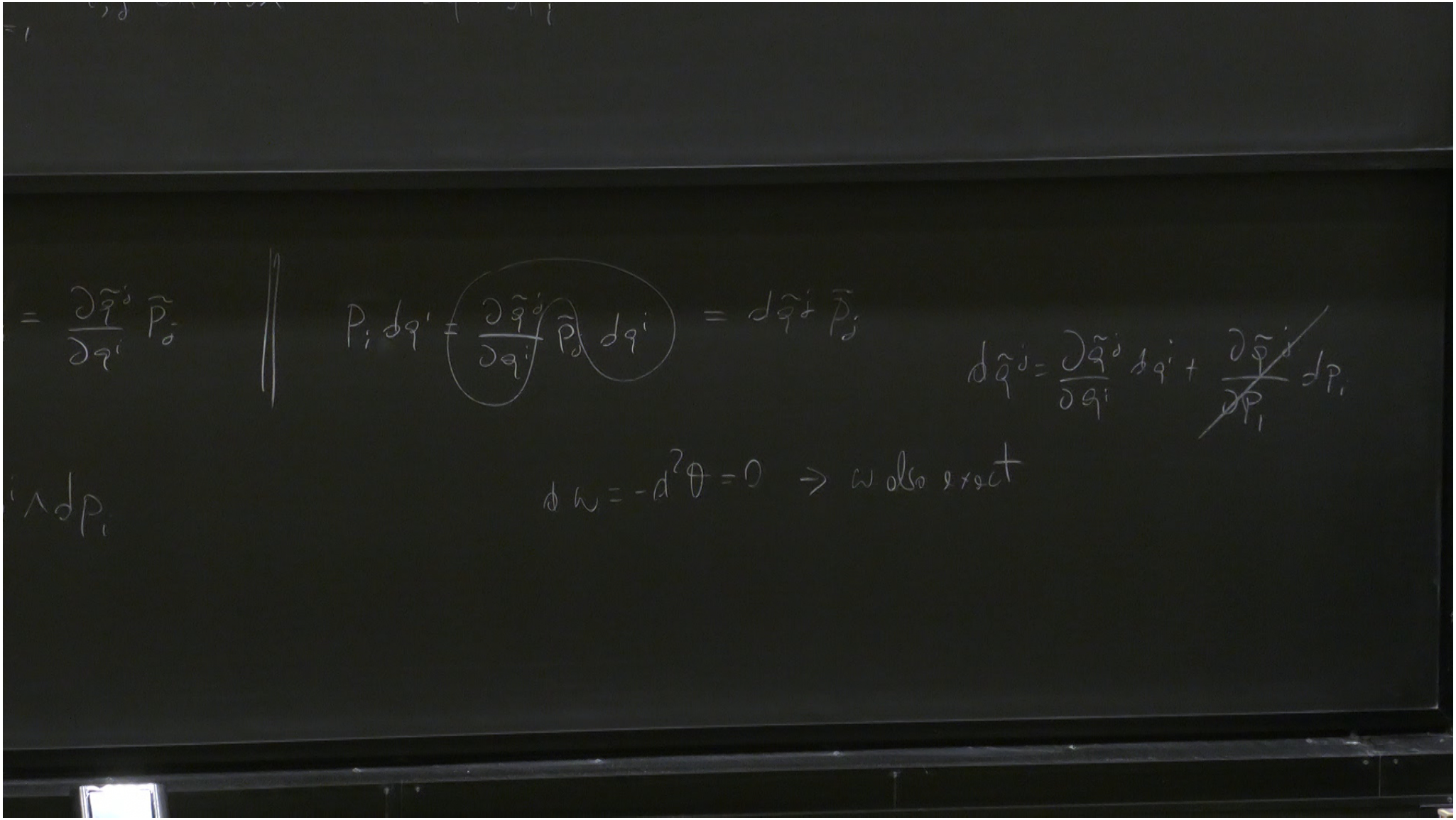
canonical 1-form: $\theta = P_i\, dq^i$ → 1-form on T^*M

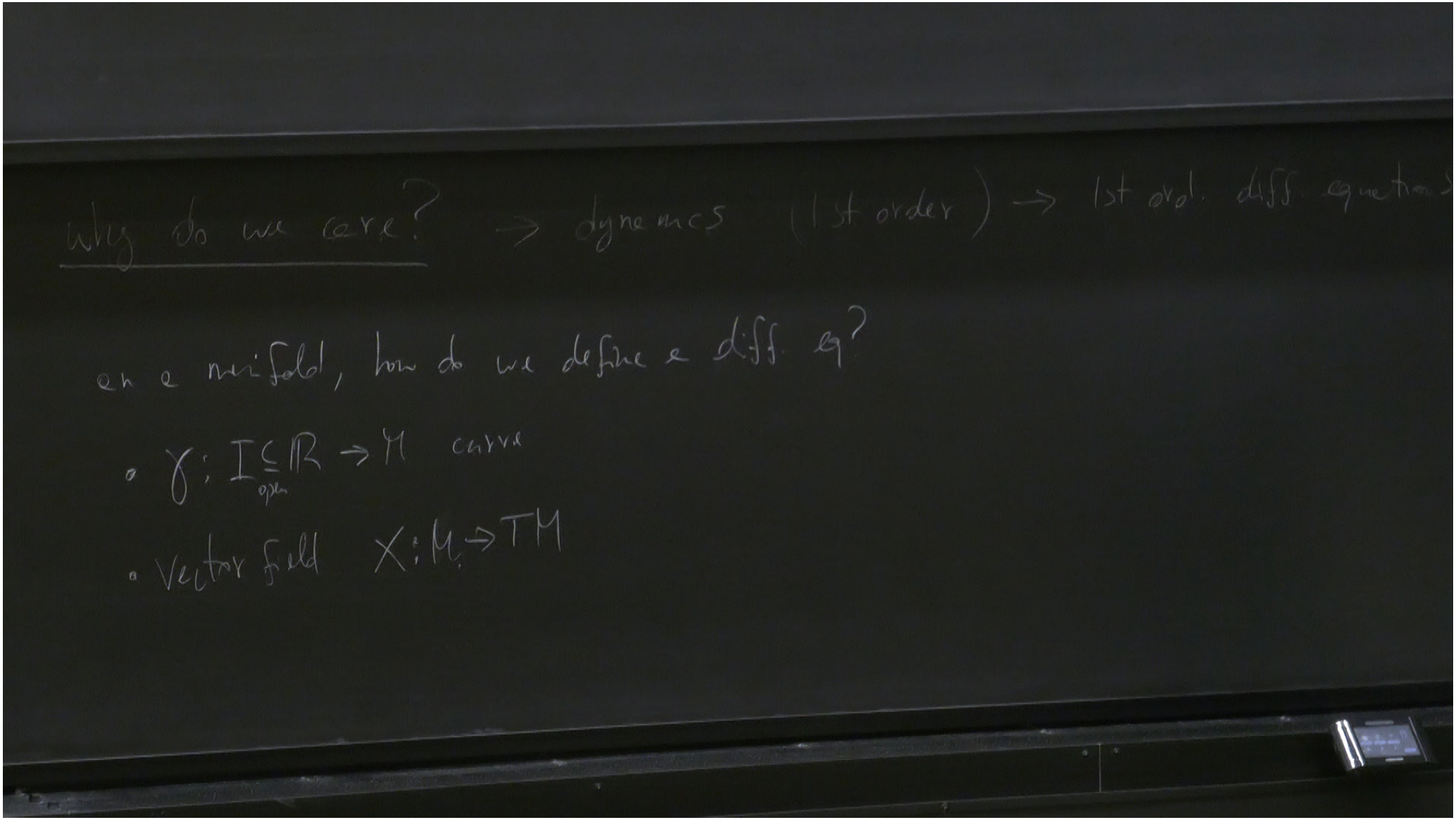
Change coordinates: $(\tilde{q}^1, \dots, \tilde{q}^n)$ on M → $(\tilde{q}^1, \dots, \tilde{q}^n, \tilde{P}_1, \dots \tilde{P}_n)$ $\quad \tilde{P}_i(q,$

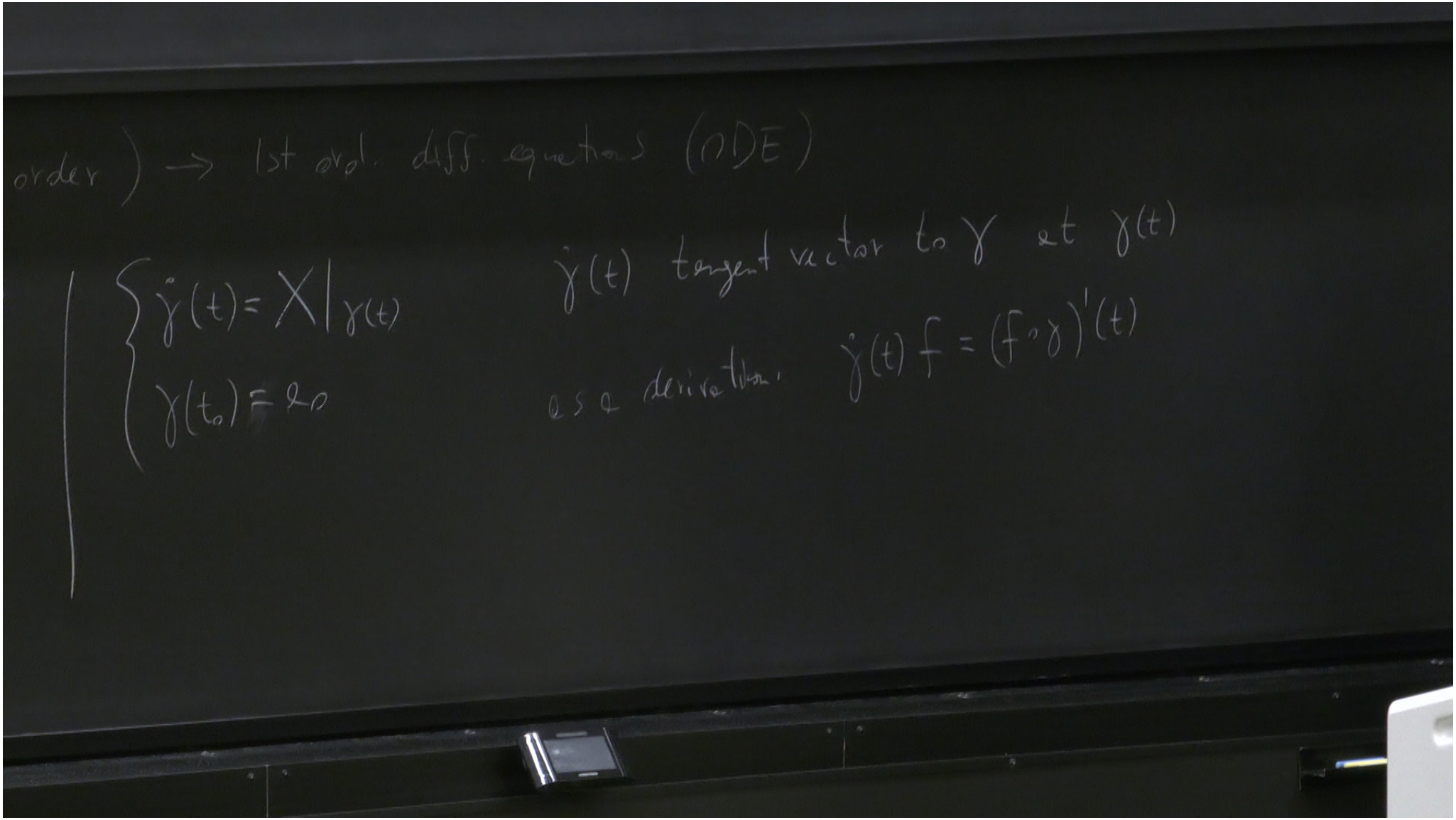

$(q,\alpha) \in T^*M$ interpret q as position, α momentum → { ideal to study; transform the right way
$P_1, \dots, P_n)$ on T^*M with $q^i(q,\alpha) = q^i(q)$
$P_i\left(q, \alpha_j, \frac{\partial q^j}{\partial q^i}\right) = \alpha_i$
$\tilde{P}_i\left(q, P_j, \frac{\partial \tilde{q}^j}{\partial q^i}\right) = P_i$

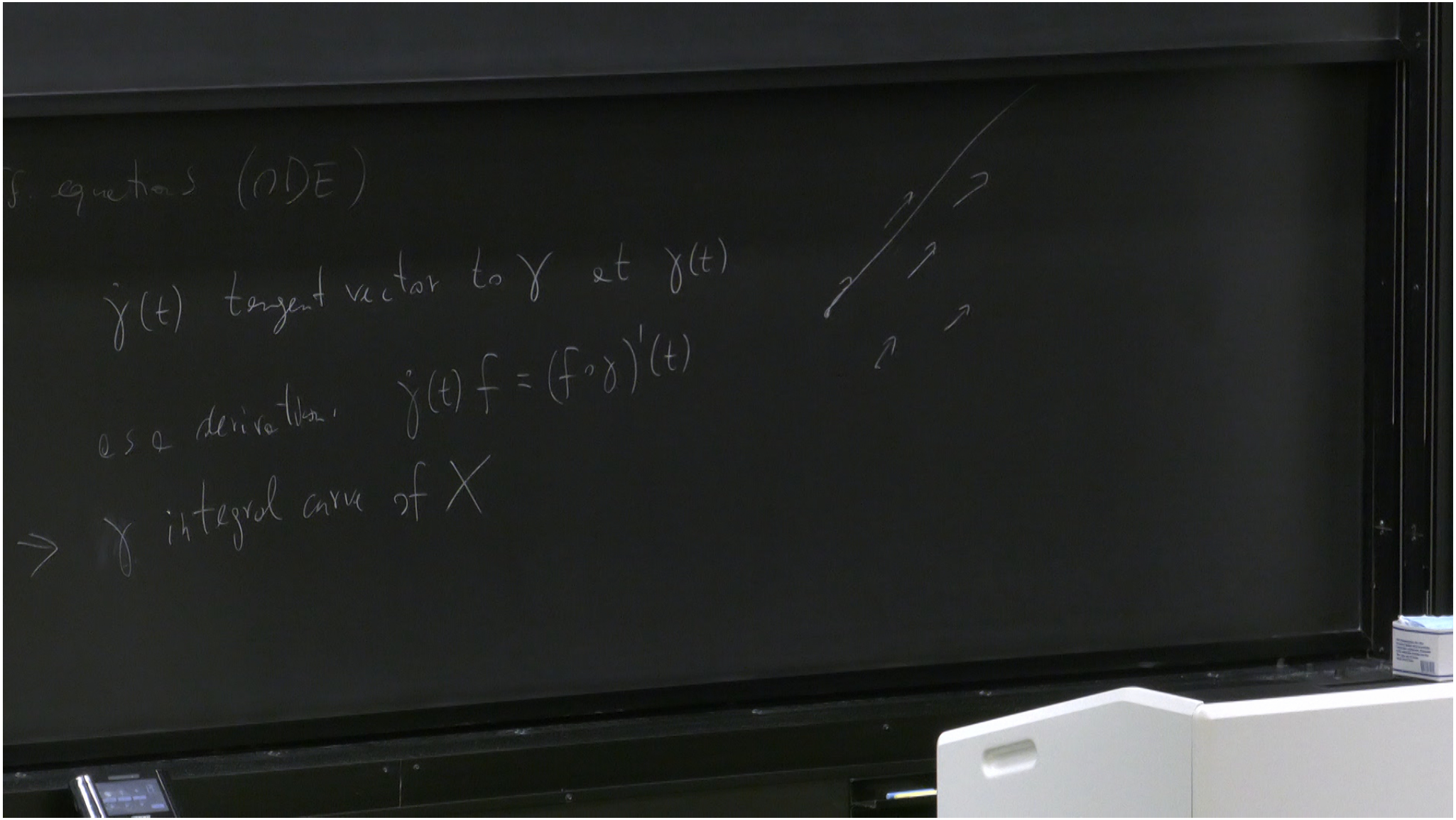
$$P_i\left(q, \alpha_j dq^j_q\right) = P_i\left(q, \alpha_j \frac{\partial \tilde{q}^j}{\partial q^k}(q) dq^k_q\right) = \alpha_j \frac{\partial \tilde{q}^j}{\partial q^i}(q) \quad \Rightarrow \quad P_i = \frac{\partial \tilde{q}^j}{\partial q^i} \tilde{P}_j$$

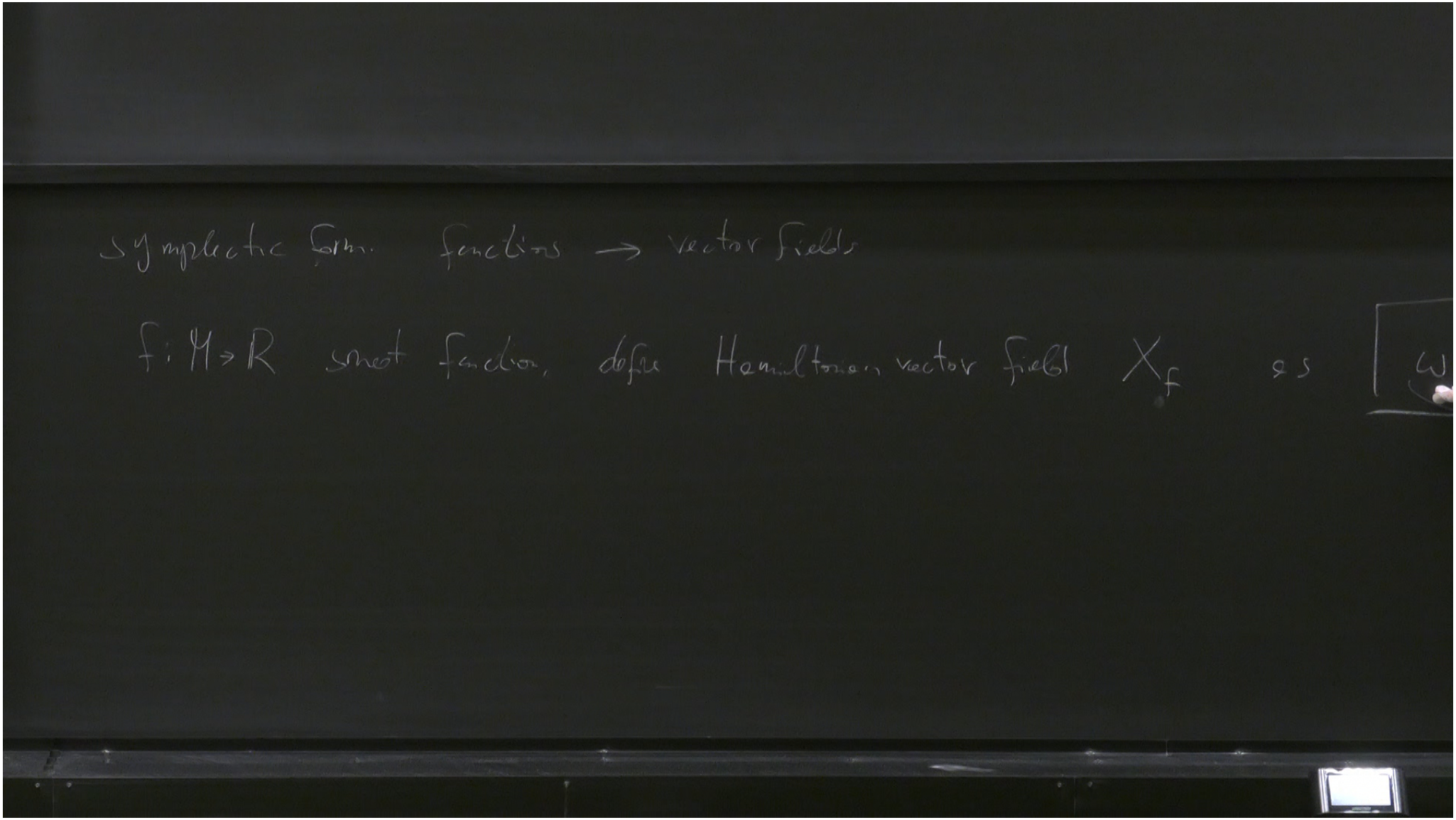
Canonical symplectic form: $\omega = -d\theta = -dp_i \wedge dq^i = dq^i \wedge dp_i$


$= \frac{\partial \tilde{q}^j}{\partial q^i} \tilde{P}_j$
$P_i dq^i = \frac{\partial \tilde{q}^j}{\partial q^i} \tilde{P}_j dq^i = d\tilde{q}^j \tilde{P}_j$
$d\tilde{q}^j = \frac{\partial \tilde{q}^j}{\partial q^i} dq^i + \frac{\partial \tilde{q}^j}{\partial P_i} dP_i$
$\wedge dP_i$
$d\omega = -d^2\theta = 0 \Rightarrow$ w obs exact


why do we care? → dynamics (1st order) → 1st ord. diff. equation
on a manifold, how do we define a diff. eq?
• $\gamma : I \subseteq_{\text{open}} \mathbb{R} \to M$ curve
• Vector field $X : M \to TM$


order) → 1st ord. diff. equations (ODE)
$\begin{cases} \dot\gamma(t) = X|_{\gamma(t)} \\ \gamma(t_0) = x_0 \end{cases}$
$\dot\gamma(t)$ tangent vector to γ at $\gamma(t)$
as a derivation: $\dot\gamma(t) f = (f \circ \gamma)'(t)$


equations (ODE)
$\dot\gamma(t)$ tangent vector to γ at $\gamma(t)$
as a derivation: $\dot\gamma(t) f = (f\circ\gamma)'(t)$
γ integral curve of X


symplectic form. functions → vector fields
f: M → R smooth function, define Hamiltonian vector field X_f s.t.

functions → vector fields

function, define Hamiltonian vector field X_f as $\boxed{\omega(X_f, \cdot) = df}$

↳ interpret as 1-form

$= dq^i \otimes dp_i - dp_i \otimes dq^i$

X_f is $\boxed{\omega(X_f, \cdot) = df}$

↳ interpret as 1-form

→ (metric g: $g(\text{grad } f, \cdot) = df$)

in coordinates $X_f = A^i \frac{\partial}{\partial q^i} + B_i \frac{\partial}{\partial p_i}$

$\omega(X_f, \cdot) = A^i dp_i - B_i dq^i$

$df = \frac{\partial \hat{f}}{\partial q^i} dq^i + \frac{\partial \hat{f}}{\partial p_i} dp_i$

$= dq^i \otimes dp_i -$

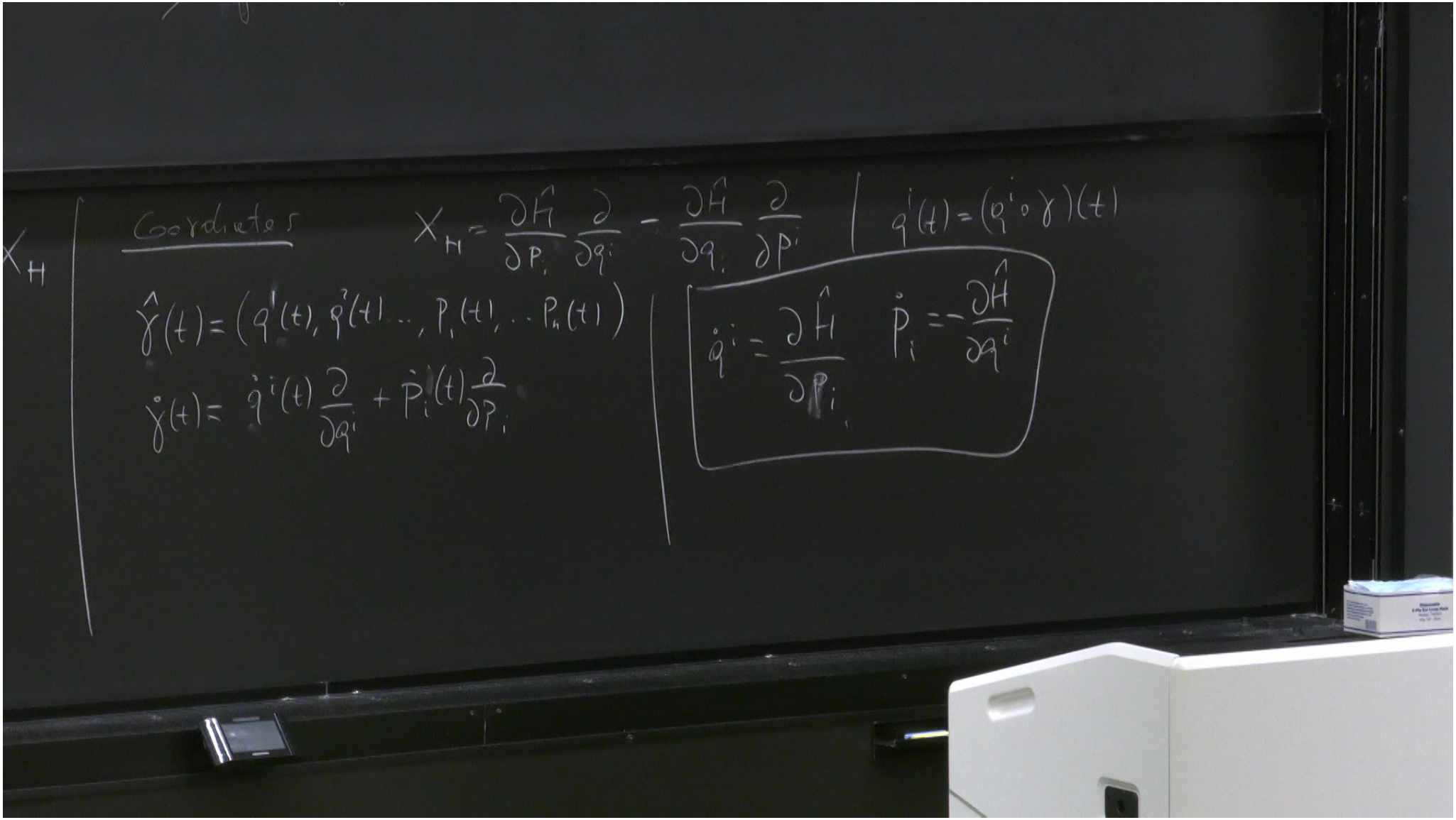
symplectic form: functions $\rightarrow$ vector fields

$f : M \to \mathbb{R}$ smooth function, define Hamiltonian vector field X_f as

$$\rightarrow \boxed{X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}}$$

Special function $H : M \to \mathbb{R}$ (Hamiltonian) $\Rightarrow$ define X_H

physical paths: solutions of $\begin{cases} \dot{\gamma}(t) = X_H|_{\gamma(t)} \\ \gamma(t_0) = q_0 \end{cases}$


X_H
Coordinates
$X_H = \frac{\partial \hat{H}}{\partial P_i}\frac{\partial}{\partial q^i} - \frac{\partial \hat{H}}{\partial q^i}\frac{\partial}{\partial P_i}$
$q^i(t) = (q^i \circ \gamma)(t)$
$\hat{\gamma}(t) = (q^1(t), q^2(t) \ldots, P_1(t), \ldots P_n(t))$
$\dot{\gamma}(t) = \dot{q}^i(t)\frac{\partial}{\partial q^i} + \dot{P}_i(t)\frac{\partial}{\partial P_i}$
$\dot{q}^i = \frac{\partial \hat{H}}{\partial P_i} \quad \dot{P}_i = -\frac{\partial \hat{H}}{\partial q^i}$