Title: Mathematical Physics Core Lecture

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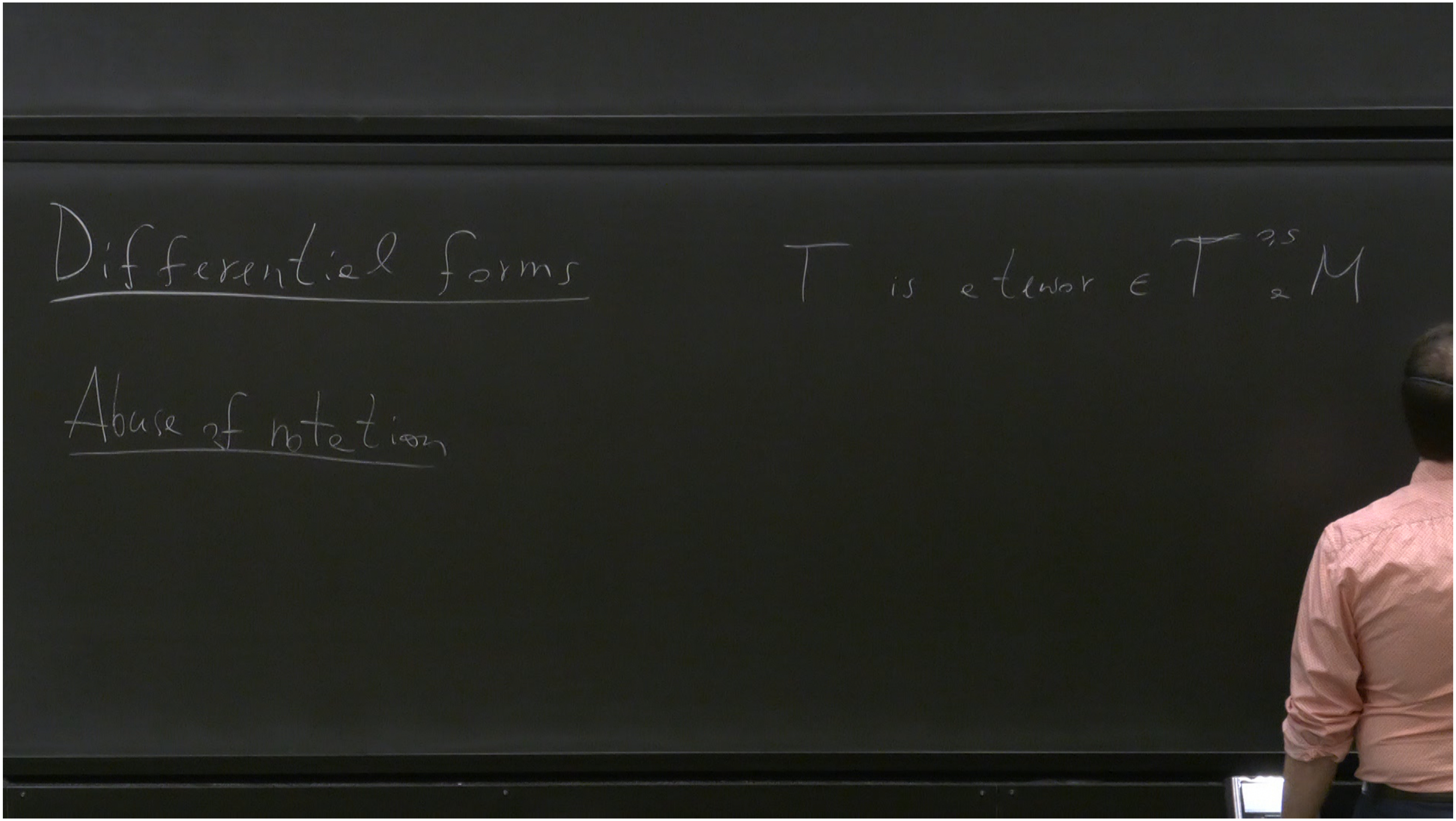

Differential forms
Abuse of notation
T is a tensor ∈ T M

$T_a$ is a tensor $\in \mathcal{T}^{0,s}_a M$ $\rightarrow$ $T_a : T_aM \times \dots \times T_aM \rightarrow \mathbb{R}$ acts on vectors

$T : U \subseteq M \rightarrow \mathcal{T}^{0,s} M$ tensor field

$X_1, \dots, X_s$ vector fields

"$T(X_1, \dots, X_s)$" is a function

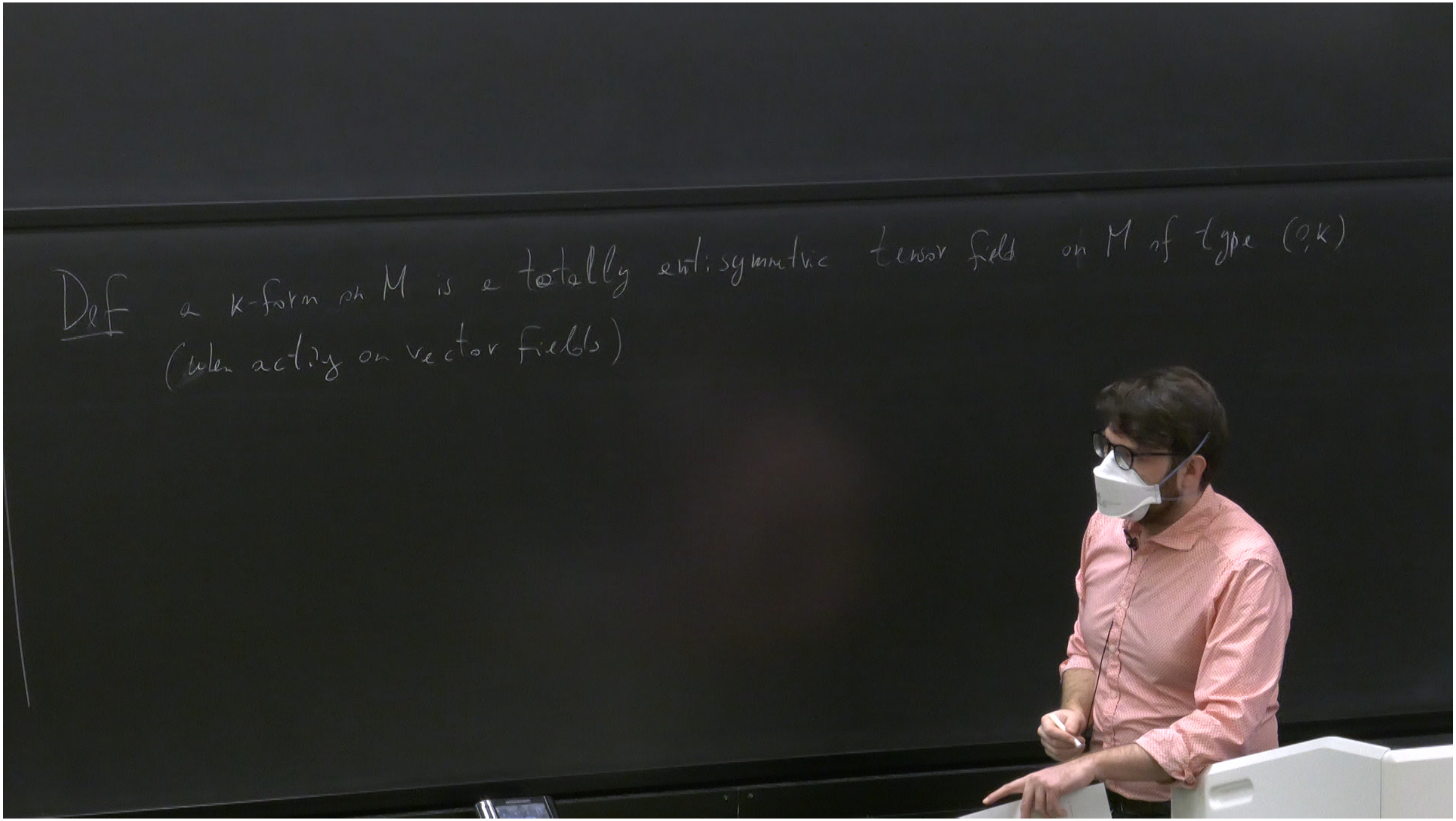
$T(X_1, \dots, X_s)(a)$

$= T_a(X_1|_a, \dots, X_s|_a) \in \mathbb{R}$

another abuse of notation

$X$ vector field

$Xf$ is function defined by $(Xf)(a) = X_a f$


Def a k-form on M is a totally antisymmetric tensor field on M of type (0,k)
(when acting on vector fields)

**Def** a $k$-form on $M$ is a totally antisymmetric tensor field on $M$ of type $(0,k)$ (when acting on vector fields)

if $\sigma \in S_k$ is a permutation

another abuse of notation

$X$ vector field

$Xf$ is function defined by $(Xf)(a) = X_a f$

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$S_k : \{ \sigma : \{1, \dots, k\} \to \{1, \dots k\} \text{ invertible} \}$

$i \mapsto \sigma_i$

**Def** a $k$-form on $M$ is a tota
(when acting on vector fiel

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if $\sigma \in S_k$ is a permutation, w

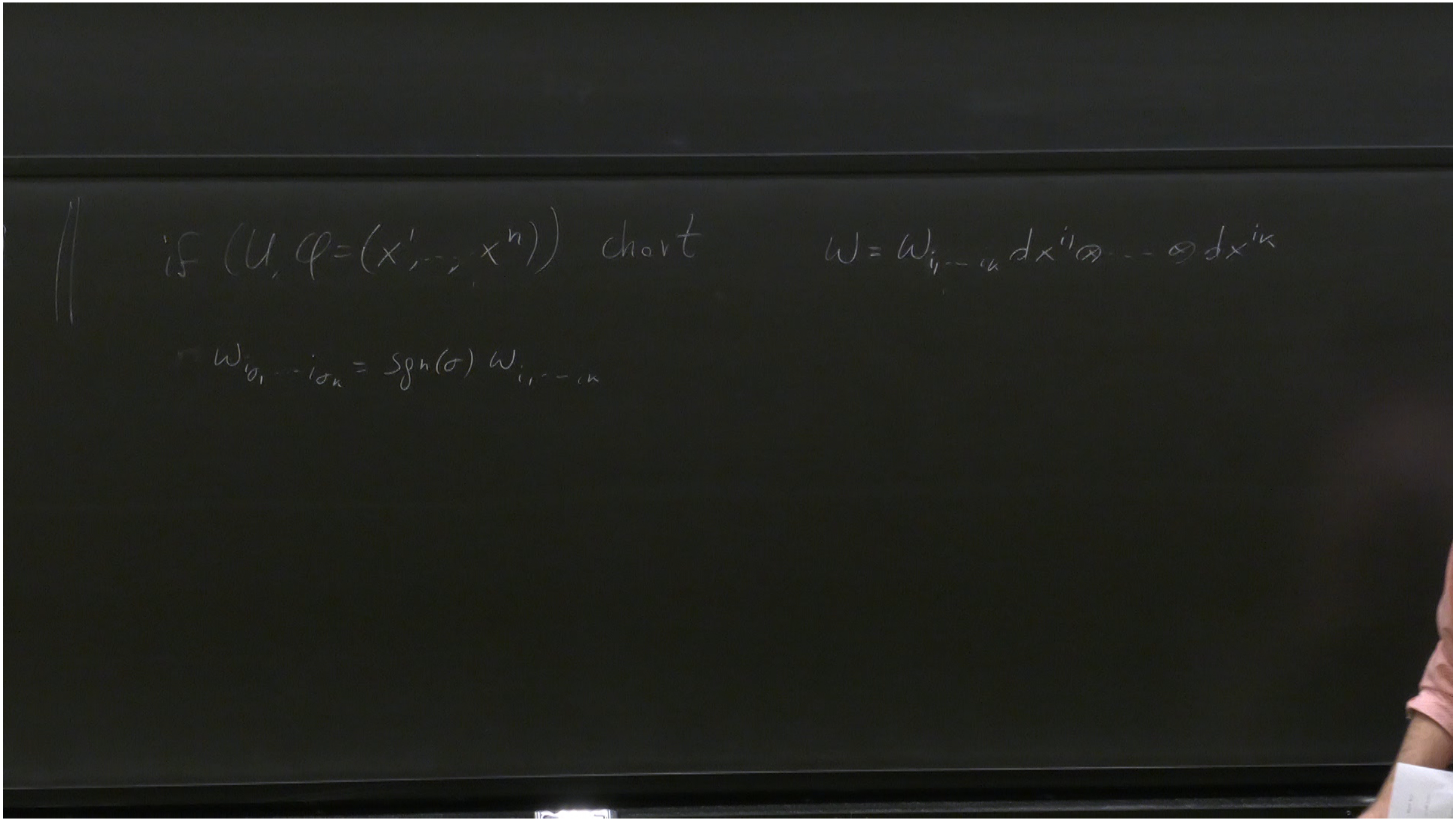
**Def** a k-form on M is a totally antisymmetric tensor field on M of type (0,k) (when acting on vector fields)

The set of all k-forms is denoted by $\Omega^k(M)$

if $\sigma \in S_k$ is a permutation, $\omega$ k-form if

$$\omega(X_{\sigma_1}, \ldots, X_{\sigma_k}) = \mathrm{sgn}(\sigma)\, \omega(X_1, \ldots, X_k)$$

$$\mathrm{sgn}(\sigma) = \begin{cases} +1 & \text{for even permutation} \\ -1 & \text{" odd "} \end{cases}$$


if $(U, \varphi = (x^1, \dots, x^n))$ chart
$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$
$\omega_{i_{\sigma_1} \dots i_{\sigma_k}} = \mathrm{sgn}(\sigma)\, \omega_{i_1 \dots i_k}$

by convention, $\Omega^0(M) = C^\infty(M, \mathbb{R})$

if $(U, \varphi = (x^1, \dots, x^n))$

$\omega_{i_{\sigma_1} \dots i_{\sigma_n}} = \mathrm{sgn}(\sigma)\, \omega_{i_1 \dots}$

Wedge product $\quad \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$

$\alpha \otimes \beta$ not always a diff. form f $(U, \varphi = (x^1, \dots, x^n))$ chart $\qquad \omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$

$\omega_{i_{\sigma_1} \dots i_{\sigma_k}} = \mathrm{sgn}(\sigma)\, \omega_{i_1 \dots i_k}$

$$dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} := \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, dx^{i_{\sigma_1}} \otimes dx^{i_{\sigma_2}} \otimes \dots \otimes dx^{i_{\sigma_k}}$$

symbols

locally generated k-forms

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{i_{\sigma_1} \dots i_{\sigma_k}} dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}} = \frac{1}{k!} \sum_{\sigma} \omega_{i_1 \dots i_k} \mathrm{sgn}(\sigma)\, dx^{i_{\sigma_1}} \otimes \dots$$

$f\ (U, \varphi = (x^1, \dots, x^n))$ chart $\qquad \omega = \omega_{i_1 \dots i_k}\, dx^{i_1} \otimes \dots \otimes dx^{i_k}$

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$$\omega = \omega_{i_1 \dots i_k}\, dx^{i_1} \otimes \dots \otimes dx^{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{i_{\sigma_1} \dots i_{\sigma_k}}\, dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}} = \frac{1}{k!} \sum_{\sigma} \omega_{i_1 \dots i_k}\, \mathrm{sgn}(\sigma)\, dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}}$$

of $(U, \varphi = (x^1, \dots, x^n))$ chart $\qquad \omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$

$$\omega_{i_{\sigma_1} \dots i_{\sigma_k}} = \mathrm{sgn}(\sigma)\, \omega_{i_1 \dots i_k}$$

$$dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} := \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, dx^{i_{\sigma_1}} \otimes dx^{i_{\sigma_2}} \otimes \dots \otimes dx^{i_{\sigma_k}}$$

symbols

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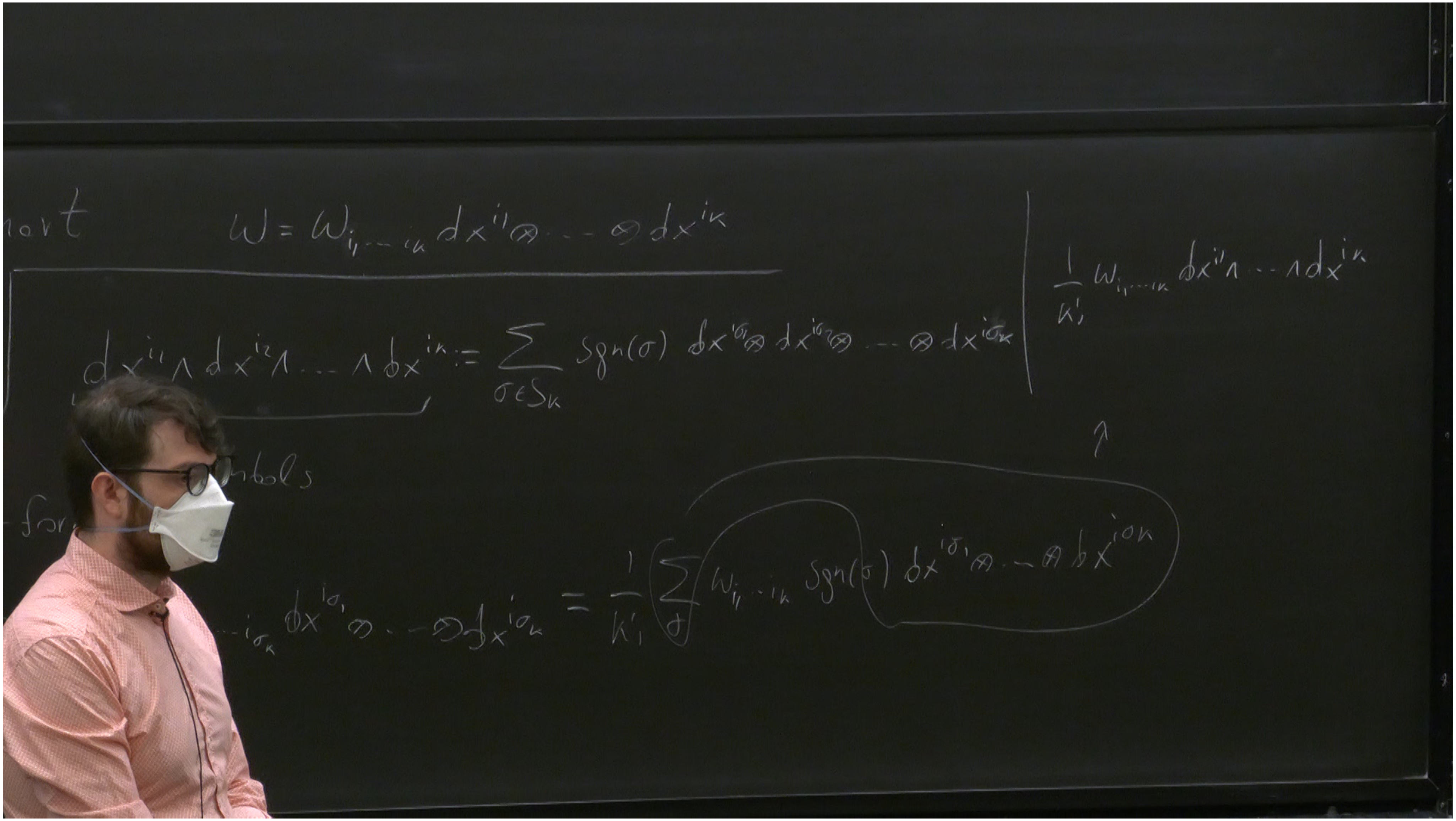
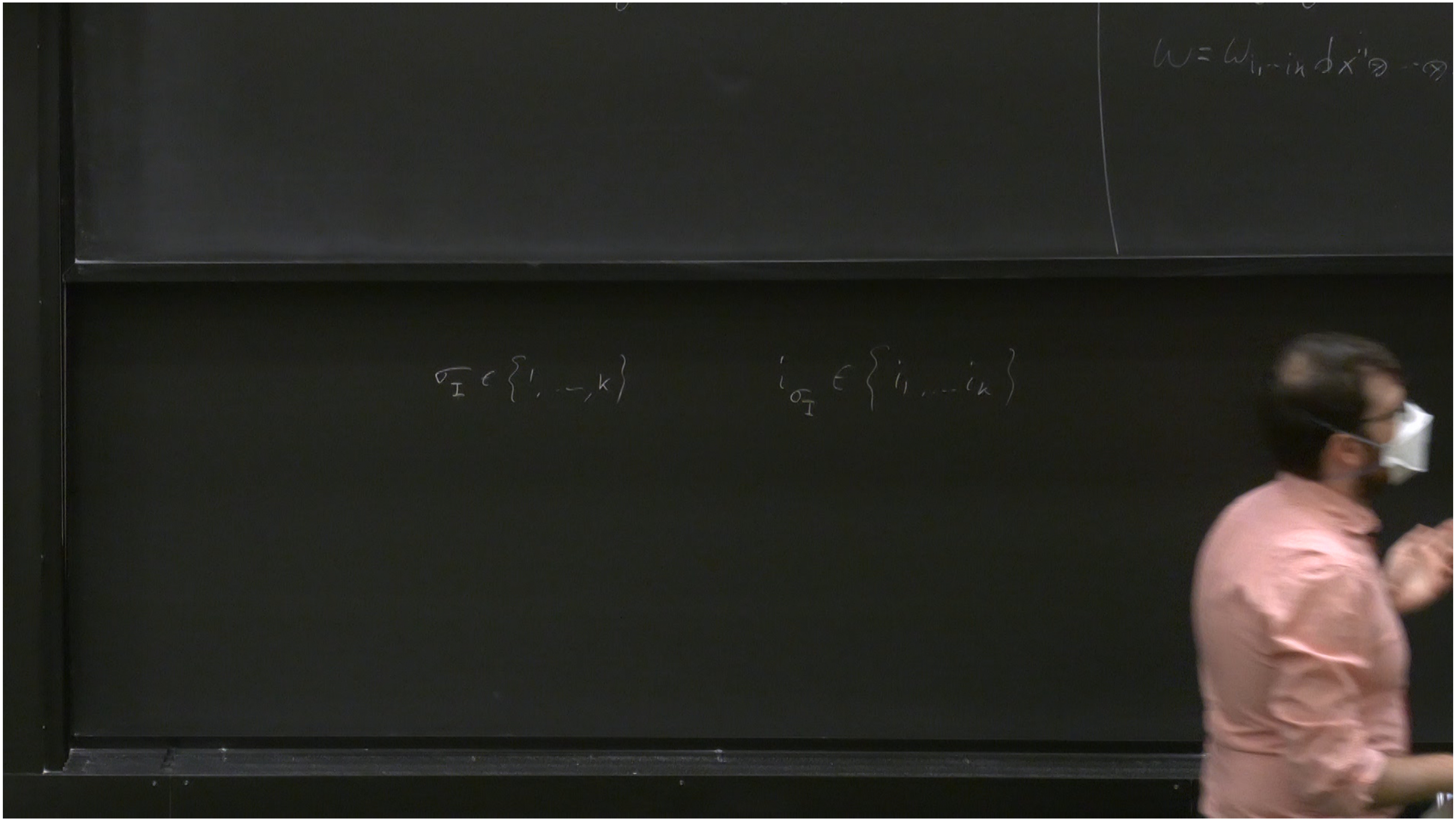
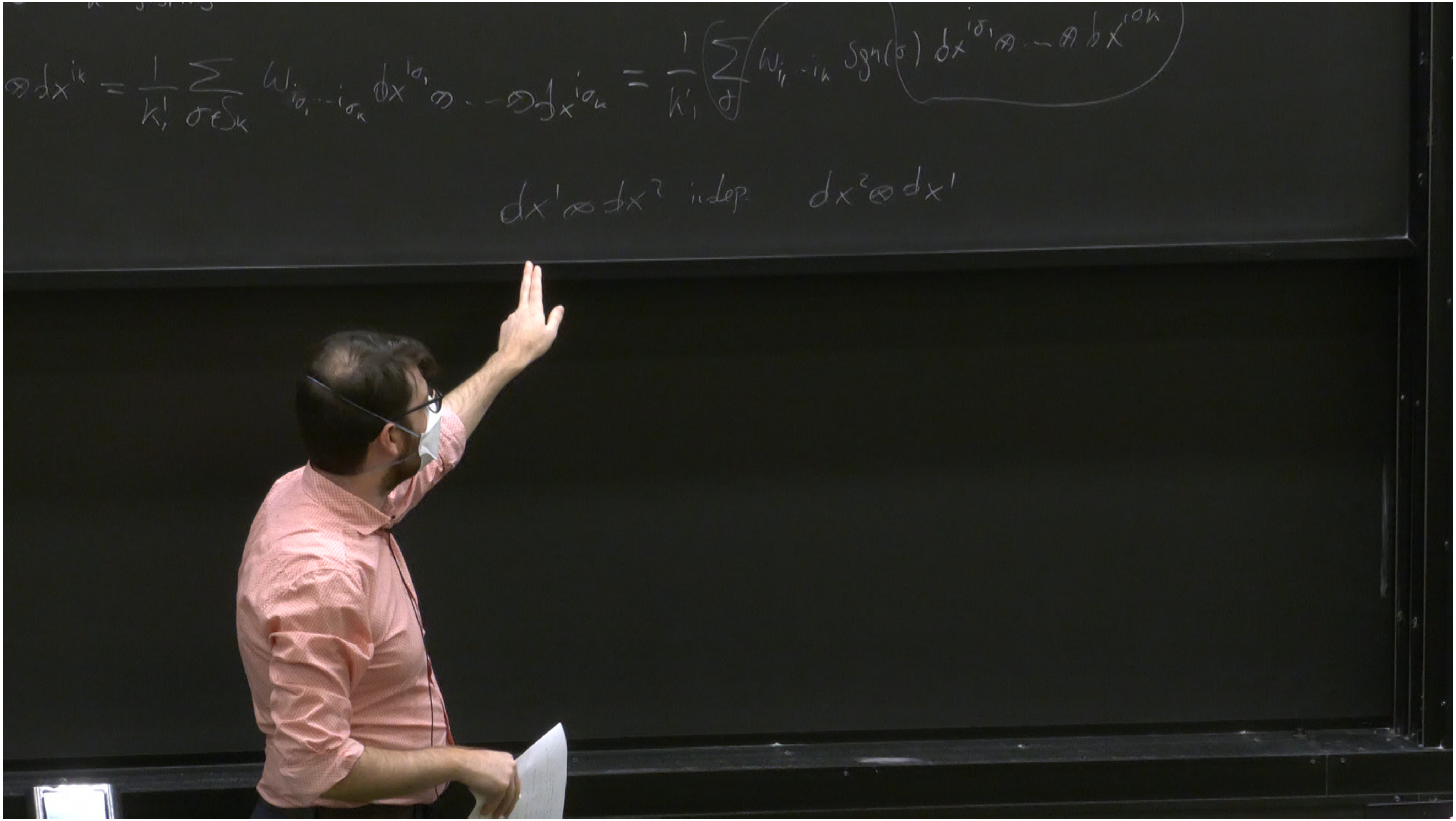
$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{i_{\sigma_1} \dots i_{\sigma_k}} dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}} = \frac{1}{k!} \sum_{\sigma} \omega_{i_1 \dots i_k}\, \mathrm{sgn}(\sigma)\, dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}}$$


$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$
$dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} := \sum_{\sigma \in S_k} sgn(\sigma)\, dx^{i_{\sigma_1}} \otimes dx^{i_{\sigma_2}} \otimes \dots \otimes dx^{i_{\sigma_k}}$
$\frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$
$= \frac{1}{k!} \sum_{\sigma} \omega_{i_1 \dots i_k} sgn(\sigma)\, dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}}$



ω = ω_{i_1…i_k} dx^{i_1} ⊗ … ⊗
σ_I ∈ {1, …, k}
i_{σ_I} ∈ {i_1, … i_k}


dx¹⊗dx² indep. dx²⊗dx¹

symbols

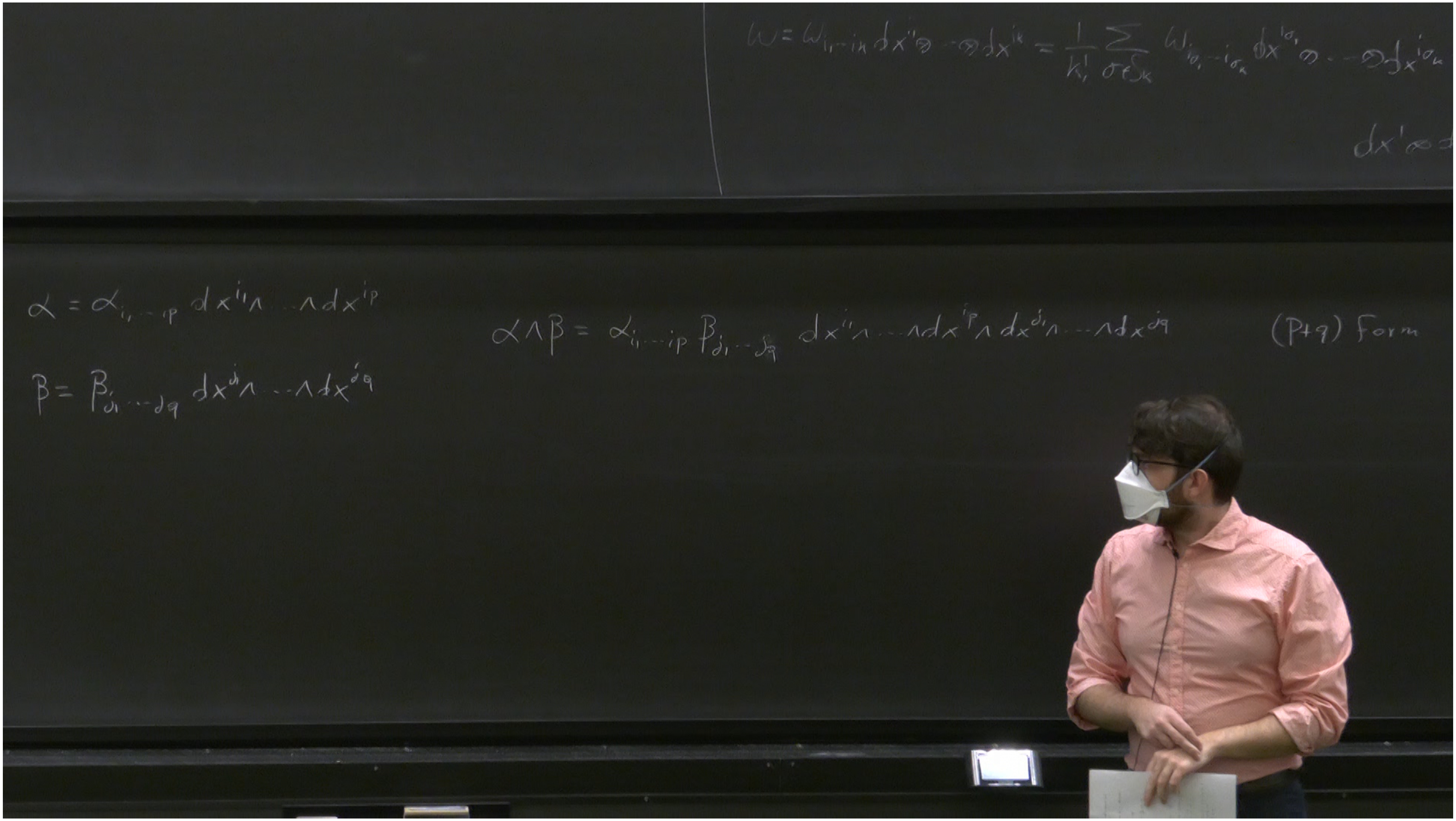
locally generated $k$-forms

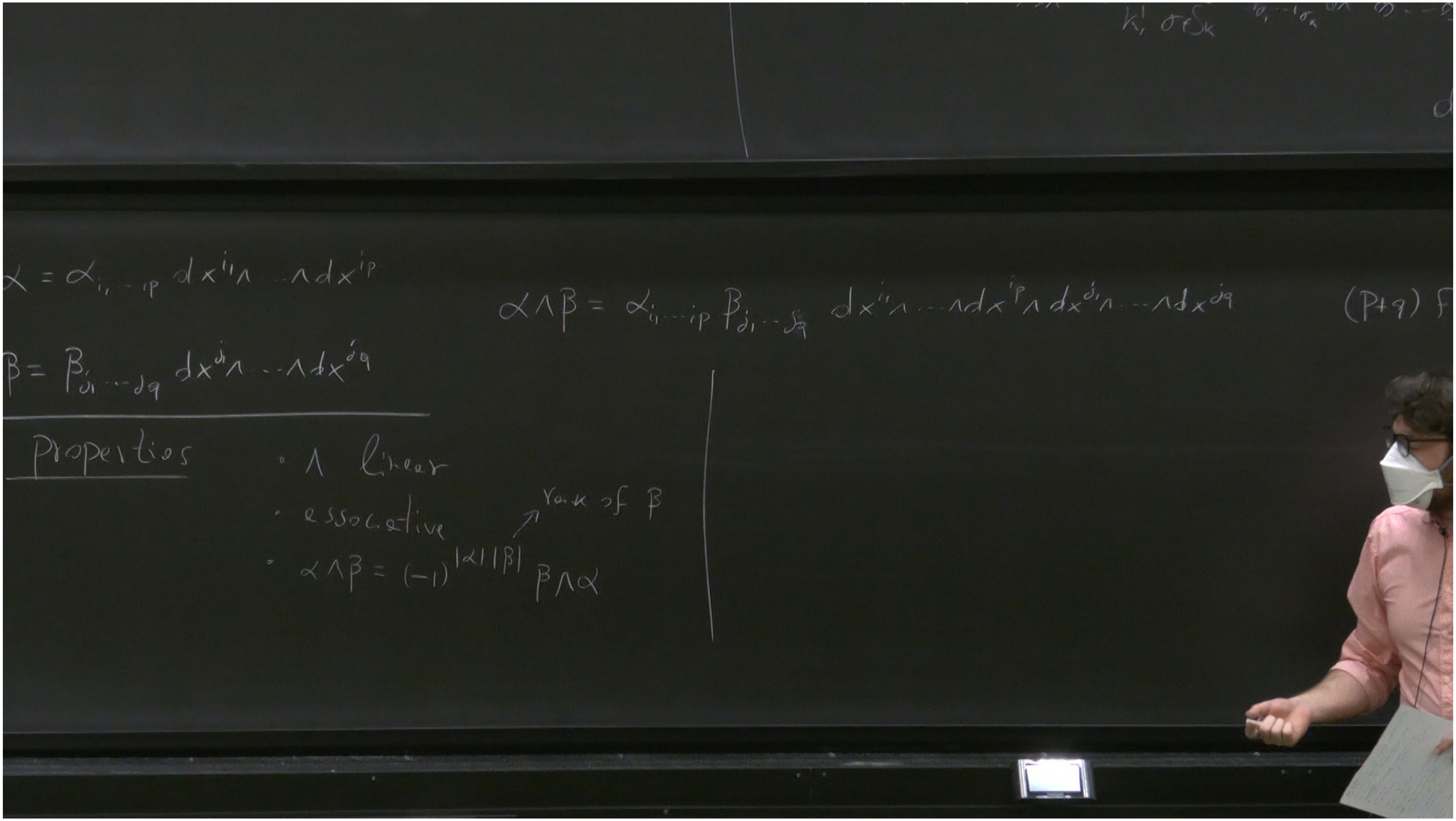
$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{i_{\sigma_1} \dots i_{\sigma_k}} dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}} = \frac{1}{k!} \sum_{\sigma} \omega_{i_1 \dots i_k} \operatorname{sgn}(\sigma)\, dx^{i_{\sigma_1}} \otimes \dots \otimes dx^{i_{\sigma_k}}$$

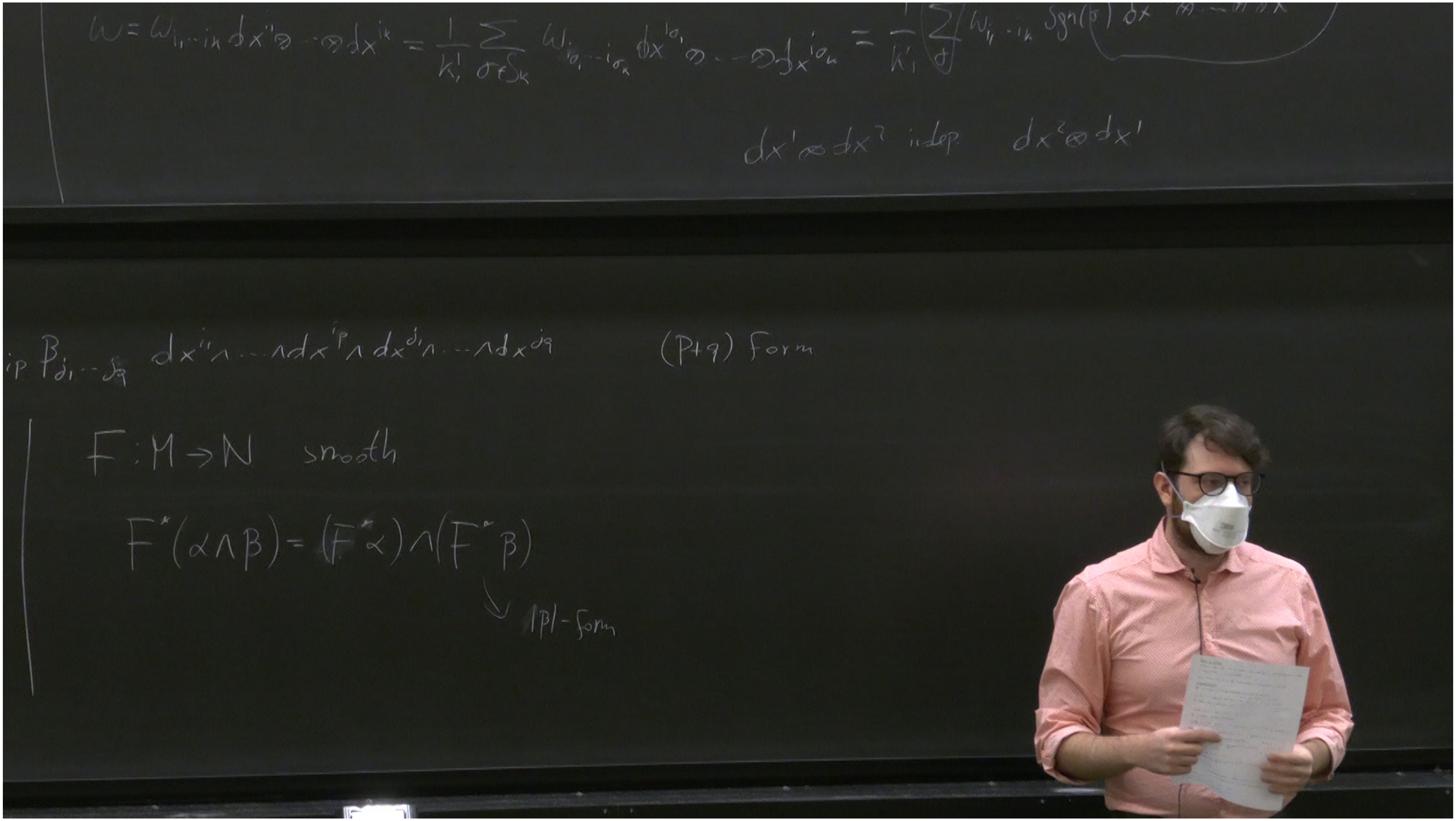
$dx^1 \otimes dx^2$ indep. $dx^2 \otimes dx^1$

independent ones: $i_1 < i_2 < \dots < i_k$ $\rightarrow$ $\binom{n}{k}$ independent ones

$k$-forms and $(n-k)$-forms have same number of generators


$\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{i_{\sigma_1} \cdots i_{\sigma_k}} dx^{i_{\sigma_1}} \otimes \cdots \otimes dx^{i_{\sigma_k}}$
$\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$
$\beta = \beta_{j_1 \cdots j_q} dx^{j_1} \wedge \cdots \wedge dx^{j_q}$
$\alpha \wedge \beta = \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$
(p+q) Form


$\alpha = \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$
$\beta = \beta_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q}$
$\alpha \wedge \beta = \alpha_{i_1 \dots i_p} \beta_{j_1 \dots j_q} dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q}$
(p+q)
Properties
· ∧ linear
· associative
rank of β
· $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$



$dx^1 \otimes dx^2$ indep. $dx^2 \otimes dx^1$
(p+q) form
F : M → N smooth
$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$
|β|-form

$\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{i_{\sigma_1} \cdots i_{\sigma_k}} dx^{i_{\sigma_1}} \otimes \cdots \otimes dx^{i_{\sigma_k}} = \frac{1}{k!} \sum_{\sigma} \omega_{i_1 \cdots i_k} \operatorname{sgn}(\sigma)\, dx \otimes \cdots \otimes x$

$dx^1 \otimes dx^2$ indep $dx^2 \otimes dx^1$

$\alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$ (p+q) form

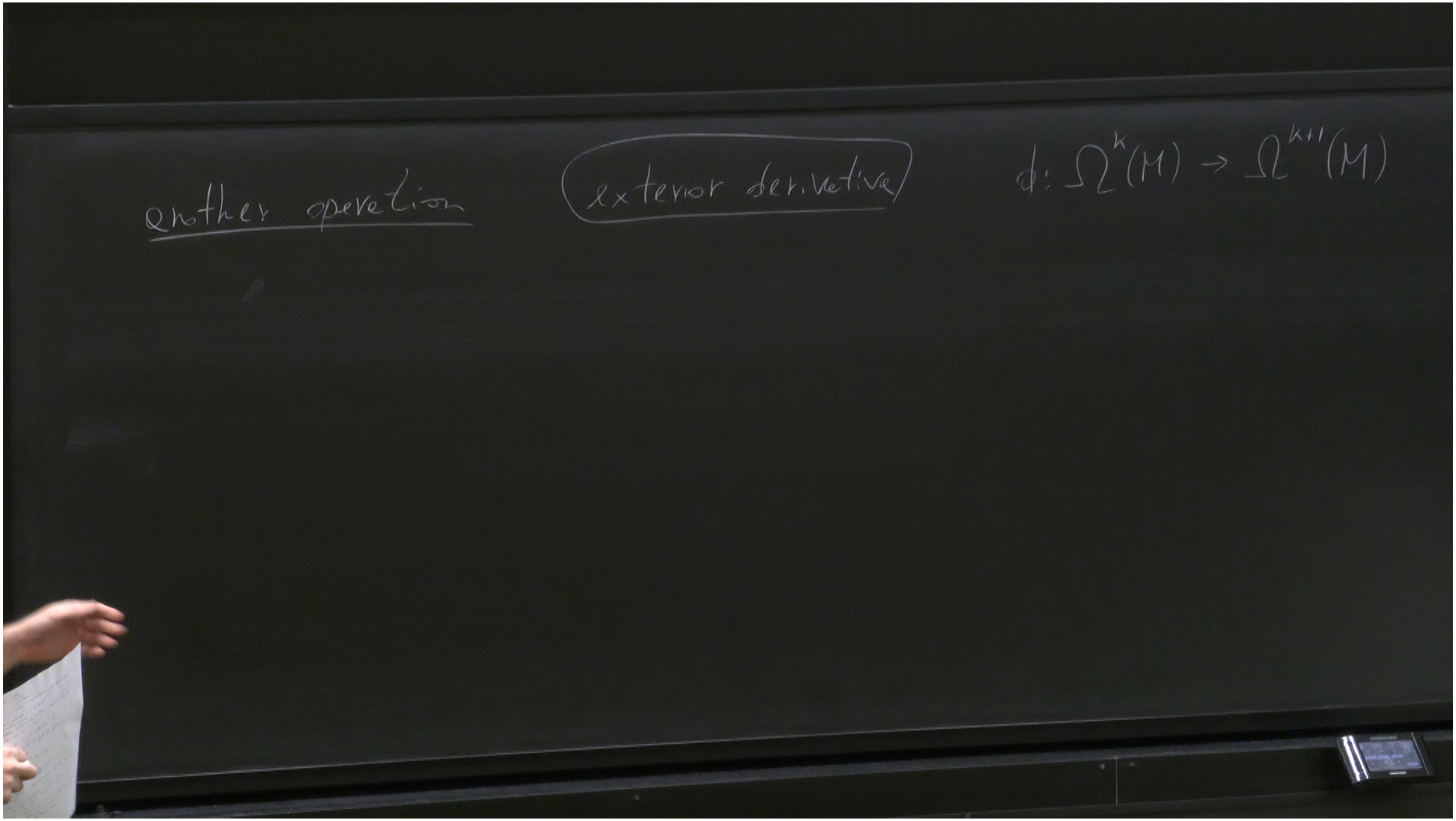
$\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ bivector

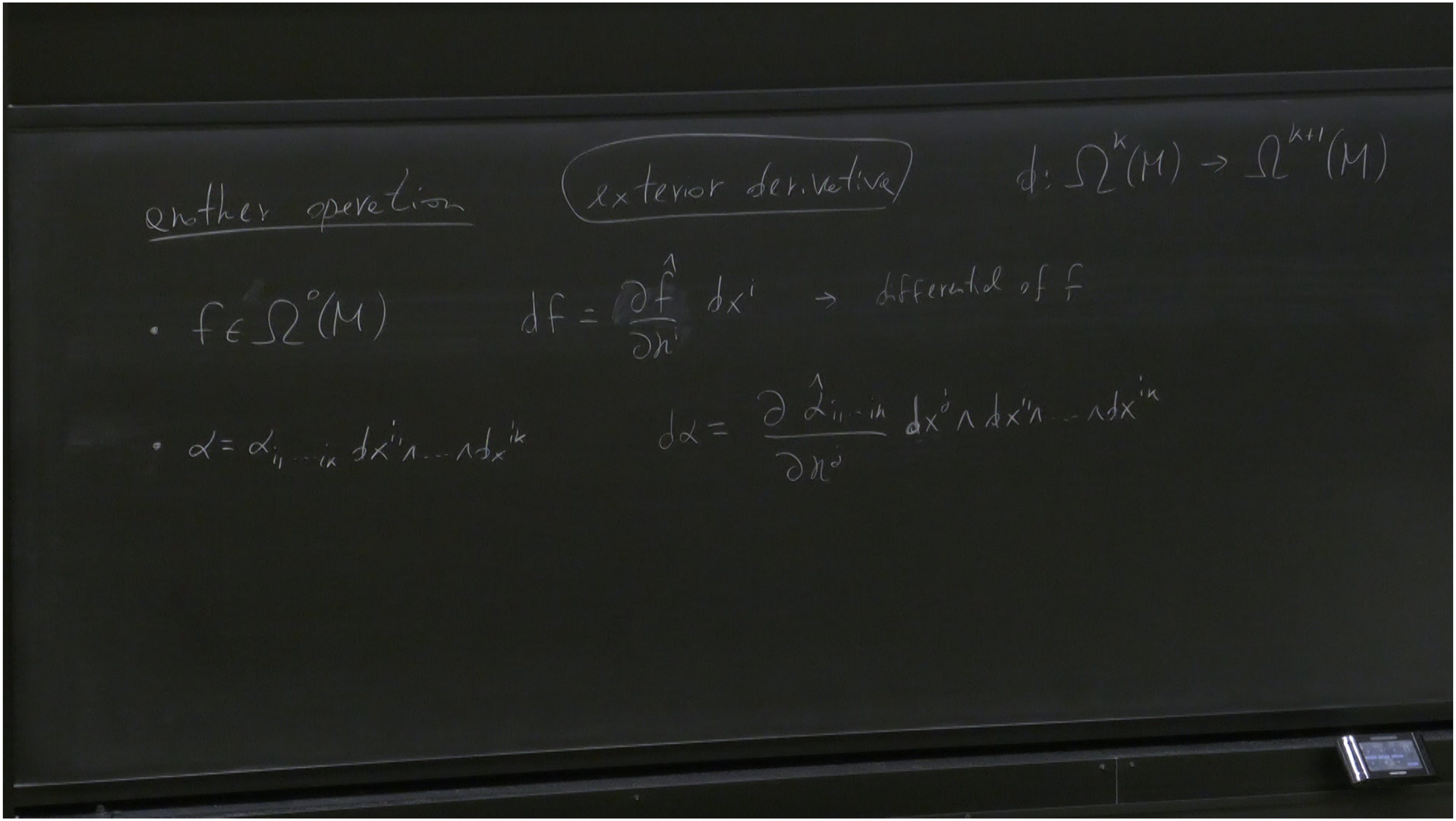
$\beta$

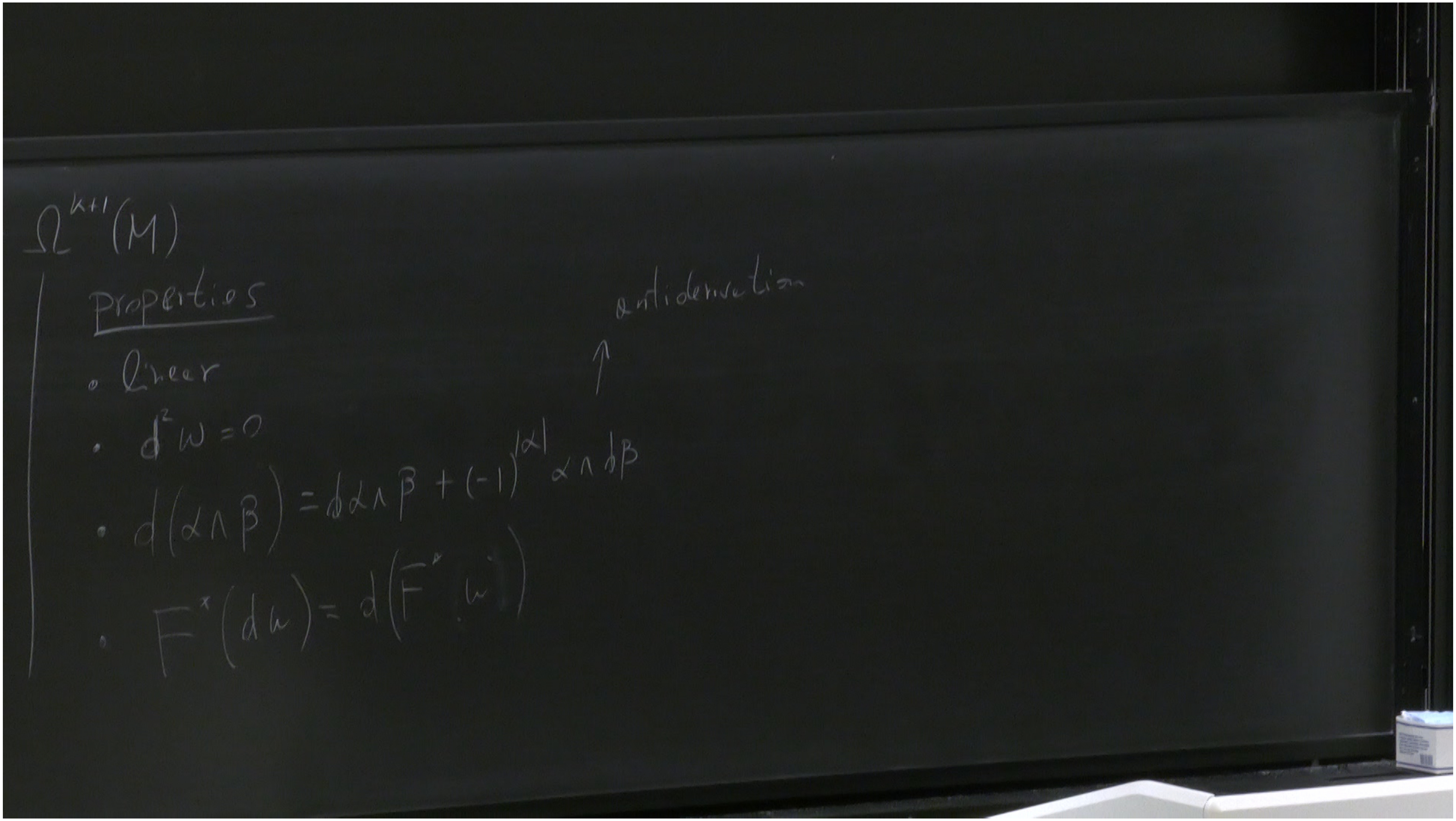
$F : M \to N$ smooth

$F^*(\alpha \wedge \beta) = (F^* \alpha) \wedge (F^* \beta)$

$|\beta|$-form


another operation
exterior derivative
d: Ω^k(M) → Ω^{k+1}(M)


another operation
exterior derivative
$d : \Omega^k(M) \to \Omega^{k+1}(M)$
• $f \in \Omega^0(M)$
$df = \frac{\partial \hat{f}}{\partial x^i} dx^i$ → differential of f
• $\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$
$d\alpha = \frac{\partial \hat{\alpha}_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$


$\Omega^{k+1}(M)$
properties
· linear
· $d^2\omega = 0$
· $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$
antiderivation
· $F^*(d\omega) = d(F^*(\omega))$

another operation — exterior derivative

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

- $f \in \Omega^0(M)$ $\qquad df = \frac{\partial \hat{f}}{\partial x^i} dx^i \to$ differential of f

- $\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ $\qquad d\alpha = \frac{\partial \hat{\alpha}_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

axioms

- d linear $\qquad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$

$(df)(X) = Xf \qquad d^2 f = 0$

properties

- linear
- $d^2 \omega = 0$
- $d(\alpha \wedge \beta$
- $F^*(d$

**another operation**

**exterior derivative** $d: \Omega^k(M) \to \Omega^{k+1}(M)$

- $f \in \Omega^0(M)$  $df = \frac{\partial \hat{f}}{\partial x^i} dx^i$ → differential of f

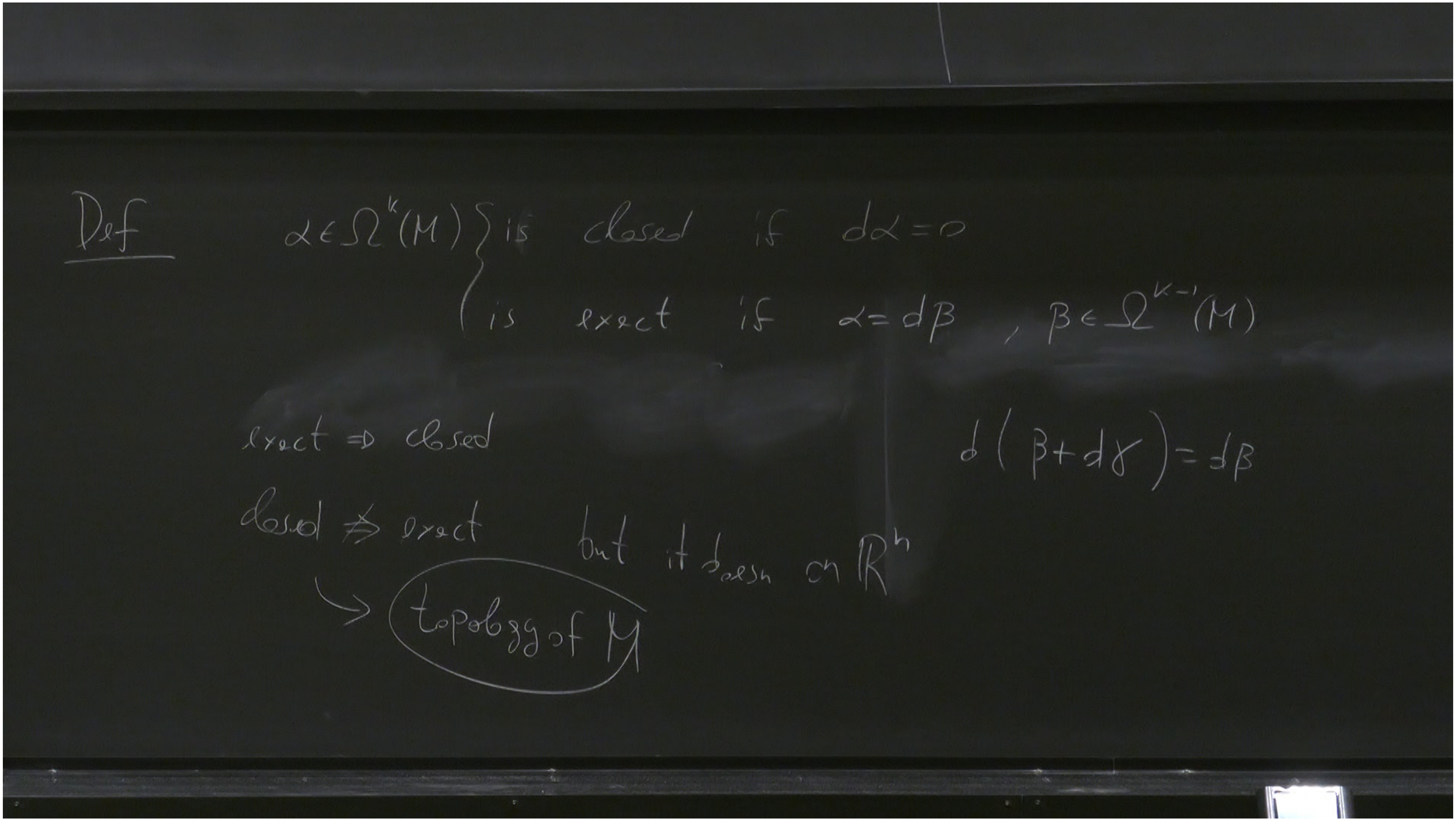
- $\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$  $d\alpha = \frac{\partial \hat{\alpha}_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

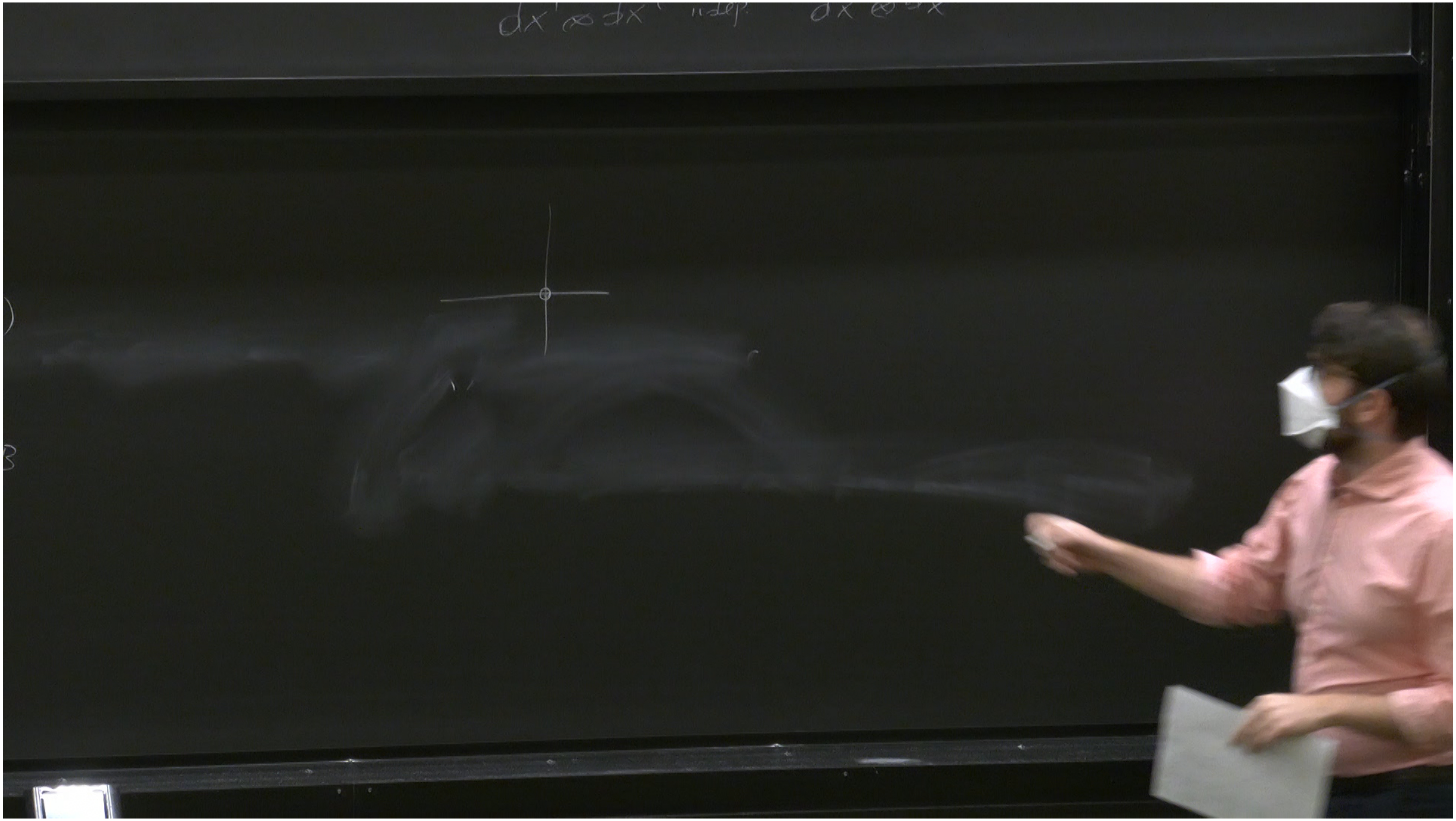
**axioms** • d linear  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$

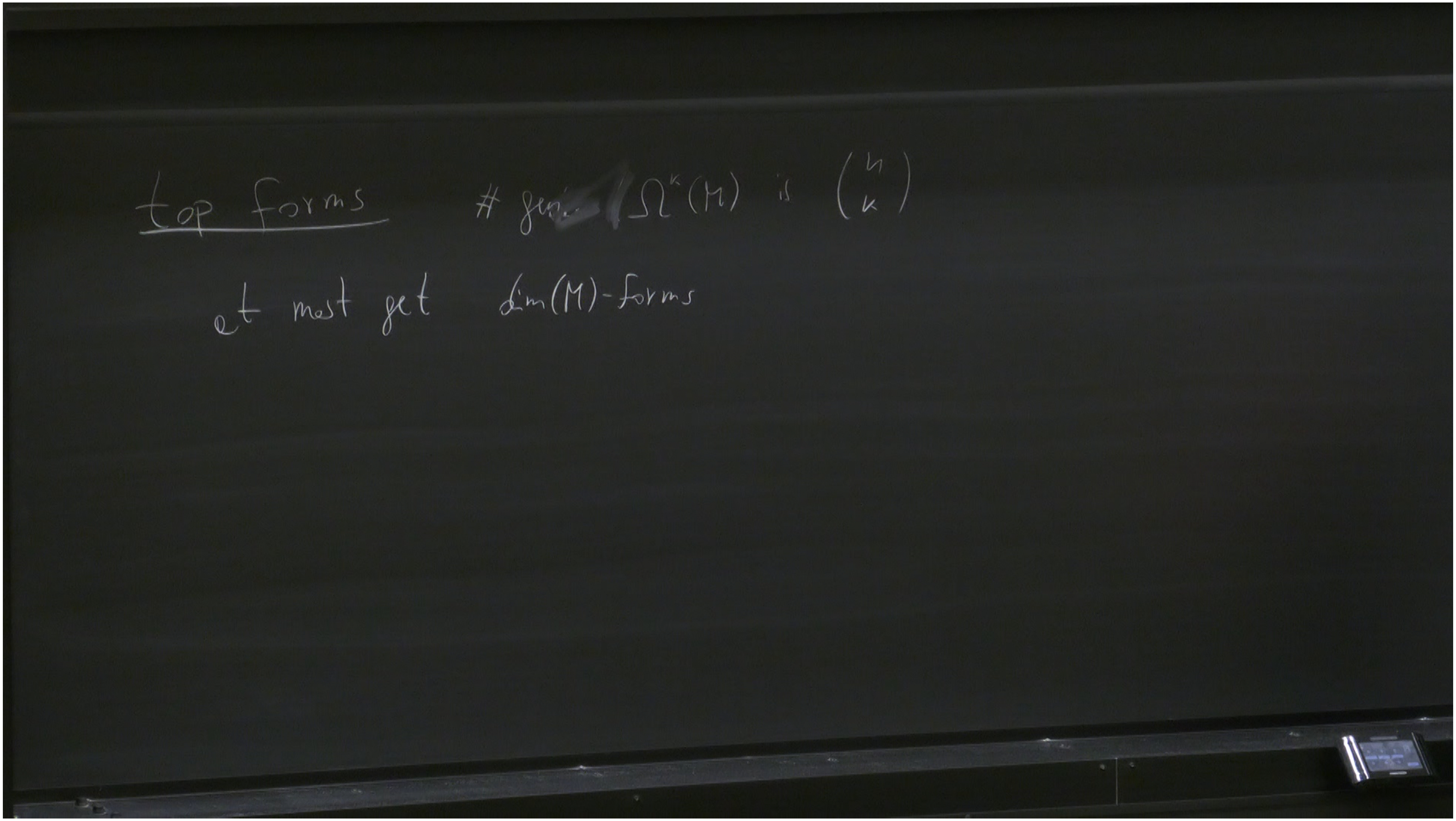
$(df)(X) = Xf$  $d^2 f = 0$

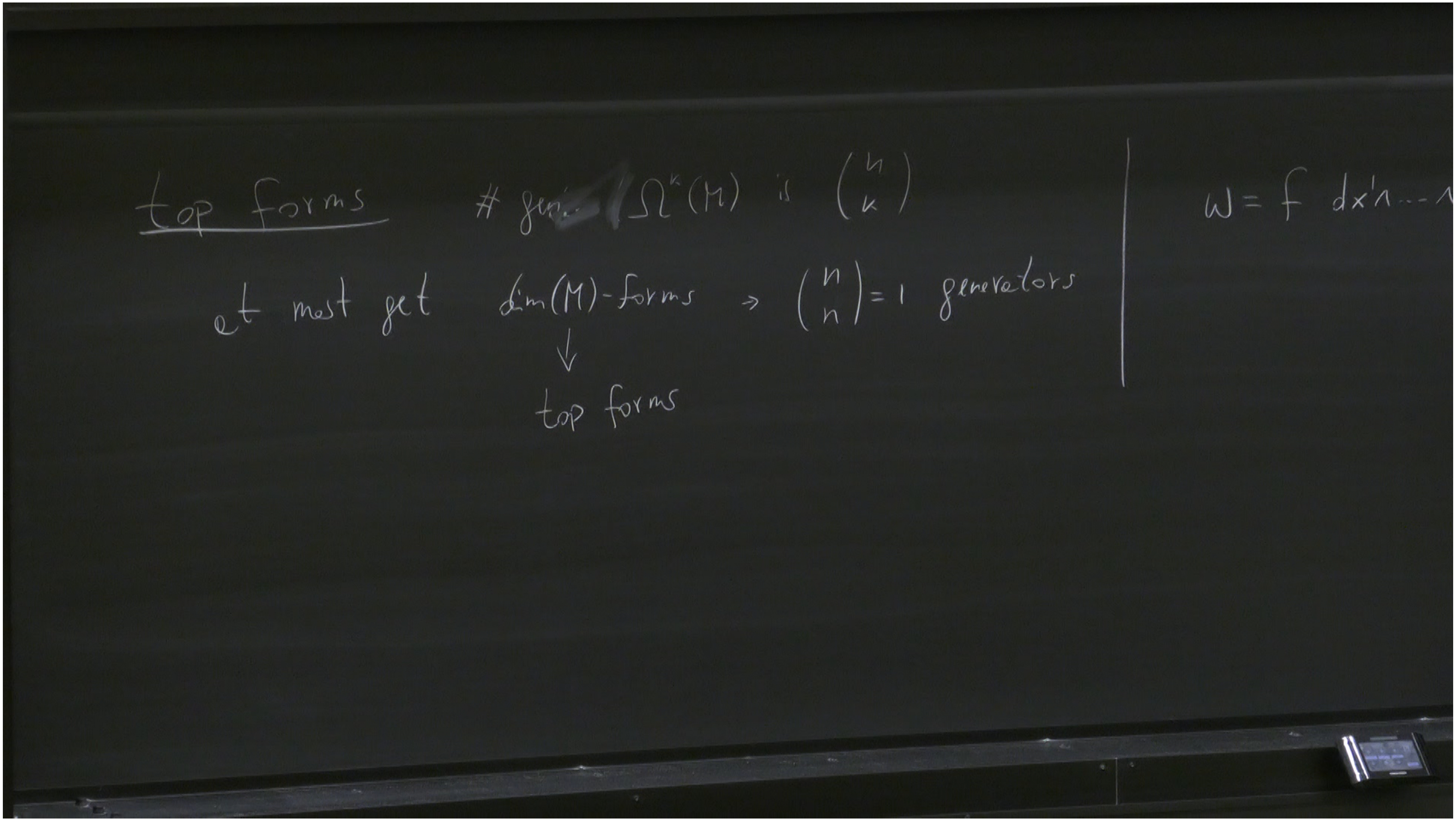
**properties**

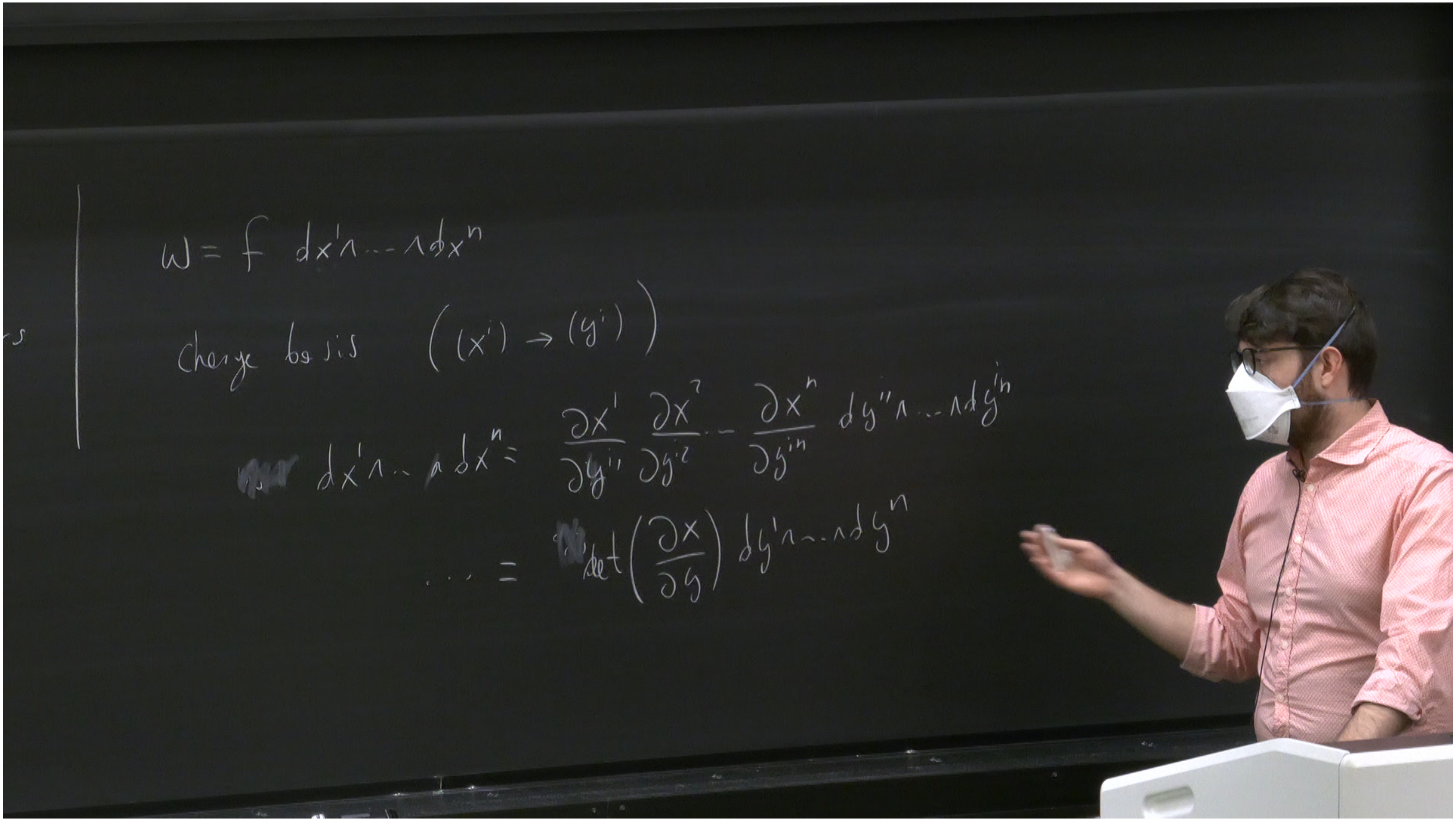
- linear
- $d^2 \omega =$
- $d(\alpha \wedge \beta$
- $F^*(d$

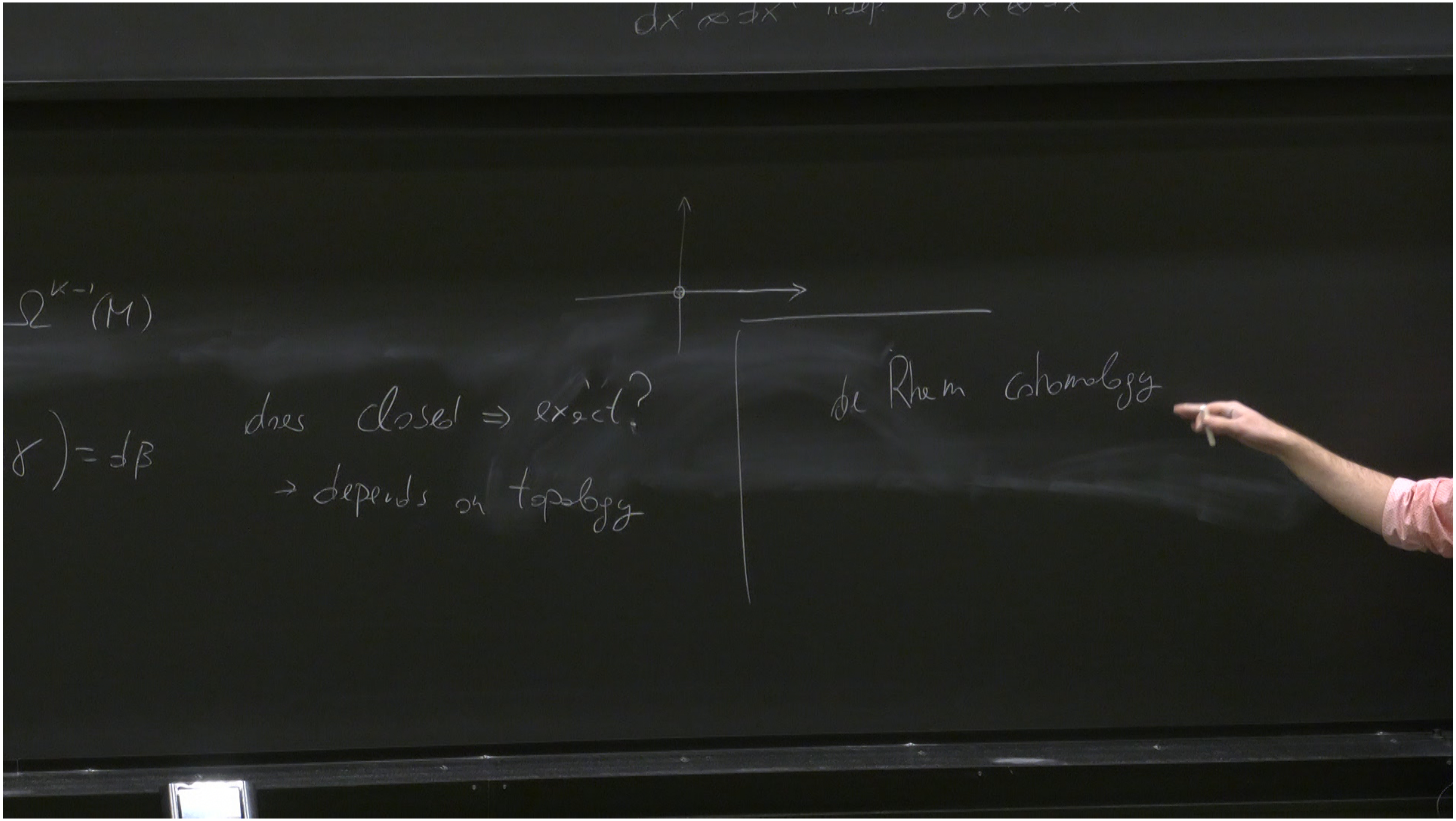

Def
$\alpha \in \Omega^k(M)$ is closed if $d\alpha = 0$
is exact if $\alpha = d\beta$, $\beta \in \Omega^{k-1}(M)$
exact ⇒ closed
$d(\beta + d\gamma) = d\beta$
closed ⇏ exact
but it does on $\mathbb{R}^n$
topology of M

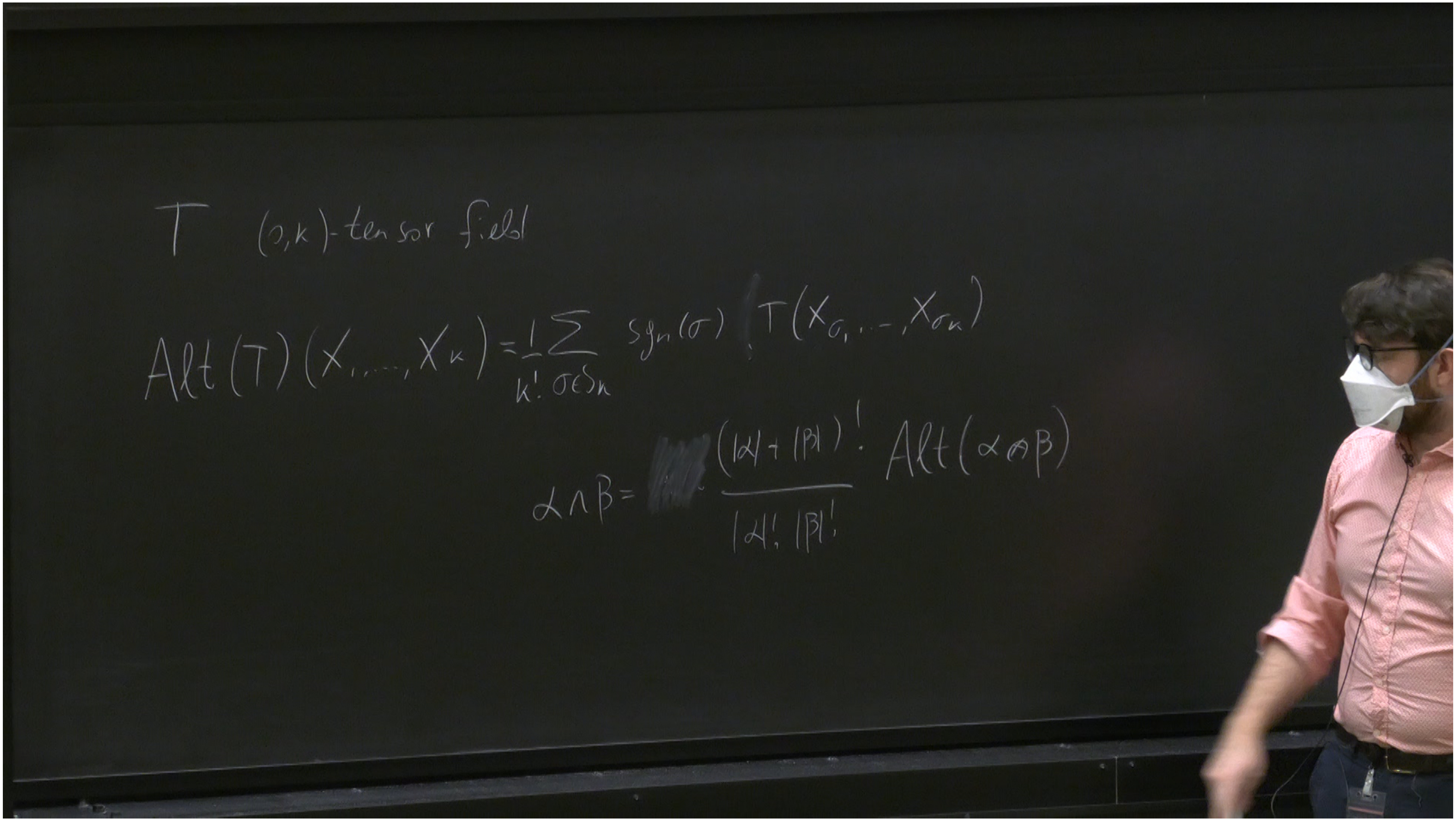


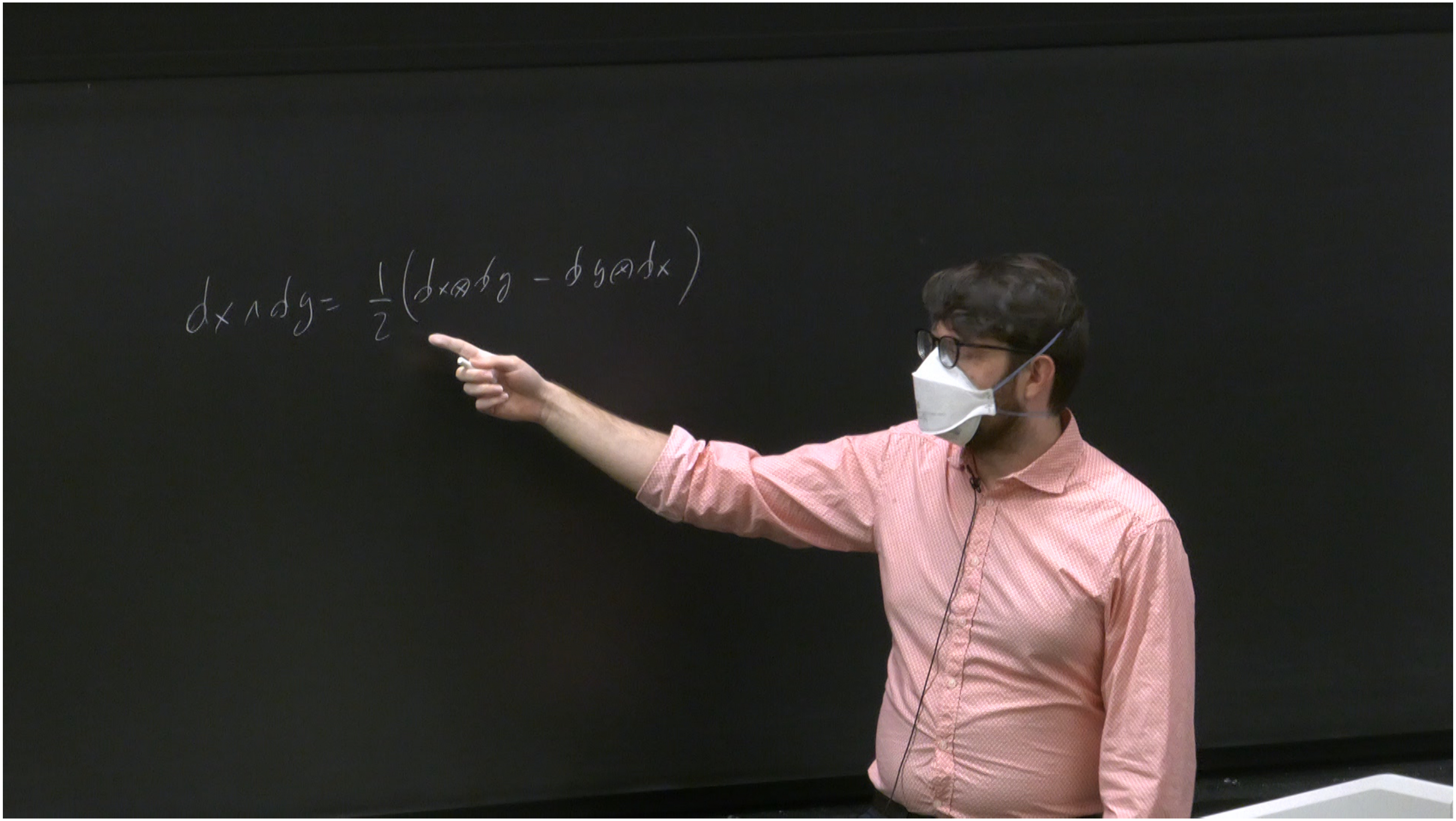

top forms
# gen. Ω^k(M) is (n k)
at most get dim(M)-forms

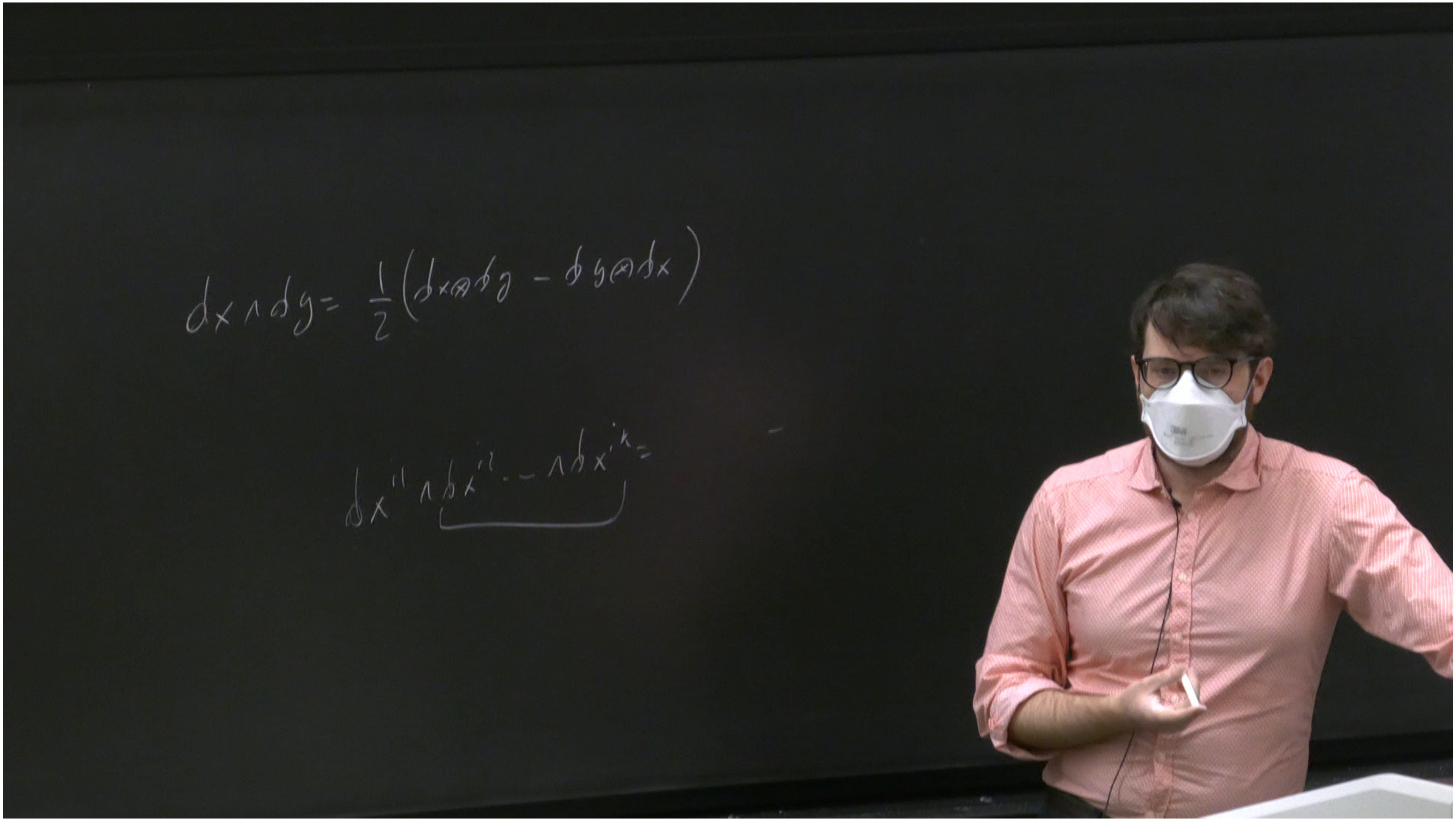

top forms
# gen... $\Omega^k(M)$ is $\binom{n}{k}$
at most get dim(M)-forms $\Rightarrow$ $\binom{n}{n} = 1$ generators
top forms
$\omega = f\, dx^1 \wedge \dots \wedge$


ω = f dx¹∧...∧dxⁿ
change basis ((x^i) → (y^i))
dx^1∧...∧dx^n = ∂x^1/∂y^{i_1} ∂x^2/∂y^{i_2} ... ∂x^n/∂y^{i_n} dy^{i_1}∧...∧dy^{i_n}
... = det(∂x/∂y) dy^1∧...∧dy^n


Ω^{k-1}(M)
γ) = dβ
does closed ⇒ exact?
→ depends on topology
de Rham cohomology


T (0,k)-tensor field
$\mathrm{Alt}(T)(X_1,\dots,X_k) = \frac{1}{k!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, T(X_{\sigma_1},\dots,X_{\sigma_n})$
$\alpha\wedge\beta = \frac{(|\alpha|+|\beta|)!}{|\alpha|!\,|\beta|!}\,\mathrm{Alt}(\alpha\otimes\beta)$


$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$


$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$
$dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k} =$