Title: Mathematical Physics Core Lecture

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$\boxed{\text{Frame fields}}$ + pushforward + pull back + differ

extend $\otimes$ to tensor fields

$(T \otimes S) : a \in M \mapsto (a, T_a \otimes S_a) \in (\ldots)$

$\{E_i\}_{i=1}^{m}$ f

T tensor

...ck + differential forms

$E_i : U \subseteq M \to TM$

$\{E_i\}_{i=1}^{m}$ frame $\{\varepsilon^i\}$ dual co-frame

T tensor field of type (r,s)

$$T = \underbrace{T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}}_{\in C^\infty(U,\mathbb{R})} E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

local chart $(U, (x^0, \dots, x^{n-1}))$

$\{dx^0, dx^1, \dots, dx^{n-1}\}$ co-frame

$$\boxed{g|_U = g_{\mu\nu}\, dx^\mu \otimes dx^\nu}$$

$$g_{\mu\nu} \in C^\infty(U, \mathbb{R})$$

$$g_a = g_{\mu\nu}(a)\, dx^\mu_a \otimes dx^\nu_a$$

$$g_a(v_1, v_2) = 0 \quad \forall v_2 \in T_a M$$

$$\Rightarrow v_1 = 0$$

$f: M \to N$ smooth. Derivative of f at $a \in M$?

$f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ differentiable at x_0 if there is a linear map

turns out $df_{x_0}(v) = v^i \dfrac{\partial f^j}{\partial x^i}(x_0)\, e_j$ → Jacobian matrix

$v = v^i e_i$

M?

liner map $df_{x_0} : \mathbb{R}^m \to \mathbb{R}^n$ s.t. $\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - df_{x_0}(x - x_0) \|}{\| x - x_0 \|} = 0$

matrix

$F : M \to N$ smooth , $a \in M$

$dF_a : T_aM \to T_{F(a)}N$ linear

$dF_a(v)$ derivation at $F(a)$

$\left(dF_a(v)\right) f = v\left(\underbrace{f \circ F}_{\in C^\infty(M,\mathbb{R})}\right) \in \mathbb{R}$

$f \in C^\infty(N,\mathbb{R})$

$\underbrace{\left(dF_a(v)\right)}_{X}$

dF_a symbol

$dx^i_a : T_aM \to \mathbb{R}$

$x^i : U \subseteq M \to \mathbb{R}$

$\ldots F) \in \mathbb{R}$

$\mathcal{C}^\infty(M,\mathbb{R})$

$$\underbrace{(dF_a(v))}_{X}(fg) = v((fg)\circ F) = v((f\circ F)(g\circ F)) \qquad v \in T_aM$$

$$= \underbrace{(f\circ F)(a)}_{f(F(a))}\, v(g\circ F) + (g\circ F)(a)\, \underbrace{v(f\circ F)}$$

$$f(F(a)) \underbrace{(dF_a(v))(g)}_{X(g)} + \ldots + g(F(a))\, X(f)$$

$F : M \to N$ smooth, $a \in M$

$dF_a : T_aM \to T_{F(a)}N$ linear

$dF_a(v)$ derivation at $F(a)$

pushforward of F at a

dF_a symbol

$x^i : U \subseteq M \to \mathbb{R}$

$dx^i_a : T_aM \to \mathbb{R}$

$$(dF_a(v))f = v(\underbrace{f \circ F}_{\in \mathcal{C}^\infty(M,\mathbb{R})}) \in \mathbb{R}$$

$\mathcal{C}^\infty(N,\mathbb{R})$

$$\underbrace{(dF_a(v))}_{X}$$

$$\ldots_{x_n}(v) = v \left(\frac{\partial F}{\partial x^i}(x_n)\right) e_j$$

$$v = v^i e_i$$

coordinate charts $\quad (U, \varphi = (x^1, \dots, x^m)) \in \mathcal{A}_M \quad (V, \psi = (y^1, \dots, y^n)) \in \mathcal{A}_N \quad$ &

$$\left(dF_a\left(\frac{\partial}{\partial x^i}\Big|_a\right)\right)(f) = \frac{\partial}{\partial x^i}\Big|_a (f \circ F) = D_i(f \circ F \circ \varphi^{-1})(\varphi(a)) = D_i(\underbrace{f \circ \psi^{-1}} \circ \overbrace{\psi \circ F \circ \varphi^{-1}})(\ldots$$

$$dF_a\left(\frac{\partial}{\partial x^i}\Big|_a\right) = \left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{a})\right) \frac{\partial}{\partial y^j}\Big|_{F(a)}$$

$(y^1,\dots,y^n)) \in \mathcal{A}_N \qquad q \in U, \quad F(q) \in V$

$$= D_i\big(f\circ\psi^{-1}\circ\psi\circ F\circ\varphi^{-1}\big)(\varphi(q)) = \boxed{D_j\big(f\circ\psi^{-1}\big)(\psi(F(q)))}\, D_i\Big(\big(\overbrace{\psi\circ F\circ\varphi^{-1}}^{\hat F}\big)^j\Big)\big(\overbrace{\varphi(q)}^{\hat q}\big)$$

$$= \frac{\partial}{\partial y^j}\Big|_{F(q)}(f)\, \frac{\partial \hat F^j}{\partial x^i}(\hat q)$$

pullback $\quad F: M \to N$ smooth $\to$ takes some fields on N

- $f: N \to \mathbb{R}$ smooth, its pullback by F is $F^* f = f \circ F : M \to \mathbb{R}$
- $\alpha: N \to TN$ covector field, $F^*\alpha : M \to TM \quad , \quad \underbrace{(F^*\alpha)_e (v)}_{\in T_e M} = \alpha_{F(e)}\big(\underbrace{dF_e(}_{\in T_{F(e)}}$

$M \to N$ smooth $\to$ takes some fields on N to fields on M

its pullback by F is $F^* f = f \circ F : M \to \mathbb{R}$

field, $F^* \alpha : M \to TM$, $(F^* \alpha)_x (v) = \alpha_{F(x)} (dF_x (v))$

$v \in T_x M$, $dF_x(v) \in T_{F(x)} N$

pushforward of F at a | $x^i : U \subseteq M \to \mathbb{R}$ $dx^i_a : T_aM \to \mathbb{R}$

pullback $F : M \to N$ smooth → takes some fields on N to fields on M

- $f : N \to \mathbb{R}$ smooth, its pullback by F is $F^*f = f \circ F : M \to \mathbb{R}$
- $\alpha : N \to TN$ covector field, $F^*\alpha : M \to TM$, $\underbrace{(F^*\alpha)_a(v)}_{\in T_aM} = \alpha_{F(a)}\big(\underbrace{dF_a(v)}_{\in T_{F(a)}N}\big)$

$v = v^i e_i$

- $T : N \to T^{0,s}N$ tensorfield , $F^*T : M \to T^{0,s}M$

$$(F^*T)_p(v_1, v_2, \dots, v_s) = T_{F(p)}(dF_p(v_1), \dots, dF_p(v_s))$$

Property: pullback is linear

in coordinates

$$F^* dy^i \qquad dy^i : V \subseteq N \to TN$$

$$\left(F^* dy^i\right)_e \left(\frac{\partial}{\partial x^j}\Big|_e\right) = dy^i_{F(e)} \left(dF_e \left(\frac{\partial}{\partial x^j}\Big|_e\right)\right) = dy^i_{F(e)} \left(\frac{\partial \hat{F}^k}{\partial x^j}(\hat{e}) \frac{\partial}{\partial y^k}\Big|_{F(e)}\right)$$

$$= \frac{\partial \hat{F}^i}{\partial x^j}(\hat{e})$$

$$\boxed{F^* dy^i = \frac{\partial \hat{F}^i}{\partial x^j} dx^j}$$

<u>other properties</u>

<u>ex</u>

- $f: N \to \mathbb{R}$, $T: N \to T^{r,s}N$ $\qquad F^*(fT) = (F^*f)(F^*T)$
- $F^*(T \otimes S) = (F^*T) \otimes (F^*S)$

ex $T = T_{ij}\, dy^i \otimes dy^j$

$$F^*T = (F^*T_{ij})(F^*dy^i)\otimes(F^*dy^j)$$

$$\downarrow$$

$$T_{ij} \circ F$$